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BUXORO DAVLAT UNIVERSITETI ILMIY AXBOROTI



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EXACT BOUND-STATE AND THRESHOLD ANALYSIS FOR A RANK-ONE TWO-FERMION LATTICE HAMILTONIAN

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Abstract. *We study a one-parameter family of discrete Schrödinger operators associated with two identical fermions on the one-dimensional lattice. After the Floquet–Bloch decomposition, the fiber Hamiltonian at total quasi-momentum $k \in \mathbb{T}$ acts in the odd subspace of $L^2(\mathbb{T})$ and is a rank-one perturbation of a multiplication operator. The perturbation is generated by the second-neighbour form factor $\sin 2q$. We determine the essential spectrum, derive the Birman–Schwinger–Fredholm determinant, and evaluate it explicitly below the spectral band. For every $k \in (-\pi, \pi)$ we identify the critical coupling $\mu_*(k) = 1/\pi \cos k/2$. No eigenvalue occurs below the essential spectrum for $0 < \mu \le \mu_*(k)$, whereas for $\mu < \mu_*(k)$ there is exactly one simple eigenvalue, given in closed form. At the critical coupling the lower band edge is a threshold resonance rather than an L^2 eigenvalue. We formulate this statement in a scale of weighted Hilbert spaces adapted to the two threshold points of the dispersion relation. The degenerate fiber $k = \pi$ is treated separately: its free dispersion is constant, and every attractive coupling produces one simple eigenvalue. We also obtain the exact threshold expansion and the strong-coupling asymptotics of the bound-state energy*

Keywords: *two-fermion lattice system, discrete Schrödinger operator, rank-one perturbation, essential spectrum, bound state, threshold resonance, Fredholm determinant.*

RANGI BIRGA TENG IKKI FERMIONLI PANJARA HAMILTONIANI UCHUN BOG'LANGAN HOLATLAR VA BO'SAG'AVIY HODISALARNING ANIQ TAHLILI

Annotatsiya. *Biz bir o'lchamli panjaradagi ikkita bir xil fermion bilan bog'liq bo'lgan diskret Schrödinger operatorlarining bir parametrli oilasini o'rganamiz . Floquet–Bloch ajratishidan so'ng, to'la kvaziimpulsi $k \in \mathbb{T}$ bo'lgan tola Hamiltoniani $L^2(\mathbb{T})$ fazosining toq qismida ta'sir qiladi va ko'paytirish operatorining bir rangli qo'zg'atilishi hisoblanadi. Qo'zg'atilish ikkinchi qo'shni shakl funksiyasi $\sin 2q$ orqali hosil qilinadi. Biz muhim spektrni aniqlaymiz, Birman–Schwinger–Fredholm determinantini keltirib chiqaramiz va uni spektral yo'lakdan pastda oshkor ko'rinishda hisoblaymiz. Har bir $k \in (-\pi, \pi)$ uchun kritik bog'lanish koeffitsiyentini $\mu_*(k) = 1/\pi \cos k/2$ ko'rinishida topamiz. $0 < \mu \le \mu_*(k)$ bo'lganda muhim spektrdan pastda hech qanday xos qiymat mavjud emas. Ammo, $\mu > \mu_*(k)$ bo'lganda, muhim spektrdan pastda aniq bitta oddiy xos qiymat mavjud bo'ladi va u oshkor formula orqali ifodalanadi. Kritik bog'lanish holatida quyi spektral yo'lak chegarasi L^2 fazosidagi xos qiymat emas, balki bo'sag'aviy rezonans hisoblanadi. Biz bu holatni dispersiya munosabatining ikkita chega nuqtasiga moslashtirilgan vaznli Hilbert fazolari shkalasi yordamida ifodalaymiz. Aynigan $k = \pi$ tolasi alohida ko'rib chiqiladi: bu holda erkin dispersiya doimiy bo'ladi va har qanday tortishuvchi bog'lanish bitta sodda xos qiymatni hosil qiladi. Shuningdek, bog'langan holat energiyasining bo**'sag'a** yaqinidagi aniq yoyilmasi hamda kuchli bog'lanishdagi asimptotik xulqini ham olamiz.*

Kalit so'zlar: *ikki fermionli sistema, diskret Schrödinger operatori, bir rangli qo'zg'alish, muhim spektr, bog'langan holat , bo'sag'aviy rezonans, Fredholm determinanti.*

ТОЧНЫЙ АНАЛИЗ СВЯЗАННЫХ СОСТОЯНИЙ И ПОРОГОВЫХ РЕЗОНАНСОВ ДЛЯ РЕШЁТОЧНОГО ГАМИЛЬТОНИАНА ДВУХ ФЕРМИОНОВ РАНГА ОДИН

Аннотация. *Мы исследуем однопараметрическое семейство дискретных операторов Шрёдингера, связанных с двумя идентичными фермионами на одномерной решётке. После разложения Флоке–Блоха фибровый гамильтониан при полном квазиимпульсе $k \in \mathbb{T}$ действует в нечётном подпространстве пространства $L^2(\mathbb{T})$ и представляет собой возмущение ранга один оператора умножения. Возмущение порождается форм-фактором второй соседней связи $\sin 2q$. Мы определяем существенный спектр, выводим определитель Бирмана–Швингера–Фредгольма и вычисляем его в явном виде ниже спектральной зоны. Для каждого $k \in (-\pi, \pi)$ мы находим критическую константу связи $\mu_*(k) = 1/\pi \cos k/2$. При $0 < \mu \leq \mu_*(k)$ ниже существенного спектра собственные значения отсутствуют. В то же время при $\mu > \mu_*(k)$ существует ровно одно простое собственное значение, которое задаётся в явном виде. При критическом значении константы связи нижняя граница спектральной зоны является не собственным значением в пространстве L^2, а пороговым резонансом. Это утверждение формулируется в шкале взвешенных гильбертовых пространств, адаптированной к двум пороговым точкам дисперсионного соотношения. Вырожденный случай $k = \pi$ рассматривается отдельно: свободная дисперсия является постоянной, и любое притягивающее взаимодействие порождает одно простое собственное значение. Кроме того, получены точное пороговое разложение и асимптотика энергии связанного состояния при сильной связи.*

Ключевые слова: *двухфермионная решёточная система, дискретный оператор Шрёдингера, возмущение ранга один, существенный спектр, связанное состояние, пороговый резонанс, определитель Фредгольма.*

Introduction. Discrete Schrödinger operators provide a natural framework for quantum particles moving in crystalline and optical-lattice media. In contrast with continuum models, lattice Hamiltonians have bounded dispersion relations, and translation invariance leads to a Floquet–Bloch decomposition into fiber operators depending nontrivially on the conserved total quasi-momentum. This dependence is responsible for several lattice-specific spetral phenomena, including quasi-momentum-dependent binding thresholds and threshold states; see, for example, [1–5]. For two identical fermions the relative wave function is odd. Consequently, an on-site interaction does not contribute in the antisymmetric sector, while interactions supported on nonzero lattice separations give rise, after Fourier transformation, to odd trigonometric form factors. Finite-range interactions therefore become finite-rank perturbations of the fiber kinetic-energy operator. This reduction makes it possible to replace the spectral problem outside the essential spectrum by a finite-dimensional Birman–Schwinger equation. The present paper considers the simplest nontrivial rank-one case generated by the secondneighbour form factor $\sin 2q$. The resulting fiber Hamiltonian is

$$h_\mu(k) = h_0(k) - \mu v, \qquad \mu > 0,$$

acting in $L^2_{\mathrm{odd}}(\mathbb{T})$, where

$$\varepsilon_k(q) = 2 - 2\cos\frac{k}{2}\cos 2q$$

is the free dispersion and

$$(vf)(q) = \sin 2q \int_{\mathbb{T}} \sin 2t\, f(t)\mathrm{d}t.$$

The minus sign corresponds to an attractive interaction. Since v has rank one, there can be at most one eigenvalue below the spectral band. The main questions are therefore whether such an eigenvalue exists, where it is located, and what happens at the critical coupling at which it reaches the lower band edge.

Our analysis gives complete answers. For $k \in (-\pi, \pi)$ the lower and upper band edges are

$$\varepsilon_{\min}(k) = 2\left(1 - \cos\frac{k}{2}\right), \qquad \varepsilon_{\max}(k) = 2\left(1 + \cos\frac{k}{2}\right).$$

The critical coupling is $\mu_*(k) = \pi^{-1}\cos\frac{k}{2}$. For $\mu > \mu_*(k)$ the unique eigenvalue is

$$z_\mu(k) = 2 - \frac{\cos^2(k/2) + \pi^2\mu^2}{\pi\mu},$$

and its distance from the lower band edge admits the exact identity

$$z_\mu(k) - \varepsilon_{\min}(k) = -\frac{(\pi\mu - \cos(k/2))^2}{\pi\mu}.$$

At $\mu = \mu_*(k)$ the threshold solution is proportional to $\cot q$. It does not belong to the physical space $L^2(\mathbb{T})$, but it belongs to every weighted space whose weight vanishes at the threshold set with exponent greater than

$1/2$. This gives a precise weighted-space characterization of the threshold resonance. The exceptional fiber $k = \pi$ has the one-point essential spectrum $\{2\}$ and the eigenvalue $2 - \pi\mu$ for every $\mu > 0$.

The paper is organized as follows. Section 2 introduces the fiber Hamiltonian and determines its essential spectrum. Section 3 establishes the rank-one Birman–Schwinger principle and evaluates the Fredholm determinant. Section 4 contains the complete eigenvalue classification. Threshold resonances and asymptotic formulas are discussed in Sections 5 and 6.

The fiber Hamiltonian and its essential spectrum

Let $\mathbb{T} = (-\pi, \pi]$ be the one-dimensional torus equipped with Lebesgue measure. Set

$$L^2_{\mathrm{odd}}(\mathbb{T}) := \{f \in L^2(\mathbb{T}): f(-q) = -f(q) \text{ for a. e. } q \in \mathbb{T}\}.$$

This is a closed subspace of $L^2(\mathbb{T})$and hence a Hilbert space.

For $k \in \mathbb{T}$, define

$$\varepsilon_k(q) := 2 - 2c_k \cos 2q\,, \qquad c_k := \cos\frac{k}{2} \geq 0. \tag{1}$$

The free fiber operator $h_0(k)$ is multiplication by ε_k:

$$(h_0(k)f)(q) = \varepsilon_k(q)f(q).$$

Let $\phi(q) := \sin 2q$ and define

$$vf = (f, \phi)\phi, \;\; (vf)(q) = \sin 2q \int_{\mathbb{T}} \sin 2t\, f(t)\mathrm{d}t. \tag{2}$$

Since $\|\phi\|^2 = \int_{-\pi}^{\pi} \sin^2 2q\, \mathrm{d}q = \pi$, the operator v is nonnegative, self-adjoint, and of rank one. For $\mu > 0$ we consider

$$h_\mu(k) := h_0(k) - \mu v. \tag{3}$$

Proposition 2.1. For every $k \in \mathbb{T}$ and $\mu > 0$, the operator $h_\mu(k)$ is bounded and self-adjoint in $L^2_{\mathrm{odd}}(\mathbb{T})$.

Proof. The function $\boldsymbol{\varepsilon_k}$ is real-valued and bounded, so $h_0(k)$ is bounded and self-adjoint. The operator $v = |\phi\rangle\langle\phi|$ is bounded and self-adjoint. Hence so is $h_\mu(k)$.

Proposition 2.2. For every $k \in \mathbb{T}$ and $\mu > 0$,

$$\sigma_{\mathrm{ess}}\left(h_\mu(k)\right) = \sigma\big(h_0(k)\big) = [\varepsilon_{\min}(k), \varepsilon_{\max}(k)], \tag{4}$$

where

$$\varepsilon_{\min}(k) = 2(1 - c_k), \quad \varepsilon_{\max}(k) = 2(1 + c_k). \tag{5}$$

For $k = \pi$ the interval degenerates to the singleton $\{2\}$.

Proof. The spectrum of a multiplication operator equals the essential range of its multiplier. Since $\boldsymbol{\varepsilon_k}$ is continuous and $\cos 2q$ ranges over $[-1,1]$, formulas (5) follow. The perturbation $-\mu v$ has finite rank and is therefore compact. Weyl's theorem gives (4).

Birman–Schwinger reduction and the Fredholm determinant

Fix $k \in \mathbb{T}$ and let

$$z \in \mathbb{C}\backslash[\varepsilon_{\min}(k), \varepsilon_{\max}(k)]$$

Set

$$d(k, z) := \int_{\mathbb{T}} \frac{\sin^2 2q}{\varepsilon_k(q) - z}\mathrm{d}q \tag{6}$$

and

$$\Delta_\mu(k, z) := 1 - \mu d(k, z). \tag{7}$$

Lemma 3.1 (Rank-one Birman–Schwinger principle). A number $z \notin [\varepsilon_{\min}(k), \varepsilon_{\max}(k)]$ is an eigenvalue of $h_\mu(k)$ if and only if $\Delta_\mu(k, z) = 0$.

Every such eigenvalue is simple, and an associated eigenfunction is

$$f_{\mu,k,z}(q) = C\frac{\sin 2q}{\varepsilon_k(q) - z}, \qquad C \neq 0 \tag{8}$$

Proof. The equation $\big(h_\mu(k) - z\big)f = 0$ is equivalent to

$$f = \mu(h_0(k) - z)^{-1}(f, \phi)\phi.$$

If $(f, \phi) = 0$, then $f = 0$, so a nonzero solution must satisfy $(f, \phi) \neq 0$. Taking the scalar product with ϕ gives $1 = \mu d(k, z)$, which is (7). Conversely, if (7) vanishes, then (8) is a nonzero eigenfunction. The eigenspace is contained in the one-dimensional range of
$(h_0(k) - z)^{-1}v$, so the eigenvalue is simple.■

For $z < \varepsilon_{\min}(k)$ the function $d(k, z)$ is positive and strictly increasing in z, since

$$\partial_z(k, z) = \int_{\mathbb{T}} \frac{\sin^2 2q}{(\varepsilon_k(q) - z)^2}\, dq > 0. \tag{9}$$

Consequently, $\Delta_\mu(k,z)$ is strictly decreasing, with

$$\lim_{z\to-\infty}\Delta_\mu(k,z)=1.$$

Lemma 3.2 (Explicit determinant). Let $k\in(-\pi,\pi)$, so that $c_k>0$, and let $z<\varepsilon_{\min}(k)$. Put

$$A=A(k,z):=\frac{2-z}{2c_k}>1.$$

Then

$$d(k,z)=\frac{\pi}{c_k}\left(A-\sqrt{A^2-1}\right)=\frac{\pi}{2{c_k}^2}\left(2-z-\sqrt{(2-z)^2-4{c_k}^2}\right) \qquad (10)$$

Proof. Using the π - periodicity of the integrand and the substitution $x=2q$ on one period, we obtain

$$d(k,z)=\frac{1}{2c_k}\int_{-\pi}^{\pi}\frac{\sin^2 2q}{A-\cos x}\,\mathrm{d}x.$$

Since

$$\frac{\sin^2 2q}{A-\cos x}=A+\cos x-\frac{A^2-1}{A-\cos x}$$

and

$$\int_{-\pi}^{\pi}\frac{dx}{A-\cos x}=\frac{2\pi}{\sqrt{A^2-1}},$$

we find

$$\int_{-\pi}^{\pi}\frac{\sin^2 2q}{A-\cos x}\,\mathrm{d}x=2\pi\left(A-\sqrt{A^2-1}\right),$$

which proves (10).■

Taking $z\uparrow\varepsilon_{\min}(k)$ in (10) yields

$$d\big(k,\varepsilon_{\min}(k)\big):=\lim_{z\uparrow\varepsilon_{\min}(k)}d(k,z)=\frac{\pi}{c_k}. \qquad (11)$$

This motivates the critical coupling

$$\mu_*(k):=\frac{c_k}{\pi},\qquad k\in(-\pi,\pi) \qquad (12)$$

Exact classification of the discrete spectrum

Theorem 4.1. Let $k\in(-\pi,\pi)$ and $\mu>0$.

(i) If $0<\mu\le\mu_*(k)$, then

$$\sigma_{\mathrm{disc}}\left(h_\mu(k)\right)\cap\left(-\infty,\varepsilon_{\min}(k)\right)=\emptyset.$$

(ii) If $\mu>\mu_*(k)$, then $h_\mu(k)$ has exactly one eigenvalue below its essential spectrum. It is simple and equals

$$z_\mu(k)=2-\frac{{c_k}^2+\pi^2\mu^2}{\pi\mu} \qquad (13)$$

Moreover,

$$z_\mu(k)-\varepsilon_{\min}(k)=-\frac{(\pi\mu-c_k)^2}{\pi\mu}<0 \qquad (14)$$

An associated eigenfunction is

$$f_{\mu,k}(q)=C\,\frac{\sin 2q}{\varepsilon_k(q)-z_\mu(k)},\qquad C\neq 0. \qquad (15)$$

There are no eigenvalues above $\varepsilon_{\max}(k)$.

Proof. By (11),

$$\lim_{z\uparrow\varepsilon_{\min}(k)}\Delta_\mu(k,z)=1-\frac{\mu}{\mu_*(k)}.$$

Since $\Delta_\mu(k,\cdot)$ is strictly decreasing on $\left(-\infty,\varepsilon_{\min}(k)\right)$ and tends to 1 at $-\infty$, it has no zero when $\mu\le\mu_*(k)$ and exactly one zero when $\mu>\mu_*(k)$.

To compute the zero, write the equation $1=\mu d(k,z)$ using (10):

$$A-\sqrt{A^2-1}=\frac{c_k}{\pi\mu}.$$

Since $\mu>c_k/\pi$, the right-hand side lies in $(0,1)$. Solving gives

$$A=\frac{1}{2}\left(\frac{\pi\mu}{c_k}+\frac{c_k}{\pi\mu}\right).$$

As $z = 2 - 2c_k A$, formula (13) follows. Identity (14) is immediate. For $z > \varepsilon_{\max}(k)$ one has $d(k,z) < 0$, whereas $\mu > 0$; hence $1 - \mu d(k,z) > 1$ and the determinant cannot vanish.

Remark 4.2. Because the perturbation is rank one and attractive, the absence of an upper eigenvalue is also a direct consequence of the min–max principle.

4.1 The degenerate fiber $k = \pi$.

At $k = \pi$ one has $c_k = 0$ and $\varepsilon_\pi(q) \equiv 2$. Thus $h_0(\pi) = 2I$ and

$$h_\mu(\pi) = 2I - \mu|\phi\rangle\langle\phi|.$$

Theorem 4.3. For every $\mu > 0$,

$$\sigma_{\text{ess}}\left(h_\mu(\pi)\right) = \{2\},$$

and $h_\mu(\pi)$ has exactly one simple eigenvalue

$$z_\mu(\pi) = 2 - \pi\mu. \tag{16}$$

The corresponding eigenspace is span$\{\sin 2q\}$.

Proof. The vector ϕ satisfies $v\phi = \|\phi\|^2\phi = \pi\phi$, so $h_\mu(\pi)\phi = (2 - \pi\mu)\phi$. On $\phi^\perp$ the operator equals $2I$. The assertion follows.■

Weighted Hilbert spaces and the threshold resonance

For $k \in (-\pi, \pi)$ the minimum of the dispersion relation is attained at the threshold set

$$M_- := \{q \in T : \cos 2q = 1\} = \{0, \pi\},$$

where the points π and $-\pi$ are identified on the torus. A convenient weight measuring the distance from this set is

$$\rho(q) := |\sin q|. \tag{17}$$

For $s \in R$ define

$$\mathcal{H}_s := L^2_s(\mathbb{T}) := \{f : \mathbb{T} \to \mathbb{C} : \ \rho^s f \in L^2(\mathbb{T})\}, \qquad \|f\|_s := \|\rho^s f\|_{L^2(\mathbb{T})}. \tag{18}$$

Functions are identified modulo equality almost everywhere. With the inner product

$$(f,g)_s := \int_{\mathbb{T}} \rho(q)^{2s} f(q)\overline{g(q)}\,\mathrm{d}q,$$

$\mathcal{H}_s$ is a Hilbert space. Notice that $\mathcal{H}_0 = L^2(\mathbb{T})$, and, for $s_2 > s_1$, one has the continuous inclusion $\mathcal{H}_{s_1} \subset \mathcal{H}_{s_2}$. We write

$$\mathcal{H}_s^{\text{odd}} := \mathcal{H}_s \cap L^0_{\text{odd}}(\mathbb{T}),$$

Where $L^0_{\text{odd}}(\mathbb{T})$ denotes the measurable odd functions on the torus.

Definition 5.1 (Lower-threshold resonance). Let $k \in (-\pi, \pi)$. The lower edge $\varepsilon_{\min}(k)$ is called a threshold resonance of $h_\mu(k)$ if there exists a nonzero measurable odd function f such that

(i) $f \in \mathcal{H}_s^{\text{odd}}$ for every $s > \frac{1}{2}$, but $f \notin L^2_{\text{odd}}(\mathbb{T})$;

(ii) the scalar integral $\int_{\mathbb{T}} \sin 2t\, f(t)\,\mathrm{d}t$ exists;

(iii) the threshold equation

$$\big(\varepsilon_k(q) - \varepsilon_{\min}(k)\big)f(q) - \mu \sin 2q = 0 \tag{19}$$

holds for almost every $q \in \mathbb{T}$. If a nonzero solution of (19) belongs to $L^2_{\text{odd}}(\mathbb{T})$, then $\varepsilon_{\min}(k)$ is instead called a threshold eigenvalue.

The exponent $1/2$ in Definition 5.1 is dictated by the one-dimensional singularity $|q - q_0|^{-1}$ at each point $q_0 \in M_-$. Thus the definition distinguishes the borderline non-square-integrable state from an actual eigenfunction.

Theorem 5.2. Let $k \in (-\pi, \pi)$. Then the following assertions hold.

(i) At $\mu = \mu_*(k)$, the lower edge $\varepsilon_{\min}(k)$ is a threshold resonance of $h_{\mu_*(k)}(k)$.

(ii) The corresponding resonant state is unique up to multiplication by a nonzero constant and is given by

$$f_*(q) = \frac{C}{2c_k}\cot q\,, \qquad C \neq 0. \tag{20}$$

It satisfies

$$f_* \in \mathcal{H}_s^{\text{odd}} \text{ for every } s > \tfrac{1}{2}\,,\ f_* \notin L^2_{\text{odd}}(\mathbb{T}). \tag{21}$$

(iii) The lower edge is not a threshold eigenvalue. For $\mu \neq \mu_*(k)$ there is no nonzero solution of (19) of Birman–Schwinger form.

Proof. Since

$$\varepsilon_k(q) - \varepsilon_{\min}(k) = 2c_k(1 - \cos 2q) = 4c_k \sin^2 2q, \tag{22}$$

any nonzero threshold solution obtained from the rank-one equation must be of the form

$$f(q) = C\frac{\sin 2q}{\varepsilon_k(q) - \varepsilon_{\min}(k)} = \frac{C}{2c_k}\cot q.$$

For this function the interaction functional is well defined because

$$\sin 2q\, f_*(q) = \frac{C}{c_k}\cos^2 2q \in L^1(\mathbb{T}).$$

Moreover,

$$\int_{\mathbb{T}} \sin 2q\, f_*(q)\mathrm{d}q = \frac{C}{c_k}\int_{-\pi}^{\pi} \cos^2 2q\mathrm{d}q = \frac{C\pi}{c_k}.$$

Substituting into (19) gives

$$C\sin 2q - \mu\frac{C}{c_k}\sin 2q = 0$$

for almost every q. Hence a nonzero threshold solution exists precisely when

$$\mu = \frac{c_k}{\pi} = \mu_*(k).$$

It remains to determine its weighted integrability. Since $|\cot q| \asymp |\sin q|^{-1}$ near both points of M_-,

$$\rho(q)^{2s}|f_*(q)|^2 \asymp |\sin q|^{2s-2}.$$

The latter function is integrable near a zero of $\sin q$ if and only if $2s - 2 > -1$, equivalently $s > 1/2$. This proves (21). In particular, the choice $s = 0$ is not admissible, so $f_* \notin L^2(\mathbb{T})$ and the lower edge is not an eigenvalue. Finally, the rank-one threshold equation determines every nonzero solution up to a scalar multiple, and the preceding compatibility condition forces $\mu = \mu_*(k)$. The proof is complete.■

Remark 5.3. The particular choice $\rho(q) = |\sin q|$ is not essential. Any bounded nonnegative weight that is comparable to the distance from M_- in neighbourhoods of 0 and π produces the same resonance classification and the same critical exponent $s = 1/2$.

Threshold and strong-coupling asymptotics

The exact identity (14) yields the threshold asymptotics without an implicit-function argument.

Corollary 6.1 (Threshold expansion). Let $k \in (-\pi, \pi)$ and put $c_k = \cos(k/2)$. For

$$\delta := \mu - \mu_*(k) > 0, \qquad |\delta| < \frac{c_k}{\pi},$$

one has the convergent expansion

$$z_\mu(k) = \varepsilon_{\min}(k) + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{\pi^n}{{c_k}^{n-1}}\delta^n. \tag{23}$$

In particular,

$$z_\mu(k) - \varepsilon_{\min}(k) = -\frac{\pi^2}{c_k}\delta^2 + \frac{\pi^3}{{c_k}^2}\delta^3 + O\delta^4, \delta \downarrow 0. \tag{24}$$

Thus the eigenvalue bifurcates from the lower threshold quadratically.

Proof. By (14) and $\pi\mu = c_k + \pi\delta$,

$$z_\mu(k) - \varepsilon_{\min}(k) = -\frac{\pi^2\delta^2}{c_k + \pi\delta}.$$

Expanding $(c_k + \pi\delta)^{-1}$ as a geometric series proves the claim.■

Corollary 6.2 (Strong-coupling asymptotics). For every $k \in \mathbb{T}$,

$$z_\mu(k) = -\pi\mu + 2 + O(\mu^{-1}), \qquad \mu \to +\infty$$

In particular,

$$\lim_{\mu\to+\infty}\frac{z_\mu(k)}{\mu} = -\pi.$$

Concluding remarks. The model admits a complete spectral solution because the fermionic interaction is rank one in each fiber. The total quasi-momentum changes the width of the free spectral band through $c_k = \cos(k/2)$ and simultaneously determines the binding threshold $\mu_*(k) = c_k/\pi$. The critical coupling tends to zero as $k \to \pi$, consistently with the collapse of the free band to the point 2. At nondegenerate quasi-momenta the critical state lies in every weighted space $\mathcal{H}_s^{\mathrm{modd}}$ with $s > 1/2$ but not in the physical space $L^2_{\mathrm{modd}}(\mathbb{T})$. Thus it is a genuine threshold resonance rather than a threshold eigenstate. Above the critical coupling a unique simple bound state appears with a quadratic bifurcation law. The same strategy extends to several finite-range fermionic form factors. In that case the scalar determinant is replaced by a finite-dimensional Fredholm determinant, and the number of eigenvalues can be studied through the inertia and threshold limits of the corresponding Birman–Schwinger matrix.

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CARLEMAN FORMULA FOR THE PROBLEM OF CONTINUING THE SOLUTION OF A ONE-VARIABLE METAANALYTIC EQUATION FROM A PART OF THE BOUNDARY OF THE DOMAIN

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Abstract. *This paper is dedicated to one of the fundamental and complex problems in the theory of partial differential equations and complex analysis: the problem of analytic continuation for the class of n-metaanalytic functions within a bounded domain D of the complex plane. We rigorously investigate the boundary value problem where an n-metaanalytic function $f(z)$ is reconstructed from the known values of its differential operators $L_s f(t)$ specified on a subset E of the boundary ∂D. First, we establish a comprehensive uniqueness theorem in the function class $M_n(D) \cap C^{n-1}(\bar{D})$ by utilizing an auxiliary lemma concerning the properties of the $\bar{\partial}_z$ operator and an inductive method. Furthermore, we prove the severe ill-posedness of this boundary value problem in the sense of Hadamard by explicitly constructing a sequence of counterexamples (Hadamard-type examples) demonstrating exponential instability in the interior of the domain despite vanishing boundary data as $m \to \infty$. To overcome this instability and construct a stable solution, we derive an explicit Carleman-type formula for n-metaanalytic functions. By leveraging V.V. Pokazeyev's generalized Cauchy integral formula and introducing a specially constructed quenching function $g_\sigma(z) = e^{\sigma h(z)}$ governed by a positive parameter σ, we establish a limiting integral representation. As $\sigma \to \infty$, this formula enables the stable recovery of the function with a controllable error estimate bounded by $O(e^{-\sigma\omega(z)})$, where $\omega(z)$ represents the harmonic measure of the set E. These results provide a robust theoretical framework for treating improperly posed inverse problems associated with higher-order meta-analytic systems.*

Keywords: *n-metaanalytic functions, analytic continuation, ill-posed problems, Carleman formula, Cauchy integral formula, harmonic measure, generalized Cauchy-Riemann system.*

BIR O'ZGARUVCHILI METAANALITIK TENGLAMA YECHIMINI SOHA CHEGARASINING QISMIDAN DAVOM ETTIRISH MASALASI UCHUN KARLEMAN FORMULASI

Annotatsiya. *Ushbu maqola differensial tenglamalar nazariyasi va kompleks analizning murakkab masalalaridan biri bo'lib, kompleks tekisligining chegaralangan D sohasidagi n-metaanalitik funksiyalar sinfi uchun analitik davom ettirish masalasiga bag'ishlangan. Biz n-metaanalitik $f(z)$ funksiyani uning differensial operatorlari $L_s f(t)$ yordamida ∂D soha chegarasidagi E to'plamda berilgan qiymatlariga ko'ra analitik davom ettirish masalasini qaraymiz. Dastlab $\bar{\partial}_z$ operatorning xossalariga oid yordamchi lemmani va induktiv usulni qo'llab, $M_n(D) \cap C^{n-1}(\bar{D})$ sinfida to'liq yagonalik teoremasini isbotlaymiz. Bundan tashqari Hadamard tipidagi misollar yordamida ushbu chegaraviy masalaning nokorrekt qo'yilganligini isbotlaymiz. Ushbu nokorrektlikni bartaraf etish uchun n-metaanalitik funksiyalar uchun Karleman tipidagi formulani keltiramiz. Shuningdek V.V Pokazeyevning umumlashgan Koshi integral formulasidan va musbat σ delta parametrga bog'liq maxsus $g_\sigma(z) = e^{\sigma h(z)}$ so'ndiruvchi funksiyadan foydalanib, $\sigma \to \infty$ da limitga o'tib, $O(e^{-\sigma\omega(z)})$ bilan chegaralangan nokorrektlikni korreksiyalash imkonini beradi, bunda $\omega(z)$ funksiya E to'plamning garmonik o'lchovini ifodalaydi. Olingan natijalar yuqori tartibli n-metaanalitik sistemalar bilan bog'liq nokorrekt qo'yilgan teskari masalalarni yechish uchun mustahkam nazariy asos yaratadi.*

Kalit so'zlar: *n-metaanalitik funksiyalar, analitik davom ettirish, nokorrekt masalalar, Karleman formulasi, Koshi tipidagi integral formulasi, garmonik o'lchov, umumlashgan Koshi-Riman sistemasi.*

ФОРМУЛА КАРЛЕМАНА ДЛЯ ЗАДАЧИ АНАЛИТИЧЕСКОГО ПРОДОЛЖЕНИЯ РЕШЕНИЯ ОДНО ПЕРЕМЕННОГО МЕТААНАЛИТИЧЕСКОГО УРАВНЕНИЯ С ЧАСТИ ГРАНИЦЫ ОБЛАСТИ

Аннотация. *Настоящая работа посвящена одной из фундаментальных и сложных проблем теории дифференциальных уравнений с частными производными и комплексного анализа — проблеме аналитического продолжения для класса n-метааналитических функций в ограниченной области D комплексной плоскости. Мы строго исследуем краевую задачу, в которой n-метааналитическая функция $f(z)$ восстанавливается по известным значениям своих дифференциальных операторов $L_s f(t)$, заданных на подмножестве E границы ∂D. Сначала мы устанавливаем полную теорему единственности в классе функций $M_n(D) \cap C^{n-1}(\bar{D})$, используя вспомогательную лемму о свойствах оператора $\bar{\partial}_z$ и индуктивный метод. Кроме того, мы доказываем сильную некорректность этой краевой задачи в смысле Адамара, явно строя последовательность контрпримеров (примеров типа Адамара), демонстрирующих экспоненциальную неустойчивость внутри области, несмотря на стремление к нулю граничных данных при $m \to \infty$. Чтобы преодолеть эту неустойчивость и построить устойчивое решение, мы выводим явную формулу типа Карлемана для n-метааналитических функций. Используя обобщённую интегральную формулу Коши В.В. Показеева и вводя специально сконструированную «гасящую» функцию $g_\sigma(z) = e^{\sigma h(z)}$, зависящую от положительного параметра σ, мы устанавливаем предельное интегральное представление. При $\sigma \to \infty$ эта формула обеспечивает устойчивое восстановление функции с контролируемой оценкой погрешности, ограниченной величиной $O(e^{-\sigma\omega(z)})$, где $\omega(z)$ представляет собой гармоническую меру множества E. Полученные результаты предоставляют надёжную теоретическую основу для решения некорректно поставленных обратных задач, связанных с n-метааналитическими системами высших порядков.*

Ключевые слова: *n-метааналитические функции, аналитическое продолжение, некорректные задачи, формула Карлемана, интегральная формула Коши, гармоническая мера, обобщённая система Коши–Римана.*

Introduction and Formulation of the Problem. The theory of generalized analytic functions, initiated and extensively developed by I.N. Vekua and other prominent mathematicians, serves as a powerful apparatus for investigating various problems in mathematical physics, continuum mechanics, and complex analysis. Among these generalized classes, ***n***-analytic and ***n***-metaanalytic functions occupy a special place due to their natural appearance in higher-order elliptic systems.

Let D be a bounded simply connected domain in the complex plane $\mathbb{C}$ with a sufficiently smooth boundary ∂D. We consider the class of n-metaanalytic functions denoted by $M_n(D)$. The primary objective of this study is to examine the analytical continuation problem: given the values of the differential operators $L_s f(t)$ on a measurable subset $E \subset \partial D$ of positive measure, recover the n-metaanalytic function $f(z)$ throughout the entire domain D.

Formally, the boundary conditions are stated as follows:

$$L_s f(t) = h_s(t), \quad (s = 0,1,2,\dots,n-1), \quad t \in E,$$

where $L_0 f(t) = f(t) = h_0(t)$, and $h_s(t)$ are prescribed continuous functions on E. When the parameters vanish identically, i.e., $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$, problem (1)-(2) naturally degenerates into the classical boundary value analytic continuation problem for n-analytic functions, which has been thoroughly analyzed in foundational works such as [1]. However, the presence of non-zero parameters λ_s introduces substantial analytical complications, requiring the development of new functional-theoretic tools.

2.Auxiliary Results and Uniqueness Theorem

To establish the uniqueness of the solution to the stated continuation problem, we first formulate and prove an essential auxiliary lemma regarding the solutions of the first-order foundational differential equation.

Lemma 1. *Let $w(z) \in C(D)$ be a continuous function in D satisfying the equation $\bar{\partial}_z w - \lambda w = 0, \quad z \in D$, where λ is a complex constant. If the boundary values of $w(z)$ vanish on a non-trivial arc $E \subset \partial D$, then $w(z) \equiv 0$ everywhere in D.*

Proof. We formally integrate equation (3) by rewriting it in terms of logarithmic derivatives:

$$\frac{\bar{\partial}_z w}{w} = \lambda \Longrightarrow \bar{\partial}_z(\ln w(z) - \lambda \bar{z}) = 0.$$

This implies that the function $f(z) = w(z)e^{-\lambda\bar{z}}$ satisfies the homogeneous Cauchy-Riemann equation $\bar{\partial}_z f = 0$ in D, meaning that $f(z)$ is an arbitrary holomorphic function in D. Consequently, the general solution of equation (3) can be represented in the form:

$$w(z) = f(z)e^{\lambda\bar{z}},$$

where $f(z) \in H(D)$. According to the hypothesis of the lemma, the boundary values satisfy $w(t) = 0$ for $t \in E$, which yields:

$$f(t)e^{\lambda\bar{t}} = 0 \Rightarrow f(t) = 0, \quad t \in E.$$

Since $f(z)$ is a holomorphic function in D and vanishes on a set of positive measure E on the boundary, by applying the generalized Riss theorem (or uniqueness theorem for analytic functions), we obtain $f(z) \equiv 0$ in D. Hence, it directly follows that $w(z) \equiv 0$ throughout D. The lemma is fully proven. □

Using this lemma, we can now state and prove the uniqueness theorem for the main problem in the class $M_n(D) \cap C^{n-1}(\bar{D})$.

Theorem 1. *Let $w(z) \in M_n(D) \cap C^{n-1}(\bar{D})$ be an n-metaanalytic function satisfying the homogeneous boundary conditions on the subset $E \subset \partial D$: $L_s w(t) = (\bar{\partial}_t - \lambda_s)w(t) = 0$, $(s = 0,1,2,\dots,n-1)$. Then, the function vanishes identically in the domain, i.e., $w(z) \equiv 0$ for all $z \in D$.*

Proof. The proof is carried out via mathematical induction with respect to the order of the operators. For $s = n-1$, consider the function $L_{n-1}w(z)$. By the definition of n-metaanalytic functions, it satisfies the first-order equation:

$$(\bar{\partial}_z - \lambda_n)L_{n-1}w(z) = 0, \quad z \in D.$$

In view of boundary condition (4), its trace on $E \subset \partial D$ is equal to zero. Applying Lemma 1 to $L_{n-1}w(z)$, we immediately conclude that:

$$L_{n-1}w(z) = 0, \quad z \in D.$$

Next, descending to the subsequent step, the function $L_{n-2}w(z)$ satisfies the equation governed by λ_{n-1} due to relation (5):

$$(\bar{\partial}_z - \lambda_{n-1})L_{n-2}w(z) = 0, \quad z \in D.$$

Since condition (4) dictates that the boundary values of $L_{n-2}w(z)$ also vanish on E, Lemma 1 ensures that $L_{n-2}w(z) = 0$ in D. Continuing this downward induction process for all indices down to $s = 0$, we ultimately arrive at $w(z) \equiv 0$ throughout the domain D. The theorem is proven.

3.Analysis of Ill-Posedness: Hadamard-Type Example

It is a well-established fact in the theory of partial differential equations that Cauchy-type problems for elliptic equations are typically improperly posed (ill-posed) in the sense of Hadamard, meaning that solutions do not depend continuously on the input data. To demonstrate that problem (1)-(2) shares this property, we construct an explicit counterexample.

Example. Consider the sequence of functions defined in the upper half-plane by:

$$w_m(z) = \frac{e^{-imz}}{m}\sum_{k=1}^{n} e^{\lambda_k\bar{z}}, \quad m = 1,2,3,\dots$$

It can be readily verified that these functions satisfy the n-metaanalytic equation system. Let us compute their s-order differential operators:

$$L_s w_m(z) = \frac{e^{-imz}}{m}\sum_{k=s+1}^{n}(\lambda_k - \lambda_s)(\lambda_k - \lambda_{s-1})\dots(\lambda_k - \lambda_1)e^{\lambda_k\bar{z}}, \quad (s = 0,1,\dots,n-1).$$

For $y = 0$ (on the real axis boundary), evaluating the limit as $m \to \infty$ yields:

$$|L_s w_m(z)|_{y=0} = \frac{e^{-imx}}{m}\sum_{k=s+1}^{n}\prod_{l=1}^{s}(\lambda_k - \lambda_l)e^{\lambda_k x} \to 0, \quad \text{as } m \to \infty.$$

Thus, the boundary data uniformly converges to zero as m grows large. However, inside the domain where $y > 0$, the behavior of $w_m(z)$ is governed by the factor $e^{-imz} = e^{-im(x+iy)} = e^{my}e^{-imx}$, which implies that:

$$|w_m(z)| \to \infty, \quad \text{as } m \to \infty \quad (\text{for } y > 0).$$

This drastic instability explicitly confirms that arbitrarily small perturbations on the boundary can lead to unbounded deviations in the interior of the domain. Consequently, problem (1)-(2) is severely ill-posed, necessitating the construction of regularized solutions via Carleman formulas.

4.Cauchy and Carleman Integral Formulas

In the theory of analytic and generalized analytic functions, integral representations play an instrumental role. For n-metaanalytic functions, the generalized Cauchy integral formula was originally established by V.V. Pokazeyev [2].

For any function $w(z) \in M_n(D) \cap C^{n-1}(\bar{D})$, the following Cauchy formula holds:

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \sum_{k=1}^{n} L_{k-1} f(\tau) \sum_{\nu=1}^{k} K_\nu (z-\tau) \prod_{l=1}^{k} \alpha_{\nu l} \frac{d\tau}{\tau - z}, \quad z \in D,$$

where $\alpha_{ll} = 1$, $\alpha_{\nu l} = (\lambda_\nu - \lambda_l)^{-1}$ for $\nu \neq l$, and the kernel function is defined as:

$$K_\nu(z-\tau) = \exp\big(\lambda_\nu \overline{(z-\tau)} - \bar{\lambda}_\nu (z-\tau)\big).$$

To construct the Carleman formula—which provides a stable analytical continuation operator from the subset $E \subset \partial D$ to the interior domain D—we utilize a specially designed quenching (damping) function. Let us introduce:

$$g_\sigma(z) = e^{\sigma h(z)},$$

where $\sigma > 0$ is a large regularization parameter, and $h(z)$ is an analytic function in D such that its real part satisfies $\operatorname{Re} h(z) = \omega(z)$, where $\omega(z)$ is the harmonic measure of the set E relative to the domain D.

For any function $w(z) \in M_n(D)$, the auxiliary product function $F_\sigma(z) = g_\sigma(z) w(z)$ also belongs to the class $M_n(D)$. Applying Cauchy's formula (6) to $F_\sigma(z)$, we write:

$$F_\sigma(z) = \frac{1}{2\pi i}\int_{\partial D} \sum_{k=1}^{n} L_{k-1} F_\sigma(\tau) \sum_{\nu=1}^{k} K_\nu (z-\tau) \prod_{l=1}^{k} \alpha_{\nu l} \frac{d\tau}{\tau - z}.$$

By transforming this expression and isolating the integration over the subset E versus the remaining boundary $\partial D \backslash E$, we evaluate the approximation error at an interior point $z \in D$:

$$|w(z) - w_\sigma(z)| = \left| \frac{1}{2\pi i}\int_{\partial D \backslash E} \sum_{k=1}^{n} L_{k-1} w(\tau) \sum_{\nu=1}^{k} K_\nu (z-\tau) \prod_{l=1}^{k} \alpha_{\nu l}\, e^{\sigma[\alpha(\tau)-\alpha(z)]} \frac{d\tau}{\tau - z} \right| \leq$$

$$\leq e^{-\sigma\omega(z)} aK,$$

where parameters a and K denote respective bounds:

$$a = \sum_{\nu=1}^{n} \prod_{l=1}^{k} |\alpha_{\nu l}|, \quad K = \max_{\tau \in \partial D \backslash E} |L_{k-1} w(\tau)|, \quad (1 \leq k \leq n).$$

Inequality (8) directly implies the main structural convergence theorem of our paper.

Theorem 2. *For any function $w(z)$ satisfying the conditions of the problem within $M_n(D) \cap C^{n-1}(\bar{D})$, the stable reconstruction formula via the limiting process holds: $w(z) = \lim_{\sigma\to\infty} \frac{1}{2\pi i}\int_E \sum_{k=1}^{n} f_{k-1}(\tau) \sum_{\nu=1}^{k} e^{\lambda_\nu \overline{(z-\tau)} - \bar{\lambda}_\nu (z-\tau)} \prod_{l=1}^{k} (\lambda_\nu - \lambda_l)^{-1} e^{\sigma[\alpha(\tau)-\alpha(z)]} \frac{d\tau}{\tau - z}$, valid for every interior point $z \in D$, ensuring uniform convergence on compact subsets of D.*

Conclusion. In this comprehensive study, the boundary value problem of analytic continuation for the class of n-metaanalytic functions in a bounded domain has been thoroughly investigated. The primary theoretical and constructive findings of this work are summarized as follows:

- **Uniqueness Criteria:** By formulating an auxiliary proposition for the first-order $\bar{\partial}_z$ operator and executing a rigorous downward mathematical induction, we established a definitive uniqueness theorem confirming that vanishing boundary data on a subset $E \subset \partial D$ forces the n-metaanalytic function to be identically zero throughout D.
- **Ill-Posedness Proof:** Through the explicit construction of a Hadamard-type counterexample sequence $w_m(z)$, we demonstrated that the studied boundary value problem is severely unstable and improperly posed in the classical sense, as boundary perturbations grow exponentially inside the domain.
- **Carleman Representation:** Utilizing Pokazeyev's generalized Cauchy formula and implementing an exponential quenching function governed by harmonic measures, we successfully derived an explicit Carleman-type formula for n-metaanalytic functions.
- **Error Estimation and Stability:** We established that the convergence rate of the proposed regularization method is controlled by an error bound of order $O(e^{-\sigma\omega(z)})$ as $\sigma \to \infty$, providing a constructive tool for solving practical inverse problems in mathematical physics.

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SPECTRAL PROPERTIES OF A TWO-PARTICLE SCHRÖDINGER OPERATOR WITH FINITE-RANGE INTERACTIONS

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Abstract. *We study the spectral properties of the operator $H_{\mu_1\mu_2\mu_3}(K)$, K is the total quasi-momentum, corresponding to a system of two particles with on-site, nearest-neighbor, and next-nearest-neighbor interactions along the coordinate axes on the three-dimensional lattice, with the strengths μ_1, μ_2 and μ_3, respectively. For $K = 0$, we identify an invariant subspace of the operator. On this subspace, the restricted operator depends only on the parameters μ_2 and μ_3. We determine the number of its eigenvalues lying below the essential spectrum. In particular, the (μ_2, μ_3)-parameter plane is partitioned by two hyperbolic curves into three unbounded regions, in each of which the number of eigenvalues below the essential spectrum remains constant and equals* 0, 1 *or* 2.

Keywords: *Two-particle system, lattice Schrödinger operator, essential spectrum, eigenvalues, Fredholm determinant, spectral analysis.*

CHEKLI TA'SIR RADIUSIGA EGA IKKI ZARRACHALI SHREDINGER OPERATORINING SPEKTRAL XOSSALARI

Annotatsiya. *Biz uch o'lchamli panjarada koordinata o'qlari bo'ylab nuqtada, eng yaqin qo'shni tugunlar va keyingi eng yaqin qo'shni tugunlarda mos ravishda μ_1, μ_2 va μ_3 kuchlar bilan ta'sirlashuvchi ikki zarrachali Shredinger operatorining spektral xossalarini o'rganamiz. $K = 0$ hol uchun berilgan operatorning invariant qism fazosi aniqlangan. Bu qism fazodagi operator toraytmasi faqat μ_2 va μ_3 parameterlarga bog'liq. Biz ushbu operator toraytmasining muhim spektrdan pastda joylashgan xos qiymatlari sonini aniqlaymiz. Xususan, ikkita giperbolik egri chiziq (μ_2, μ_3)-parametrlar tekisligini uchta chegaralanmagan sohaga ajratadi. Har bir sohada muhim spektrdan pastda joylashgan xos qiymatlar soni o'zgarmas bo'lib, mos ravishda 0, 1 yoki 2 ga teng.*

Kalit so'zlar: *ikki zarrachali sitema, Shredinger operatori, muhim spektr, xos qiymatlar, Fredgolm determinanti, spektral tahlil.*

СПЕКТРАЛЬНЫЕ СВОЙСТВА ДВУХЧАСТИЧНОГО ОПЕРАТОРА ШРЁДИНГЕРА С ВЗАИМОДЕЙСТВИЯМИ КОНЕЧНОГО РАДИУСА

Аннотация. *Исследуются спектральные свойства оператора $H_{\mu_1\mu_2\mu_3}(K)$, где K- полный квазиимпульс, соответствующего системе двух частиц с одноузельным взаимодействием, а также взаимодействиями между ближайшими и следующими за ближайшими соседями вдоль координатных осей с интенсивностями μ_1, μ_2 и μ_3 соответственно. При $K = 0$ выделяется инвариантное подпространство оператора. На этом подпространстве сужение оператора зависит только от параметров μ_2 и μ_3. Определяется число его собственных значений, лежащих ниже существенного спектра. В частности, плоскость параметров (μ_2, μ_3)-разбивается двумя гиперболическими кривыми на три неограниченные области, в каждой из которых число собственных значений, лежащих ниже существенного спектра, остаётся постоянным и равно 0, 1 или 2.*

Ключевые слова: *двухчастичная система, решётчатый оператор Шрёдингера, существенный спектр, собственные значения, определитель Фредгольма, спектральный анализ.*

Introduction. The spectral analysis of lattice Schrödinger operators is a central topic in mathematical physics. As discrete analogues of continuum Schrödinger operators, lattice Hamiltonians provide an effective framework for studying spectral transitions, threshold effects and bound states; see, e.g., [1, 2].

Few-body lattice Hamiltonians with finite-range interactions can often be treated as finite-rank or compact perturbations of the free lattice Hamiltonian. This allows many spectral problems to be reduced to

finite-dimensional matrix-valued functions and Fredholm determinants. Such methods have been successfully applied to two- and three-particle lattice systems in [3, 4, 5, 6, 7, 8, 9].

Unlike the continuum case, lattice Hamiltonians are neither rotationally invariant nor unbounded, and their dispersion relations generate finite spectral bands. This leads to new spectral phenomena, most notably the possibility of bound states not only below but also *above* the essential spectrum, as observed in [5, 10].

In this paper, we study the three-parameter family of two-particle lattice Schrödinger operators

$$H_{\mu_1\mu_2\mu_3}(K) = H_0(K) + V_{\mu_1\mu_2\mu_3},$$

acting in $L^{2,\mathrm{e}}(\mathbb{T}^3)$, where $K \in \mathbb{T}^3$ is the total quasi-momentum and $V_{\mu_1\mu_2\mu_3}$ describes on-site (μ_1), nearest-neighbor (μ_2), and next-nearest-neighbor (μ_3) interactions.

For $K = 0$, the operator has an invariant subspace and we determine the number of eigenvalues of the restriction of $H_{\mu_1\mu_2\mu_3}(0)$ to this subspace, lying below the essential spectrum. In particular, this restriction depends only on μ_2 and μ_3.

We show that the parameter plane (μ_2, μ_3) is partitioned by two hyperbolic curves into regions with a constant number of eigenvalues below the essential spectrum.

These results extend earlier one- and two-dimensional studies [11, 12] to the three-dimensional case. Moreover, in [13], the discrete spectrum of the same operator was studied in another invariant subspace.

The paper is organized as follows. Section 2 introduces the model and describes its essential spectrum and symmetry reduction. In Section 3, the Fredholm determinant and its threshold asymptotics are studied. The main spectral results are proved in Section 4.

Hamiltonian of a two-boson system on the lattice $\mathbb{Z}^3$

2.1 The position-space representation

Let $\mathbb{Z}^3 := \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ be the three-dimensional integer lattice and let $\ell^{2,s}(\mathbb{Z}^3 \times \mathbb{Z}^3)$ denote the Hilbert space of square-summable symmetric functions on $\mathbb{Z}^3 \times \mathbb{Z}^3$. We study two identical spinless bosons on the three-dimensional lattice $\mathbb{Z}^3$ with finite-range pair interactions supported on the on-site, nearest-neighbor, and next-to-nearest-neighbor sites along the coordinate axes, with coupling constants μ_1, μ_2, and μ_3, respectively. The corresponding Hamiltonian in the position-space representation is introduced on $\ell^{2,s}(\mathbb{Z}^3 \times \mathbb{Z}^3)$ as

$$\widehat{\mathbb{H}}_{\mu_1\mu_2\mu_3} = \widehat{\mathbb{H}}_0 + \widehat{\mathbb{V}}_{\mu_1\mu_2\mu_3}, \qquad \mu_1, \mu_2, \mu_3 \in \mathbb{R}.$$

Here, $\widehat{\mathbb{H}}_0$ denotes the Hamiltonian of the corresponding non-interacting two-boson system:

$$(\widehat{\mathbb{H}}_0\hat{f})(x_1, x_2) = \sum_{y\in\mathbb{Z}^3} \hat{\varepsilon}(x_1 - y)\hat{f}(y, x_2) + \sum_{y\in\mathbb{Z}^3} \hat{\varepsilon}(x_2 - y)\hat{f}(x_1, y), \quad x_1, x_2 \in \mathbb{Z}^3,$$

where

$$\hat{\varepsilon}(s) = \begin{cases} 6 & \text{if} |s| = 0, \\ -1 & \text{if} |s| = 1, \\ 0 & \text{if} |s| > 1, \end{cases}$$

and $|s| = |s_1| + |s_2| + |s_3|$ for $s = (s_1, s_2, s_3) \in \mathbb{Z}^3$. The interaction operator $\widehat{\mathbb{V}}_{\mu_1\mu_2\mu_3}$ acts as multiplication by a function $\hat{v}_{\mu_1\mu_2\mu_3}(\cdot)$,

$$(\widehat{\mathbb{V}}_{\mu_1\mu_2\mu_3}\hat{f})(x, y) = \hat{v}_{\mu_1\mu_2\mu_3}(x - y)\hat{f}(x, y),$$

where

$$\hat{v}_{\mu_1\mu_2\mu_3}(x) = \begin{cases} \mu_1 & |x| = 0, \\ \mu_2 & |x| = 1, \\ \mu_3 & x \in \{(\pm 2,0,0), (0, \pm 2,0), (0,0, \pm 2)\}, \\ 0 & \text{otherwise.} \end{cases}$$

2.2 The quasimomentum-space representation.

By $\mathbb{T}^3$ we denote the three-dimensional torus, $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3 \equiv [-\pi, \pi)^3$. The torus $\mathbb{T}^3$ coincides with the Pontryagin dual group of $\mathbb{Z}^3$, equipped with the Haar measure $\mathrm{d}p$. Let $L^{2,s}(\mathbb{T}^3 \times \mathbb{T}^3)$ be the Hilbert space of square-integrable symmetric functions on $\mathbb{T}^3 \times \mathbb{T}^3$. Let $\mathcal{F}$ denote the standard Fourier transform from $\ell^2(\mathbb{Z}^3)$ to $L^2(\mathbb{T}^3)$,

$$(\mathcal{F}\hat{f})(p) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{x\in\mathbb{Z}^3} \hat{f}(x)e^{ip\cdot x}, \qquad p \in \mathbb{T}^3,$$

where $p \cdot x := p_1x_1 + p_2x_2 + p_3x_3$. Its inverse is given by

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{T}^3} f(p)e^{-ip\cdot x}dp.$$

Under the unitary map $\mathcal{F} \otimes \mathcal{F}$, the Hamiltonian becomes

$$\mathbb{H}_{\mu_1\mu_2\mu_3} := (\mathcal{F} \otimes \mathcal{F})\widehat{\mathbb{H}}_{\mu_1\mu_2\mu_3}(\mathcal{F} \otimes \mathcal{F})^{-1} := \mathbb{H}_0 + \mathbb{V}_{\mu_1\mu_2\mu_3}.$$

The free Hamiltonian is the multiplication operator:

$$\mathbb{H}_0 f(p,q) = [\varepsilon(p) + \varepsilon(q)]f(p,q),$$

where

$$\varepsilon(p) := 2\sum_{i=1}^{3} (1 - \cos p_i), \quad p = (p_1, p_2, p_3) \in \mathbb{T}^3,$$

is the *dispersion relation* of a single boson.

The interaction $\mathbb{V}_{\mu_1\mu_2\mu_3}$ is given by

$$\mathbb{V}_{\mu_1\mu_2\mu_3} f(p,q) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} v_{\mu_1\mu_2\mu_3}(p-u)f(u, p+q-u)du,$$

where $v_{\mu_1\mu_2\mu_3}$ is the Fourier transform of $\hat{v}_{\mu_1\mu_2\mu_3}$:

$$v_{\mu_1\mu_2\mu_3}(p) = \mu_1 + 2\mu_2 \sum_{i=1}^{3} \cos p_i + 2\mu_3 \sum_{i=1}^{3} \cos 2p_i.$$

2.3 Decomposition with respect to the total quasimomentum

Introducing the total quasimomentum

$$K := p + q \in \mathbb{T}^3,$$

we obtain the von Neumann decomposition

$$\mathbb{H}_{\mu_1\mu_2\mu_3} = \int_{\mathbb{T}^3}^{\oplus} H_{\mu_1\mu_2\mu_3}(K)\, dK.$$

Define the unitary map

$$\Upsilon: L^{2,s}(\mathbb{T}^3 \times \mathbb{T}^3) \to \int_{\mathbb{T}^3}^{\oplus} L^{2,\mathrm{e}}(\mathbb{T}^3)\, dK, \qquad (\Upsilon f)(K,k) := f\left(\frac{K}{2} + k, \frac{K}{2} - k\right),$$

where $k \in \mathbb{T}^3$ is the relative momentum. The fiber operator $H_{\mu_1\mu_2\mu_3}(K)$, $K \in \mathbb{T}^3$, is a self-adjoint operator on $L^{2,\mathrm{e}}(\mathbb{T}^3)$ of the form

$$H_{\mu_1\mu_2\mu_3}(K) := H_0(K) + V_{\mu_1\mu_2\mu_3},$$

where

$$(H_0(K)f)(p) = \mathcal{E}_K(p)f(p), \qquad \mathcal{E}_K(p) := 4\sum_{i=1}^{3} \left(1 - \cos\frac{K_i}{2}\cos p_i\right)$$

and

$$\left(V_{\mu_1\mu_2\mu_3} f\right)(p) = \frac{\mu_1}{8\pi^3}\int_{\mathbb{T}^3} f(q)dq + \frac{\mu_2}{4\pi^3}\sum_{i=1}^{3} \cos p_i \int_{\mathbb{T}^3} \cos q_i f(q)dq + \frac{\mu_3}{4\pi^3}\sum_{i=1}^{3} \cos 2p_i \int_{\mathbb{T}^3} \cos 2q_i f(q)dq. \tag{1}$$

Both the operators $H_0(K)$ and $V_{\mu_1\mu_2\mu_3}$, which is a rank-seven operator, are bounded and self-adjoint.

2.4 The essential spectrum of the fiber Schrödinger operators

Depending on $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ the rank of $V_{\mu_1\mu_2\mu_3}$ varies but never exceeds seven. Hence, by the classical Weyl theorem (see [14, Theorem XIII.14]), for any $K \in \mathbb{T}^3$, the essential spectrum $\sigma_{\mathrm{ess}}(H_{\mu_1\mu_2\mu_3}(K))$ of the fiber Schrödinger operator $H_{\mu_1\mu_2\mu_3}(K)$ coincides with the spectrum of $H_0(K)$:

$$\sigma_{\text{ess}}(H_{\mu_1\mu_2\mu_3}(K)) = \sigma(H_0(K)) = [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)],$$

where

$$\mathcal{E}_{\min}(K) := \min_{p\in\mathbb{T}^3} \mathcal{E}_K(p) = 4\sum_{i=1}^{3}(1-\cos\frac{K_i}{2}) \geq \mathcal{E}_{\min}(0) = 0,$$

$$\mathcal{E}_{\max}(K) := \max_{p\in\mathbb{T}^3} \mathcal{E}_K(p) = 4\sum_{i=1}^{3}(1+\cos\frac{K_i}{2}) \leq \mathcal{E}_{\max}(0) = 24.$$

2.5 Invariant subspace of the lattice Schrödinger operator $H_{\mu_1\mu_2\mu_3}(0)$

Lemma 2.1 *The subspace $L^{2,\text{e,m}}(\mathbb{T}^3) \subset L^{2,\text{e}}(\mathbb{T}^3)$ is invariant under $H_{\mu_1\mu_2\mu_3}(0)$, where*

$$L^{2,\text{e,m}}(\mathbb{T}^3) = \{f \in L^{2,\text{e}}(\mathbb{T}^3): f(p_1,p_2,p_3) = f(p_2,p_1,p_3), \quad f = f_1 + f_2,$$
$$f_1(p_1,p_2,p_3) = -f_1(p_3,p_2,p_1), \quad f_2(p_1,p_2,p_3) = -f_2(p_1,p_3,p_2)\}.$$

The restriction of $H_{\mu_1\mu_2\mu_3}(0)$ onto this subspace is:

$$H^{\text{m}}_{\mu_2\mu_3}(0) := H_{\mu_1\mu_2\mu_3}|_{L^{2,\text{e,m}}(\mathbb{T}^3)} = H_0(0) + V^{\text{m}}_{\mu_2\mu_3},$$

with

$$V^{\text{m}}_{\mu_2\mu_3} := V_{\mu_1\mu_2\mu_3}|_{L^{2,\text{e,m}}(\mathbb{T}^3)} = \mu_2(\alpha^{\text{m}}_1,\cdot)\alpha^{\text{m}}_1 + \mu_3(\alpha^{\text{m}}_2,\cdot)\alpha^{\text{m}}_2,$$

where

$$\alpha^{\text{m}}_n(p) = \frac{\cos(np_1)+\cos(np_2)-2\cos(np_3)}{\sqrt{24\pi^3}}, n = 1,2.$$

Proof. Let $f \in L^{2,\text{e,m}}(\mathbb{T}^3)$. Since $H_0(0)$ is the multiplication operator by the symmetric function $\mathcal{E}_0(p)$ under all permutations of (p_1,p_2,p_3), it follows immediately that the subspace $L^{2,\text{e,m}}(\mathbb{T}^3)$ is invariant under $H_0(0)$.

We now consider the operator $V_{\mu_1\mu_2\mu_3}$. We know that f is symmetric under the transposition $p_1 \leftrightarrow p_2$ and $f = f_1 + f_2$, where f_1 and f_2 are antisymmetric with respect to the transposition $p_1 \leftrightarrow p_3$ and $p_2 \leftrightarrow p_3$, respectively. Therefore,

$$\int_{\mathbb{T}^3} f(q)\mathrm{d}q = 0.$$

Using the symmetry property of f, we obtain

$$\int_{\mathbb{T}^3}\cos(nq_1)f(q)\mathrm{d}q = \int_{\mathbb{T}^3}\cos(nq_2)f(q)\mathrm{d}q, \quad n = 1,2. \tag{2}$$

Moreover, the antisymmetry properties of f_1 and f_2 imply that

$$\int_{\mathbb{T}^3}\cos(nq_1)f_2(q)\mathrm{d}q = 0, \quad \int_{\mathbb{T}^3}\cos(nq_2)f_1(q)\mathrm{d}q = 0, \quad n = 1,2.$$

Using these equalities together with the decomposition $f = f_1 + f_2$ and the antisymmetry properties of f_1 and f_2, we obtain

$$\int_{\mathbb{T}^3}\cos(nq_3)f(q)\mathrm{d}q = -2\int_{\mathbb{T}^3}\cos(nq_1)f(q)\mathrm{d}q, \quad n = 1,2. \tag{3}$$

Substituting the identities (2) and (3) into the definition of $V_{\mu_1\mu_2\mu_3}$, we obtain

$$\left(V_{\mu_1\mu_2\mu_3}f\right)(p) = \frac{\mu_2}{4\pi^3}(\cos p_1 + \cos p_2 - 2\cos p_3)\int_{\mathbb{T}^3}\cos q_1 f(q)\mathrm{d}q \tag{4}$$
$$+\frac{\mu_3}{4\pi^3}(\cos(2p_1)+\cos(2p_2)-2\cos(2p_3))\int_{\mathbb{T}^3}\cos(2q_1)f(q)\mathrm{d}q.$$

Since $\cos(np_1)+\cos(np_2)-2\cos(np_3) \in L^{2,\text{e,m}}(\mathbb{T}^3)$, $n = 1,2$, it follows that $L^{2,\text{e,m}}(\mathbb{T}^3)$ is invariant under $V_{\mu_1\mu_2\mu_3}$. Thus, the subspace $L^{2,\text{e,m}}(\mathbb{T}^3)$ is invariant under $H_{\mu_1\mu_2\mu_3}(0) = H_0(0) + V_{\mu_1\mu_2\mu_3}$. Furthermore, the relations (2) and (3) yield

$$\int_{\mathbb{T}^3}\cos(nq_1)f(q)\mathrm{d}q = \frac{1}{6}\int_{\mathbb{T}^3}(\cos(nq_1)+\cos(nq_2)-2\cos(nq_3))f(q)\mathrm{d}q.$$

Combining this identity with (4), we obtain the restriction $V^{\text{m}}_{\mu_2\mu_3}$ of $V_{\mu_1\mu_2\mu_3}$ onto $L^{2,\text{e,m}}(\mathbb{T}^3)$:

$$(V^{\text{m}}_{\mu_2\mu_3}f)(p) = \mu_2(\alpha^{\text{m}}_1,f)\alpha^{\text{m}}_1(p) + \mu_3(\alpha^{\text{m}}_2,f)\alpha^{\text{m}}_2(p). \tag{5}$$

Therefore,

$$H^{\text{m}}_{\mu_2\mu_3}(0) = H_0(0) + V^{\text{m}}_{\mu_2\mu_3}.$$

This completes the proof. ■

Lippmann–Schwinger operator and asymptotics of its determinant

3.1 The Lippmann–Schwinger operator

For $z \in \mathbb{C}\backslash[0,24]$ we define the Lippmann-Schwinger operator (see, e.g., [15]) as

$$B^{\mathrm{m}}_{\mu_2\mu_3}(0,z) = -V^{\mathrm{m}}_{\mu_2\mu_3} R_0(0,z),$$

where $R_0(0,z) := [H_0(0) - zI]^{-1}, z \in \mathbb{C}\backslash[0,24]$, is the resolvent of the operator $H_0(0)$.

Lemma 3.1 *Let $\mu_2, \mu_3 \in \mathbb{R}$. A number $z \in \mathbb{R}\backslash[0,24]$ is an eigenvalue of the self-adjoint operator $H^m_{\mu_2\mu_3}(0)$ if and only if z is a zero of the function*

$$\Delta_{\mu_2\mu_3}(z) := \det[I - B^{\mathrm{m}}_{\mu_2\mu_3}(0,z)] = \begin{vmatrix} 1+\mu_2 a_{11}(z) & \mu_3 a_{12}(z) \\ \mu_2 a_{12}(z) & 1+\mu_3 a_{22}(z) \end{vmatrix},$$

where

$$a_{ij}(z) := \int_{\mathbb{T}^3} \frac{\alpha_i^{\mathrm{m}}(p)\alpha_j^{\mathrm{m}}(p)\mathrm{d}p}{\varepsilon_0(p) - z}, \ i,j = 1,2 \tag{6}$$

Proof. Let $z \in \mathbb{R}\backslash[0,24]$. The eigenvalue equation

$$H^{\mathrm{m}}_{\mu_2\mu_3}(0) f(p) = z f(p)$$

is equivalent to the Lippmann-Schwinger equation

$$f_z(p) = -R_0(0,z) V^{\mathrm{m}}_{\mu_2\mu_3} f(p).$$

Define $g_z := V^{\mathrm{m}}_{\mu_2\mu_3} f_z = \mu_2 c_1(z)\alpha_1^{\mathrm{m}} + \mu_3 c_2(z)\alpha_2^{\mathrm{m}}$, with $c_i = (f_z, \alpha_i^{\mathrm{m}}), \ i = 1,2$. Substituting g_z into Lippmann-Schwinger equation yields

$$(I - B^{\mathrm{m}}_{\mu_2\mu_3}(0,z)) g_z = 0$$

which reduces to a 2×2 homogeneous linear system for $(c_1(z), c_2(z))$:

$$\begin{cases} (1+\mu_2 a_{11}(z))c_1(z) + \mu_3 a_{12}(z) c_2(z) = 0 \\ \mu_2 a_{12}(z) c_1(z) + (1+\mu_3 a_{22}(z)) c_2(z) = 0 \end{cases} \tag{7}$$

The system (7) has a non-trivial solution if and only if $\Delta_{\mu_2\mu_3}(z) = 0$. ■

Proposition 3.2 *Let $a_{ij}(z), \ i,j = 1,2$ be defined by (6). Then the following properties hold:*

- All functions $a_{ij}(z)$ are real-valued and analytic in $\mathbb{R}\backslash[0,24]$.

- The functions $a_{ij}(\cdot), i,j = 1,2$ are strictly increasing on the interval $(-\infty, 0]$, and satisfy:

$$a_{ij}(z) > 0, \quad a_{11}(z)a_{22}(z) - a_{12}^2(z) > 0 \quad \text{for} \quad z \le 0,$$

where $a_{ij}(0) := \lim_{z \mathrm{Z} 0} a_{ij}(z)$.

- The following asymptotic relations hold:

$$a_{ij}(z) = a_{ij}(0) + \mathcal{O}(|z|), \quad z \mathrm{Z} 0.$$

Proof. The proof of Proposition 3.2 proceeds similarly to the proof of [13, Lemma 3]. ■

Lemma 3.3 *The function $\Delta_{\mu_2\mu_3}(z)$ is real-valued and analytic for $z \in \mathbb{R}\backslash[0,24]$. Moreover, the following asymptotic relations hold:*

- $$\lim_{|z|\to\infty} \Delta_{\mu_2\mu_3}(z) = 1.$$

- $$\Delta_{\mu_2\mu_3}(z) = A^-(\mu_2,\mu_3) + \mathcal{O}(|z|), \quad z \mathrm{Z} 0,$$

where

$$A^-(\mu_2,\mu_3) = 1 + a_{11}(0)\mu_2 + a_{22}(0)\mu_3 + (a_{11}(0)a_{22}(0) - a_{12}^2(0))\mu_2\mu_3, \tag{8}$$

The next lemma describes the geometry of the zeros of the polynomials $A^-(\mu_2,\mu_3)$ and the induced partition of the coupling-constant plane.

Lemma 3.4

The zero set of $A^-(\mu_2,\mu_3)$ is the graph of the function

$$\mu_3^-(\mu_2) := -\frac{1+a_{11}(0)\mu_2}{a_{22}(0)+(a_{11}(0)a_{22}(0)-a_{12}^2(0))\mu_2},$$

with the horizontal asymptote $\mu_3=-\mu_3^0$ and the vertical asymptote $\mu_2=-\mu_2^0$, where

$$\mu_3^0=\frac{a_{11}(0)}{a_{11}(0)a_{22}(0)-a_{12}^2(0)}>0,\quad \mu_2^0=\frac{a_{22}(0)}{a_{11}(0)a_{22}(0)-a_{12}^2(0)}>0.$$

This graph consists of two disjoint connected branches, given by

$$\tau_0^-=\{(\mu_2,\mu_3):\mu_3=\mu_3^-(\mu_2),\quad \mu_2>-\mu_2^0\},\quad \tau_1^-=\{(\mu_2,\mu_3):\mu_3=\mu_3^-(\mu_2),\quad \mu_2<-\mu_2^0\},$$

The branches $\tau_0^-\cup\tau_1^-$ divide the plane $\mathbb{R}^2$ into three unbounded regions

$$\mathbb{R}^2\backslash(\tau_0^-\cup\tau_1^-)=\mathcal{A}_0^-\cup\mathcal{A}_1^-\cup\mathcal{A}_2^-,$$

where

$$\begin{aligned}\mathcal{A}_0^- &= \{(\mu_2,\mu_3):\mu_3>\mu_3^-(\mu_2),\ \mu_2>-\mu_2^0\},\\ \mathcal{A}_1^- &= \{(\mu_2,\mu_3):\mu_3<\mu_3^-(\mu_2),\mu_2>-\mu_2^0\}\cup\{(-\mu_2^0,\mu_3)\in\mathbb{R}^2\}\\ &\quad\cup\{(\mu_2,\mu_3):\mu_3>\mu_3^-(\mu_2),\quad \mu_2<-\mu_2^0\},\\ \mathcal{A}_2^- &= \{(\mu_2,\mu_3):\mu_3<\mu_3^-(\mu_2),\quad \mu_2<-\mu_2^0\}.\end{aligned}\tag{9}$$

Proof. By the representation (8), the equation $A^-(\mu_2,\mu_3)=0$ can be solved uniquely for μ_3, which defines the function $\mu_3=\mu_3^-(\mu_2)$. Since the function $\mu_3^-(\mu_2)$ is defined on $(-\infty,-\mu_2^0)\cup(-\mu_2^0,\infty)$, its graph consists of two disjoint unbounded connected branches τ_0^- and τ_1^-.

These branches divide the plane $\mathbb{R}^2$ into three unbounded regions $\mathcal{A}_j^-$, $j=0,1,2$, which are characterized by the sign of the polynomial $A^-(\mu_2,\mu_3)$ and are given explicitly by (9). ■

The main results and their proofs

Recall that the main purpose of this work is to determine the exact number of eigenvalues of $H^{\mathrm{m}}_{\mu_2\mu_3}(0)$ below its essential spectrum.

We will now determine the exact number of eigenvalues of the operator $H^{\mathrm{m}}_{\mu_2\mu_3}(0)$ lying below the essential spectrum within the regions $\mathcal{A}_\zeta^-$, $\zeta=0,1,2$.

Theorem 4.1 *Let $\mu_2,\mu_3\in\mathbb{R}$ and $\zeta=0,1,2$. If $(\mu_2,\mu_3)\in\mathcal{A}_\zeta^-$, the operator $H^{\mathrm{m}}_{\mu_2\mu_3}(0)$ has ζ eigenvalues below its essential spectrum.*

Proof. The Fredholm determinant $\Delta_{\mu_2\mu_3}(z)$ is analytic in $z\in\mathbb{C}\backslash[0,24]$ and jointly analytic in the coupling constants (μ_2,μ_3). Hence, by Rouché's theorem, the number of its zeros in $(-\infty,0)$ is locally constant. Since each region $\mathcal{A}_\zeta^-$, $\zeta=0,1,2$, is connected, this number is constant throughout $\mathcal{A}_\zeta^-$.

It therefore suffices to compute the number of negative eigenvalues $n_-(H^{\mathrm{m}}_{\mu_2\mu_3}(0))$ for a single representative point in each region.

Case $\mathcal{A}_0^-$. Lemma 3.4 yields that $(0,0)\in\mathcal{A}_0^-$. Since $H^{\mathrm{m}}_{00}(0)=H_0(0)$, the operator $H^{\mathrm{m}}_{00}(0)$ has no eigenvalues below the essential spectrum. Therefore $n_-(H^{\mathrm{m}}_{\mu_2\mu_3}(0))=0$ for all $(\mu_2,\mu_3)\in\mathcal{A}_0^-$.

Case $\mathcal{A}_1^-$. For $(-\mu_2^0,0)\in\mathcal{A}_1^-$, the determinant reduces to

$$\Delta^{\mathrm{m}}_{(-\mu_2^0)0}(z)=1-\mu_2^0a_{11}(z),$$

which is strictly decreasing on $(-\infty,0)$ and changes sign at the endpoints. Thus it has exactly one zero in $(-\infty,0)$. By Lemma 3.1, the operator $H^{\mathrm{m}}_{(-\mu_2^0)0}(0)$ has exactly one negative eigenvalue and therefore $n_-(H^{\mathrm{m}}_{\mu_2\mu_3}(0))=1$ for all $(\mu_2,\mu_3)\in\mathcal{A}_1^-$.

Case $\mathcal{A}_2^-$. Let $(\mu_2^{(1)},\mu_3^{(1)})$ be a point in $\mathcal{A}_2^-$. Then by the definition of $\mathcal{A}_2^-$, we have

$$\mu_2^{(1)}<-\mu_2^{(0)}<0,\quad \mu_3^{(1)}<-\mu_3^{(0)}<0$$

and

$$A^-(\mu_2^{(1)},\mu_3^{(1)})>0.$$

Since $(\mu_2^{(1)}, 0) \in \mathcal{A}_1^-$, the determinant $\Delta_{\mu_2^{(1)}0}(z)$ has a unique zero $z(\mu_2^{(1)})$ in $(-\infty, 0)$. Hence

$$1 + \mu_2^{(1)} a_{11}(z(\mu_2^{(1)})) = 0.$$

Using this relation, we obtain

$$\Delta_{\mu_2^{(1)}\mu_3^{(1)}}(z(\mu_2^{(1)})) = -\mu_2^{(1)}\mu_3^{(1)} a_{12}^2(z(\mu_2^{(1)})) < 0.$$

Moreover,

$$\lim_{z\to-\infty} \Delta_{\mu_2^{(1)}\mu_3^{(1)}}(z) = 1, \quad \lim_{z\mathrm{Z}0} \Delta_{\mu_2^{(1)}\mu_3^{(1)}}(z) = A^-(\mu_2^{(1)}, \mu_3^{(1)}) > 0.$$

Thus, $\Delta_{\mu_2^{(1)}\mu_3^{(1)}}(z)$ has at least two zeros in $(-\infty, 0)$. Hence, by Lemma 3.1, the operator $H^{\mathrm{m}}_{\mu_2^{(1)}\mu_3^{(1)}}(0)$ has at least two eigenvalues below the essential spectrum. Since the perturbation is of rank two, $H^{\mathrm{m}}_{\mu_2^{(1)}\mu_3^{(1)}}(0)$ has exactly two negative eigenvalues.

This completes the proof. ■

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QANDLI DIABETDA GLYUKOZA-INSULIN TIZIMINING MATEMATIK MODELLASHTIRILISHI: DALLA MAN MODELI ASOSIDA ILMIY SHARH VA MATEMATIK TAHLIL

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Annotatsiya. *Qandli diabet zamonaviy tibbiyotning eng dolzarb muammolaridan biri bo‘lib, uning samarali nazorati va davolash usullarini ishlab chiqishda matematik modellashtirish muhim ilmiy yo‘nalish hisoblanadi. Glukoza–insulin tizimini matematik modellashtirish organizmda kechadigan metabolik jarayonlarni miqdoriy tavsiflash, fiziologik parametrlarni aniqlash hamda uzluksiz glukoza monitoringi (CGM) tizimlarini takomillashtirish imkonini beradi.*

Mazkur maqolada Dalla Man va hammualliflari tomonidan taklif etilgan ovqatdan keyingi glukoza–insulin tizimining matematik modeli ilmiy sharh asosida tahlil qilinadi. Modelning umumiy tuzilishi, asosiy kichik tizimlari hamda matematik xususiyatlari differensial tenglamalar nazariyasi nuqtayi nazaridan yoritiladi. Shuningdek, modelning holatlar fazosi, parametrlar tizimi, musbatlik xossasi, yechimning mavjudligi va yagonaligi, barqarorlik hamda parametr sezgirligi kabi nazariy masalalar muhokama qilinadi.

Maqolada Dalla Man modelining glukoza kinetikasi, insulin kinetikasi, endogen glukoza ishlab chiqarilishi, glukozaning utilizatsiyasi va insulin sekretsiyasini tavsiflovchi modullari ilmiy jihatdan baholanadi. Modelning afzalliklari va cheklovlari ko‘rsatilib, uning zamonaviy diabetologiyada, xususan, uzluksiz glukoza monitoringi va sun’iy oshqozon osti bezi tizimlarini ishlab chiqishdagi ahamiyati tahlil qilinadi.

Mazkur sharh matematik analiz, biomatematika va tibbiy modellashtirish yo‘nalishlarida ilmiy izlanish olib borayotgan tadqiqotchilar uchun nazariy manba bo‘lib xizmat qilishi mumkin.

Kalit so‘zlar: *qandli diabet, matematik modellashtirish, glukoza–insulin tizimi, Dalla Man modeli, differensial tenglamalar, biomatematika, dinamik tizimlar, parametr identifikatsiyasi, CGM, matematik tahlil.*

MATHEMATICAL MODELING OF THE GLUCOSE–INSULIN SYSTEM IN DIABETES MELLITUS: A SCIENTIFIC REVIEW AND MATHEMATICAL ANALYSIS BASED ON THE DALLA MAN MODEL

Abstract. *Diabetes mellitus is one of the most significant challenges in modern medicine, and mathematical modeling has become an important scientific approach for developing effective methods of its monitoring and treatment. Mathematical modeling of the glucose–insulin system enables the quantitative description of metabolic processes occurring in the human body, the identification of physiological parameters, and the improvement of continuous glucose monitoring (CGM) systems.*

This article presents a scientific review and mathematical analysis of the postprandial glucose–insulin model proposed by Dalla Man and co-authors. The general structure of the model, its principal subsystems, and its mathematical properties are examined from the perspective of the theory of differential equations. Furthermore, theoretical aspects of the model, including the state space, parameter system, positivity property, existence and uniqueness of solutions, stability, and parameter sensitivity, are discussed.

The study also evaluates the modules describing glucose kinetics, insulin kinetics, endogenous glucose production, glucose utilization, and insulin secretion within the Dalla Man model. The advantages and limitations of the model are highlighted, and its significance in modern diabetology, particularly in the development of continuous glucose monitoring (CGM) technologies and artificial pancreas systems, is analyzed.

This review may serve as a theoretical reference for researchers conducting scientific investigations in the fields of mathematical analysis, biomathematics, and medical modeling.

Keywords: *diabetes mellitus, mathematical modeling, glucose–insulin system, Dalla Man model, differential equations, biomathematics, dynamical systems, parameter identification, continuous glucose monitoring (CGM), mathematical analysis.*

МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ СИСТЕМЫ «ГЛЮКОЗА–ИНСУЛИН» ПРИ САХАРНОМ ДИАБЕТЕ: НАУЧНЫЙ ОБЗОР И МАТЕМАТИЧЕСКИЙ АНАЛИЗ НА ОСНОВЕ МОДЕЛИ DALLA MAN

Аннотация. *Сахарный диабет является одной из наиболее актуальных проблем современной медицины, а математическое моделирование представляет собой важное научное направление для разработки эффективных методов его контроля и лечения. Математическое моделирование системы «глюкоза–инсулин» позволяет количественно описывать метаболические процессы, происходящие в организме человека, определять физиологические параметры и совершенствовать системы непрерывного мониторинга глюкозы (Continuous Glucose Monitoring, CGM).*

В данной статье представлен научный обзор и математический анализ модели системы «глюкоза–инсулин» в постпрандиальном состоянии, предложенной Dalla Man и соавторами. Рассмотрены общая структура модели, её основные подсистемы и математические свойства с точки зрения теории дифференциальных уравнений. Кроме того, обсуждаются такие теоретические аспекты модели, как пространство состояний, система параметров, свойство положительности, существование и единственность решения, устойчивость и чувствительность параметров.

В статье также проведена научная оценка модулей модели Dalla Man, описывающих кинетику глюкозы, кинетику инсулина, эндогенную продукцию глюкозы, утилизацию глюкозы и секрецию инсулина. Показаны преимущества и ограничения модели, а также проанализировано её значение в современной диабетологии, в частности при разработке систем непрерывного мониторинга глюкозы (CGM) и систем искусственной поджелудочной железы.

Настоящий обзор может служить теоретическим источником для исследователей, работающих в области математического анализа, биоматематики и медицинского моделирования.

Ключевые слова: *сахарный диабет, математическое моделирование, система «глюкоза–инсулин», модель Dalla Man, дифференциальные уравнения, биоматематика, динамические системы, идентификация параметров, непрерывный мониторинг глюкозы (CGM), математический анализ.*

Kirish. Qandli diabet metabolik kasalliklar ichida eng ko‘p uchraydigan va dunyo bo‘yicha millionlab insonlarning hayot sifatiga salbiy ta’sir ko‘rsatayotgan surunkali kasalliklardan biridir. Kasallikning asosiy belgisi qonda glyukoza miqdorining fiziologik me’yordan chetga chiqishi bo‘lib, bu insulin sekretsiyasi yoki insulinning biologik ta’sir mexanizmlaridagi buzilishlar natijasida yuzaga keladi. Jahon sog‘liqni saqlash tashkiloti va Xalqaro diabet federatsiyasi ma’lumotlariga ko‘ra, diabet bilan kasallanganlar soni yil sayin ortib bormoqda, bu esa kasallikni modellashtirish va boshqarishning yangi matematik usullarini ishlab chiqishni dolzarb masalaga aylantirmoqda.

So‘nggi o‘n yilliklarda glukoza–insulin tizimini matematik modellashtirish diabetologiyaning mustaqil ilmiy yo‘nalishiga aylandi. Bunday modellar yordamida organizmda ovqat qabul qilingandan keyingi metabolik jarayonlar miqdoriy tavsiflanadi, fiziologik parametrlar baholanadi hamda davolash strategiyalarining samaradorligi oldindan prognoz qilinadi. Ayniqsa, uzluksiz glukoza monitoringi (CGM) texnologiyalarining rivojlanishi ushbu modellarni yanada takomillashtirish zaruratini yuzaga keltirdi.

Glukoza–insulin tizimini modellashtirish bo‘yicha dastlabki ishlardan biri Bergman minimal modeli bo‘lib, u vena ichiga yuborilgan glyukozaga organizm javobini tavsiflash uchun ishlab chiqilgan. Keyinchalik Sorensen modeli, Hovorka modeli va Dalla Man modeli kabi murakkabroq modellar taklif qilindi. Ushbu modellar orasida Dalla Man modeli ovqatdan keyingi metabolik jarayonlarni fiziologik jihatdan batafsil aks ettirgani sababli alohida o‘rin egallaydi.

Mazkur maqolaning maqsadi Dalla Man modelini matematik analiz nuqtayi nazaridan ilmiy sharhlash, uning nazariy asoslarini tahlil qilish hamda modelning matematik xossalarini tizimli ravishda yoritishdan iborat. Maqolada differensial tenglamalar nazariyasi, dinamik tizimlar, invariant sohalar, mavjudlik va yagonalik, barqarorlik hamda parametr sezgirligi kabi masalalar Dalla Man modeli misolida ko‘rib chiqiladi.

Adabiyotlar tahlili. Qandli diabetda glyukoza–insulin tizimini matematik modellashtirish sohasidagi ilmiy tadqiqotlar o‘tgan asrning ikkinchi yarmidan boshlab jadal rivojlana boshladi. Dastlabki matematik modellar glyukoza va insulin o‘rtasidagi o‘zaro bog‘liqlikni soddalashtirilgan differensial tenglamalar yordamida tavsiflashga qaratilgan bo‘lib, keyinchalik fiziologik jarayonlarni yanada aniqroq aks ettiruvchi ko‘p bo‘limli modellar ishlab chiqildi.

Glyukoza–insulin tizimini matematik tavsiflash bo‘yicha eng muhim ishlardan biri Bergman va hammualliflari tomonidan taklif etilgan **minimal model** hisoblanadi. Ushbu model intravenoz glyukoza

tolerantlik testi (IVGTT) ma'lumotlari asosida insulin sezgirligi va glyukozaning insulin mustaqil utilizatsiyasi kabi muhim fiziologik parametrlarni baholash imkonini berdi. Modelning soddaligi va parametrlar sonining kamligi uni ilmiy tadqiqotlarda keng qo'llashga imkon yaratdi. Biroq, model ovqat qabul qilingandan keyingi (postprandial) metabolik jarayonlarni to'liq tavsiflay olmasligi sababli uning qo'llanish doirasi ma'lum darajada cheklangan [1].

Keyingi bosqichda Sorensen modeli ishlab chiqildi. Ushbu model organizmdagi glyukoza almashinuvida ishtirok etuvchi asosiy organlar va to'qimalarni alohida kompartmentlar orqali ifodalashi bilan ajralib turadi. Sorensen modeli yuqori fiziologik aniqlikka ega bo'lsa-da, differensial tenglamalar va parametrlar sonining juda ko'pligi uni parametr identifikatsiyasi hamda amaliy hisoblashlarda murakkablashtiradi. Shu sababli u ko'proq nazariy va virtual bemor modellarini yaratishda qo'llaniladi.

Sun'iy oshqozon osti bezi tizimlarini yaratish bilan bog'liq tadqiqotlarda Hovorka modeli muhim o'rin egallaydi. Ushbu model glyukoza va insulin kinetikasini nazorat algoritmlari nuqtayi nazaridan tavsiflashga mo'ljallangan bo'lib, ayniqsa yopiq siklli insulin terapiyasi (closed-loop control) tizimlarini ishlab chiqishda keng qo'llaniladi. Biroq model oshqozon-ichak tizimidagi glyukozaning fiziologik so'rilish mexanizmlarini soddalashtirilgan holda ifodalaydi.

Mazkur cheklovlarni bartaraf etish maqsadida Chiara Dalla Man, Robert Rizza va Claudio Cobelli tomonidan 2007-yilda ovqatdan keyingi glyukoza–insulin tizimini tavsiflovchi yangi matematik model ishlab chiqildi. Ushbu model oshqozonning qattiq va suyuq fazalari, ichak orqali glyukozaning so'rilishi, glyukoza kinetikasi, insulin kinetikasi, endogen glyukoza ishlab chiqarilishi hamda insulin sekretsiyasi kabi asosiy fiziologik jarayonlarni yagona nolinear differensial tenglamalar tizimi orqali ifodalaydi. Modelning asosiy afzalligi uning yuqori fiziologik realizmga ega bo'lishi bilan birga parametrlarining eksperimental ma'lumotlar asosida identifikatsiya qilinishidir. g'

So'nggi yillarda Dalla Man modeli uzluksiz glyukoza monitoringi (CGM), virtual bemorlarni yaratish, sun'iy oshqozon osti bezi tizimlarini sinovdan o'tkazish, insulin yuborish algoritmlarini ishlab chiqish hamda qaror qabul qilishni qo'llab-quvvatlovchi tizimlarda keng qo'llanilmoqda. Xususan, **UVA/Padova** simulyatori tarkibida ushbu modeldan foydalanilishi uning diabetologiya va biotibbiy muhandislikdagi amaliy ahamiyatini yanada oshirdi.

Mavjud ilmiy adabiyotlar tahlili shuni ko'rsatadiki, Dalla Man modeli bo'yicha olib borilgan tadqiqotlarning asosiy qismi modelni klinik qo'llash, parametrlarni identifikatsiya qilish va boshqaruv algoritmlarini ishlab chiqishga bag'ishlangan. Biroq modelning matematik xossalari, jumladan musbat invariantlik, yechimning mavjudligi va yagonaligi, lokal barqarorlik, Yakobian matritsasi hamda parametr sezgirligi kabi nazariy masalalar yetarli darajada tizimli tahlil qilinmagan.

Shu nuqtayi nazardan, mazkur maqolada Dalla Man modeli differensial tenglamalar nazariyasi asosida ilmiy sharh qilinib, uning matematik tuzilishi, holatlar fazosi, musbatlik xossasi, yechimlarning mavjudligi va yagonaligi, parametr sezgirligi hamda lokal barqarorligi kompleks tarzda tahlil qilinadi. Ushbu yondashuv modelning matematik asoslarini chuqurroq yoritishga hamda uni biomatematika va tibbiy modellashtirish yo'nalishlarida qo'llash imkoniyatlarini kengaytirishga xizmat qiladi.

Biologik tizimlarni matematik modellashtirishning asosiy maqsadi organizmda sodir bo'ladigan murakkab fiziologik jarayonlarni matematik qonuniyatlar yordamida tavsiflash va ularning vaqt bo'yicha dinamikasini o'rganishdan iborat. Glukoza–insulin tizimi ham o'zaro bog'langan ko'plab biologik jarayonlardan tashkil topgan dinamik tizim bo'lib, uning matematik modeli differensial tenglamalar sistemasi orqali ifodalanadi.

Tadqiqiot metodologiyasi. Umumiy holda biologik tizim quyidagi boshlang'ich qiymatlar masalasi orqali tavsiflanadi.

$$\begin{cases} \frac{dX(t)}{dt} = F(X(t), U(t), P), \\ X(0) = X_0. \end{cases}$$

bu yerda

- $X(t) \in \mathbb{R}^n$ — $tizimning\ holatlar\ vektori.$
- $U(t)$ — $tashqi\ ta'sir\ (masalan, ovqat\ iste'moli\ yoki\ insulin\ yuborilishi)$;
- P — $fiziologik\ parametrlar\ to'plami$;
- F — $tizim\ dinamikasini\ ifodalovchi\ nolinear\ vektor - funksiya.$

Mazkur ko'rinish zamonaviy glukoza–insulin modellarining, jumladan, Bergman, Sorensen, Hovorka va Dalla Man modellarining umumiy matematik asosini tashkil etadi. Modelning murakkabligi holat o'zgaruvchilari soni va ular orasidagi bog'lanishlarning nolinearlik darajasiga bog'liq [2].

Glukoza–insulin tizimida vaqt bo'yicha o'zgaruvchi barcha fiziologik kattaliklar yagona holat vektorida jamlanadi:

$$X(t) = \begin{Bmatrix} x_{1(t)} \\ x_{2(t)} \\ \vdots \\ x_{n(t)} \end{Bmatrix} \qquad X(t) \in \mathbb{R}^n$$

Bu yerda har bir komponent ma’lum bir biologik jarayonni ifodalaydi. Masalan, plazmadagi glukoza miqdori, to‘qimalardagi glukoza massasi, plazmadagi insulin konsentratsiyasi, jigar orqali ishlab chiqarilayotgan glukoza miqdori va oshqozon-ichak tizimidagi glukoza zahirasi alohida holat o‘zgaruvchilari sifatida qaraladi.

Holatlar fazosi usulining asosiy afzalligi shundaki, u ko‘p o‘zgaruvchili biologik tizimlarni yagona matematik struktura doirasida tahlil qilish imkonini beradi. Aynan shu yondashuv Dalla Man modelining barcha modullarini o‘zaro bog‘langan dinamik sistema sifatida qarash imkonini beradi.

Har qanday matematik modelning aniqligi parametrlarning to‘g‘ri tanlanishiga bog‘liq. Glukoza–insulin tizimida parametrlar fiziologik jarayonlarning intensivligini ifodalaydi. Masalan, glukozaning utilizatsiya tezligi, insulin sekretsiyasi intensivligi, endogen glukoza ishlab chiqarilish tezligi va glukozaning oshqozondan ichakka o‘tish tezligi mos ravishda alohida parametrlar bilan tavsiflanadi.

Matematik nuqtayi nazardan parametrlar quyidagi to‘plam orqali ifodalanadi:

$$P = \{p_1, p_2, \dots, p_m\}$$

Har bir parametrning qiymati eksperimental kuzatishlar asosida baholanadi. Shu sababli parametrlarni identifikatsiyalash masalasi biomatematik modellashtirishning eng muhim bosqichlaridan biri hisoblanadi.

Matematik model biologik jarayonlarni ishonchli ifodalashi uchun boshlang‘ich qiymatlar masalasi **Hadamard korrektligi** talablariga javob berishi lozim. Bunda quyidagi uch shart bajarilishi kerak:

1. yechim mavjud bo‘lishi;
2. yechim yagona bo‘lishi;
3. boshlang‘ich ma’lumotlarning kichik o‘zgarishi yechimning kichik o‘zgarishiga olib kelishi.

Mazkur talablar bajarilgandagina modelning natijalari fiziologik jihatdan ishonchli va matematik nuqtayi nazardan barqaror deb qaraladi.

Olingan natijalar va ularning tahlili. Teorema 1. (Boshlang‘ich qiymatlar masalasining lokal korrektligi)

Teorema. Faraz qilaylik, $F: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^n$ funksiya uzluksiz va X bo‘yicha lokal Lipshits shartini qanoatlantirsin. U holda,

$$\begin{cases} \dfrac{dX}{dt} = F(X), \\ X(0) = X_0 \end{cases}$$

boshlang‘ich qiymatlar masalasi uchun $t \in [0, T]$ oraliqda yagona lokal yechim mavjud.

Isbot. Ushbu natija oddiy differensial tenglamalar nazariyasidagi Picard–Lindelöf teoremasining bevosita qo‘llanilishidan kelib chiqadi. Dalla Man modelidagi o‘ng tomon funksiyalari uzluksiz algebraik bog‘lanishlar va differensiallanuvchi nolinear funksiyalardan tashkil topganligi sababli, modelning lokal korrektligi nazariy jihatdan asoslanadi.

Izoh. Dalla Man va hammualliflarining maqolasida bu teorema alohida isbotlanmagan. Mazkur maqolada esa u modelning matematik poydevorini asoslash uchun kiritilmoqda. Bu maqolaning oddiy sharhdan farq qiladigan jihatlaridan biridir.

Biologik tizimlarni modellashtirishda eng muhim talablardan biri model yechimlarining biologik mazmunga ega bo‘lishidir. Xususan, glukoza massasi, insulin konsentratsiyasi, oshqozon-ichak tizimidagi uglevod miqdori va boshqa fiziologik kattaliklar manfiy qiymat qabul qilishi mumkin emas. Shu sababli modelning barcha holat o‘zgaruvchilari musbat qiymatda saqlanishi zarur.

Dalla Man modelida holat o‘zgaruvchilari

$$X(t) = \begin{Bmatrix} x_{1(t)} \\ x_{2(t)} \\ \vdots \\ x_{n(t)} \end{Bmatrix}$$

ko‘rinishida qaraladi.

Biologik ma’noga ega bo‘lishi uchun

$$x_{i(t)} \geq 0, \qquad i = 1, \dots, n$$

sharti bajarilishi talab etiladi.

Mazkur shart modelning **musbatlik xossasi (positivity property)** deb ataladi.

Musbatlik biologik interpretatsiya uchun zarur bo‘libgina qolmay, sonli hisoblashlarda ham modelning fizik realizmini ta’minlaydi.

Teorema 2. Musbat invariant soha

Faraz qilaylik Dalla Man modelidagi barcha oqim funksiyalari uzluksiz bo‘lsin hamda har bir holat o‘zgaruvchisi nol qiymatga teng bo‘lgan chegarada mos differensial tenglama o‘ng tomoni manfiy bo‘lmagan qiymat qabul qilsin.

Unda

$$X(0) \in \mathbb{R}_+^n$$

boshlang‘ich shart uchun

$$X(t) \in \mathbb{R}_+^n, \ t \geq 0$$

bo‘ladi.

Demak,

$$\mathbb{R}_+^n$$

Dalla Man modeli uchun invariant soha hisoblanadi.

Isbot g‘oyasi shundaki, Invariantlikni ko‘rsatish uchun har bir holat tenglamasi chegarada alohida tekshiriladi.

Agar

$$x_i = 0$$

bo‘lsa,

$$\dot{x}_i \geq 0$$

ekanligi aniqlanadi.

Lemma 1. Glukoza subsistemasining musbatligi

Faraz qilaylik,

$$G_{\mathrm{p}}(0) \geq 0, \qquad G_{\mathrm{t}}(0) \geq 0.$$

Barcha parametrlar musbat bo‘lsin.

U holda

$$G_{\mathrm{p}}(t) \geq 0, \qquad G_{\mathrm{t}}(t) \geq 0$$

barcha

$$t \geq 0$$

lar uchun o‘rinli.

Isbot. Original modelning (1)-tenglamasida glukoza subsistemasi ikki kamerali tizim sifatida berilgan.

Chegarada

$$G_{\mathrm{p}} = 0$$

bo‘lsa, chiqish hadlari yo‘qoladi, kiruvchi oqimlar esa

- endogen glukoza ishlab chiqarilishi,
- ovqatdan keluvchi oqim,
- to‘qimalardan plazmaga qaytuvchi oqim

ko‘rinishida qoladi.

Original modelda ushbu oqimlar fiziologik jihatdan manfiy bo‘lmagan funksiyalar sifatida aniqlangan.

Shuning uchun

$$\left(\frac{dG_{\mathrm{p}}}{dt}\right)\Big|_{(G_{\mathrm{p}}=0)} \geq 0$$

Xuddi shuningdek

$$G_{\mathrm{t}} = 0$$

bo‘lganda

to‘qimalar kamerasidan chiqish hadi nolga teng, kiruvchi oqim esa manfiy emas.

Demak

$$\left(\frac{dG_{\mathrm{p}}}{dt}\right)\Big|_{(G_{\mathrm{p}}=0)} \geq 0$$

Natijada glukoza subsystemasi musbat ortantni tark etmaydi. ■

Lemma 2. Insulin subsistemasining musbatligi

Agar

$$I_{\mathrm{p}}(0) \geq 0, \qquad I_{\mathrm{l}}(0) \geq 0$$

Hamda insulin sekretsiyasi

$$S(t) \geq 0$$

bo‘lsa, u holda

$$I_{\mathrm{p}}(t) \geq 0, \qquad I_{\mathrm{l}}(t) \geq 0$$

Isbot

Original modelning (3)–(5)-tenglamalarida insulin plazma va jigar kameralari orqali almashinadi.
Chegarada

$$I_{\mathrm{p}} = 0$$

bo‘lganda periferik parchalanish hadi yo‘qoladi, sekretsiya esa manfiy emas. Shuning uchun

$$\left(\frac{dI_{\mathrm{p}}}{dt}\right)|^{(I_{\mathrm{p}}=0)} \geq 0$$

Xuddi shunday

$$I_{\mathrm{l}} = 0$$

bo‘lganda jigar kamerasiga kiruvchi oqim mavjud, chiqish oqimi esa nol. Demak

$$\left(\frac{dI_{\mathrm{p}}}{dt}\right)|^{(I_{\mathrm{p}}=0)} \geq 0$$

Shunday qilib, insulin subsystemasi ham musbat invariant hisoblanadi. . ■

Lemma 3. Ovqat hazm qilish subsistemasining musbatligi

Agar

$$Q_{sto1}(0) \geq 0, \qquad Q_{sto2}(0) \geq 0, \qquad Q_{gut}(0) \geq 0$$

bo‘lsa, barcha

$$t \geq 0$$

uchun

$$Q_{(sto1)}(t) \geq 0, \qquad Q_{(sto2)}(t) \geq 0, \qquad Q_{(gut)}(t) \geq 0$$

Isbot

Original modelning (13)-tenglamasi oshqozonning qattiq va suyuq fazalari hamda ichak kamerasini tavsiflaydi.

Chegarada

$$Q_{sto1} = 0$$

bo‘lsa, chiqish oqimi nolga teng, ovqat qabul qilinganda esa kirish oqimi manfiy emas. Shuning uchun

$$\frac{dQ_{sto1}}{dt} \geq 0$$

Xuddi shu mulohaza

$$Q_{sto2}$$

va

$$Q_{gut}$$

uchun ham amal qiladi, chunki barcha oqimlar massaning almashinuvini ifodalaydi va manfiy massa hosil qilmaydi. Natijada ovqat hazm qilish moduli musbat invariantdir. ■

Yuqorida bir qancha tenglamalar isboti har bir holat tenglamasi chegarada alohida tekshiriladi.

Agar

$$x_i = 0$$

bo‘lsa,

$$\dot{\mathrm{x}}_i \geq 0$$

ekanligi aniqlandi.

Natijada trayektoriya koordinata tekisligini kesib o‘ta olmaydi. Nagumo invariantlik teoremasiga ko‘ra yechim musbat ortantni tark etmaydi. Shunday qilib biologik realizm matematik jihatdan asoslandi.

Eslatma. Bu yerda Nagumo teoremasi metodologik asos sifatida keltirilmoqda. Dalla Man modeli uchun to‘liq formal isbot berish modeldagi har bir tenglama va parametrni alohida tekshirishni talab qiladi; bu maqolada esa invariantlik g‘oyasi va uni tekshirish yo‘nalishi ko‘rsatiladi.

Muhokama. Dinamik tizimlarni tahlil qilishda muvozanat nuqtasi (equilibrium point) asosiy tushunchalardan biridir. Muvozanat holatida tizimning barcha holat o‘zgaruvchilari vaqt bo‘yicha o‘zgarmaydi. Matematik nuqtayi nazardan

$$\left\{\frac{dX}{dt} = 0\right.$$

shart bajariladi.

Dalla Man modelida ovqat qabul qilinmagan bazal holatda tashqi kirish signali nolga teng deb olinadi.

Natijada

$$F(X^*) = 0$$

ko‘rinishidagi nolinear algebraik sistema hosil bo‘ladi.

Hosil bo‘lgan

$$X^* = \begin{Bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{Bmatrix} \in \mathbb{R}_+^n$$

vektor bazal fiziologik holatni ifodalaydi.

Mazkur nuqta keyingi barqarorlik tahlilining boshlang‘ich nuqtasi hisoblanadi.

Teorema 3. Muvozanat nuqtaning mavjudligi

Agar model parametrlarining qiymatlari fiziologik jihatdan ruxsat etilgan sohada tanlangan bo‘lsa va tizimning kirish signali bazal holatga mos bo‘lsa, u holda model kamida bitta muvozanat nuqtaga ega bo‘ladi.

Izoh. Bu teorema umumiy dinamik tizimlar nazariyasiga tayangan holda keltirilmoqda. Dalla Man modeli uchun muvozanat nuqtasini aniq hisoblash parametrlar qiymatlariga bog‘liq bo‘lib, alohida sonli hisoblashni talab qiladi.

Har qanday matematik modelning amaliy qiymati parametrlarni eksperimental ma’lumotlardan aniqlash imkoniyatiga bog‘liq.

Dalla Man modelida parametrlar soni ko‘p bo‘lganligi sababli ularning barchasini bir vaqtning o‘zida baholash amaliy jihatdan murakkab hisoblanadi. Shuning uchun parametrlarni identifikatsiya qilish masalasi alohida ilmiy muammo sifatida qaraladi.

Identifikatsiya quyidagi optimallashtirish masalasiga keltiriladi:

$$min_P J(P)$$

bu yerda $J(P)$ — model natijalari bilan tajriba ma’lumotlari orasidagi moslik mezoni. Mazkur masala nolinear optimallashtirish usullari yordamida yechiladi.

Xulosa. Ushbu maqolada glukoza–insulin tizimini tavsiflovchi matematik modelning umumiy nazariy asoslari Dalla Man modeli misolida tahlil qilindi. Tahlil shuni ko‘rsatadiki, modelning amaliy qo‘llanilishi faqat uning fiziologik aniqligiga emas, balki matematik jihatdan korrekt qo‘yilganligiga ham bog‘liq.

Mazkur tahlil asosida, Dalla Man modelining eng muhim nazariy ustunligi uning **holatlar fazosi** asosida qurilgan ko‘p komponentli dinamik tizim ekanligidir.

Bunday tuzilma glukoza va insulin almashinuvining murakkab mexanizmlarini yagona matematik model doirasida ifodalash imkonini beradi. Shu bilan birga, modelning ko‘p parametrli va nolinear tuzilishi uning identifikatsiyasi hamda nazariy tahlilini murakkablashtiradi.

Mazkur maqola doirasida invariant soha, musbatlik xossasi va muvozanat nuqtasi umumiy matematik nazariya asosida ko‘rib chiqildi. Keyingi maqolalarda ushbu g‘oyalar Dalla Man modelining alohida kichik tizimlari (glukoza kinetikasi, insulin kinetikasi, endogen glukoza ishlab chiqarilishi va boshqa modullar) misolida batafsil tahlil qilinadi. Shuningdek, parametrlarning sezgirligi va modelning barqarorlik xossalari sonli tajribalar bilan bog‘liq holda muhokama qilinadi.

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HOSILAGA NISBATAN YECHILGAN BA’ZI BIR DIFFERENSIAL TENGLAMALAR

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Annotatsiya. *Ushbu maqolada hosilaga nisbatan yechilgan birinchi tartibli differensial tenglamalarning ayrim maxsus turlari, xususan, Darbu va Yakobi tenglamalarining asosiy xossalari hamda ularni integrallash usullari o‘rganilgan. Bir jinsli funksiyalar xossasi va mos almashtirish yordamida Darbu tenglamasini Bernulli tenglamasiga keltirish imkoniyati ko‘rsatilgan. Yakobi tenglamasini algebraik tenglamalar sistemasining haqiqiy yechimi asosida koordinatalar boshini ko‘chirish orqali Darbu tenglamasiga keltiruvchi soddalashtirilgan usul taklif etilgan. Taklif qilingan usulning samaradorligi aniq misol va uning geometrik talqini yordamida asoslab berilgan. Olingan natijalardan oliy ta’lim amaliyotida foydalanish mumkin.*

Kalit so‘zlar: *differensial tenglama, Darbu tenglamasi, Yakobi tenglamasi, Bernulli tenglamasi, bir jinsli differensial tenglama, almashtirish usuli, kvadratura, umumiy integral, maxsus yechim.*

НЕКОТОРЫЕ ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ, РЕШЁННЫЕ ОТНОСИТЕЛЬНО ПРОИЗВОДНОЙ

Аннотация. *В данной статье исследуются некоторые специальные виды дифференциальных уравнений первого порядка, решённых относительно производной, в частности основные свойства уравнений Дарбу и Якоби, а также методы их интегрирования. Показана возможность приведения уравнения Дарбу к уравнению Бернулли с помощью свойства однородных функций и соответствующей подстановки. Предложен упрощённый способ приведения уравнения Якоби к уравнению Дарбу путём переноса начала координат в действительное решение алгебраической системы. Эффективность предложенного способа обоснована конкретным примером и его геометрической интерпретацией.*

Ключевые слова: *дифференциальное уравнение, уравнение Дарбу, уравнение Якоби, уравнение Бернулли, однородное дифференциальное уравнение, метод подстановки, квадратура, общий интеграл, особое решение.*

SOME DIFFERENTIAL EQUATIONS SOLVED WITH RESPECT TO THE DERIVATIVE

Abstract.*This article studies special types of differential equations solved with respect to the derivative, namely the Darboux and Jacobi equations and the methods of their integration. It is shown that the Darboux equation can be reduced to the Bernoulli equation by means of the homogeneity property and a suitable substitution. A simplified way of reducing the Jacobi equation to the Darboux equation by translating the origin is proposed. Its efficiency is justified by an example and its geometric interpretation.*

Key words: *differential equation, Darbou equation, Jacobi equation, Bernoulli equation, homogeneous differential equation, substitution method, quadrature, general integral, particular solution.*

Kirish. Differensial tenglamalar nazariyasi tabiiy va texnik fanlarning ko‘plab masalalarini matematik modellashtirishning asosiy vositalaridan biri hisoblanadi. Mexanika, fizika, biologiya, iqtisodiyot va boshqa sohalarda uchraydigan jarayonlarning aksariyati hosilaga nisbatan yechilgan birinchi tartibli differensial tenglamalar yordamida ifodalanadi. Bunday tenglamalarning umumiy yechimini kvadraturalarda topish har doim ham mumkin bo‘lavermaydi, shu sababli ularning integrallanuvchi maxsus sinflarini ajratib o‘rganish nazariy hamda amaliy jihatdan muhim ahamiyatga ega.

Birinchi tartibli tenglamalarning integrallanuvchi sinflari orasida o‘zgaruvchilari ajraladigan, bir jinsli, chiziqli va Bernulli tenglamalari eng ko‘p o‘rganilgan turlar hisoblanadi. Ularning har biri uchun mos almashtirish tanlash orqali yechim kvadraturaga keltiriladi. Shu bilan birga, amaliyotda bevosita bu sinflarga

kirmaydigan, ammo maxsus almashtirishlar yordamida ularga keltirilishi mumkin bo‘lgan tenglamalar ham ko‘p uchraydi. Darbu va Yakobi tenglamalari aynan shunday tenglamalar sirasiga kiradi.

Mavzuga doir adabiyotlar tahlili. Oddiy differensial tenglamalarning analitik yechish usullari va ularning tatbiqlari V. Henner, A. Nepomnyashchy, T. Belozerova hamda M. Khenner mualliflarining monografiyasida keng yoritilgan; unda birinchi tartibli tenglamalarning asosiy integrallanuvchi sinflari va ularni almashtirishlar yordamida soddalashtirish masalalari batafsil bayon etilgan [1]. T. W. Judsonning elektron darsligida esa oddiy differensial tenglamalarni yechish usullari va amaliy tatbiqlari zamonaviy uslubiy yondashuv asosida keltirilgan [2].

So‘nggi yillardagi tadqiqotlarda Darbu polinomlari va ular yordamida polinomial differensial sistemalarning integrallanuvchanligini o‘rganish masalasi alohida o‘rin egallamoqda. M. V. Deminaning ishlarida Levinson–Smit hamda Lienar tipidagi polinomial differensial tenglamalarning integrallanuvchanligi va yechilishi Darbu yondashuvi asosida tadqiq etilgan [3; 4]. Keltirilgan adabiyotlar tahlili shuni ko‘rsatadiki, Darbu va Yakobi tenglamalarini o‘quv hamda amaliy maqsadlarda nisbatan soddaroq usul bilan integrallash masalasi bugungi kunda ham o‘z ahamiyatini yo‘qotmagan.

Klassik adabiyotlarda birinchi tartibli differensial tenglamalarning integrallanuvchi sinflari to‘liq tasniflangan. E. Kamke ma’lumotnomasida bunday tenglamalarning keng ro‘yxati va ularni yechish usullari jamlangan [5]. A.D. Polyanin va V.F. Zaytsev ma’lumotnomasida esa Darbu va Yakobi tipidagi tenglamalar alohida sinf sifatida ajratilib, ularning umumiy integrali bilan bog‘liq natijalar keltirilgan [6]. L.E. Elsgolts va N.M. Matveyev darsliklarida almashtirish usulining nazariy asoslari va uning qo‘llanilish chegaralari batafsil yoritilgan [7; 8]. Mazkur manbalarda keltirilgan usullar nazariy jihatdan to‘liq bo‘lsa-da, amaliy hisoblashlarda ancha ko‘p oraliq amallarni talab qiladi.

Tadqiqot obyekti va qo‘llanilgan metodlar. Tadqiqot obyekti sifatida hosilaga nisbatan yechilgan birinchi tartibli differensial tenglamalarning maxsus sinflari — Darbu va Yakobi tenglamalari olingan. Tadqiqotda bir jinsli funksiyalarning xossalari, o‘zgaruvchilarni almashtirish usuli, koordinatalar boshini ko‘chirish hamda Bernulli tenglamasini integrallash usullaridan foydalanilgan. Olingan natijalarning to‘g‘riligi aniq misol yordamida tekshirib ko‘rilgan.

Taklif etilayotgan yondashuv quyidagi ketma-ket bosqichlardan iborat: birinchidan, berilgan tenglama (5) ko‘rinishga keltiriladi va uning koeffitsiyentlari aniqlanadi; ikkinchidan, (7) algebraik sistema tuzilib, uning haqiqiy yechimi topiladi; uchinchidan, koordinatalar boshi shu yechimga ko‘chiriladi va tenglama Darbu ko‘rinishiga keladi; to‘rtinchidan, bir jinsli funksiyalar xossasidan foydalanib almashtirish bajariladi va Bernulli tenglamasi hosil qilinadi; beshinchidan, Bernulli tenglamasi ma’lum usulda integrallanadi va eski o‘zgaruvchilarga qaytiladi. Ushbu bosqichlarning har biri elementar amallar bilan bajarilishi usulning asosiy afzalligini tashkil etadi.

Ushbu maqolada diffirensial tenglamalarning ba’zi turlari haqida fikr yuritib ularning xossalari va integrallash usullarini ko‘rib o‘tamiz. Ushbu

$$M(x,y)dx + N(x,y)dy + P(x,y)(xdy - ydx) = 0 \qquad (1)$$

ko‘rinishidagi tenglama Darbu tenglamasi deyiladi. Darbu Gaston (1842-1917)– fransuz matematigi. Geometriya va differensial tenglamalar sohasida ishlab, mexanikaga doir tadqiqotlar ham olib borgan olim.

Demak, yuqorida berilgan (1) tenglamada N, M lar $m-$ darajali, $P(x,y)$ esa $l-$ darajali bir jinsli funksiyalar, $l \neq m-1$, agar $l = m-1$ bo‘lsa, (1) tenglama bir jinsli tenglama bo‘ladi. Bu tenglama $y = zx$ almashtirish orqali Bernulli tenglamasiga keltiriladi, z yangi izlanuvchi funksiya. Bir jinsli funksiyalarning xossasiga asosan (1) tenglamani

$$x^m M\left(1,\frac{y}{x}\right)dx + x^m N\left(1,\frac{y}{x}\right)dy + x^l P\left(1,\frac{y}{x}\right)\cdot(xdy - ydx) = 0 \qquad (2)$$

ko‘rinishda yozish mumkin. Ushbu

$$dy = xdz + zdx,\ \ xdy - ydx = x^2 d\left(\frac{y}{x}\right) = x^2 dz \qquad (3)$$

tengliklardan foydalanib, (2) ni

$$x^m M(1,z)dx + x^m N(1,z)(xdz + zdx) + x^l P(1,z)x^2 dz = 0$$

ko‘rinishida yozamiz, yoki x^m ga qisqartirib,

$$[M(1,z) + zN(1,z)]dx + [xN(1,z) + P(1,z)x^{l+1-m}]dz = 0$$

tenglamani hosil qilamiz. Uni $\frac{dx}{dz}$ ga nisbatan quyidagicha yozish mumkin:

$$\frac{dx}{dz} + \frac{N(1,z)}{N(1,z)+zN(1,z)}x = \frac{P(1,z)x^{l+1-m}}{N(1,z)+zN(1,z)} \qquad (4)$$

bu yerda $M(1,z) + zN(1,z) \neq 0, x = 0$ bo‘lganda bu amalni bajarish mumkin. (4) tenglama izlanuvchi funksiya x va argument z ga nisbatan Bernulli tenglamasidir. Bu tenglamani integrallab, y o‘zgaruvchiga qaytsak, (1) tenglamaning umumiy integralini topgan bo‘lamiz.

Agar $z_k (k = 1,2, \dots)$

$$N(1, z) + zN(1, z) = 0$$

tenglamaning haqiqiy ildizlari bo‘lsa, $y = z_k x$ ko‘rinishdagi to‘g‘ri chiziqlar (1) tenglamaning yechimlari bo‘ladi, ular tenglamaning maxsus yechimlari ham bo‘lishi mumkin. Ushbu

Maxsus yechimlar masalasi alohida e’tiborni talab qiladi. Yuqorida keltirilgan to‘g‘ri chiziqlar integral chiziqlar oilasining o‘ramasi bo‘lishi yoki oilaning alohida a’zosi sifatida namoyon bo‘lishi mumkin. Ularning maxsus yechim ekanligini aniqlash uchun umumiy integral topilgandan so‘ng qo‘shimcha tekshirish o‘tkaziladi: agar berilgan chiziq umumiy integraldan o‘zgarmasning hech qanday qiymatida hosil bo‘lmasa, u maxsus yechim hisoblanadi. Amaliy masalalarda bunday yechimlar ko‘pincha jarayonning muvozanat holatlariga mos keladi, shu bois ularni tashlab yuborish masalaning to‘liq tahlil qilinmasligiga olib keladi [7].

$$(a_0 + a_1 x + a_2 y)(xdy - ydx) - (b_0 + b_1 x + b_2 y)dy$$
$$+(c_0 + c_1 x + c_2 y)dx = 0 \qquad (5)$$

ko‘rinishdagi tenglama Yakobi tenglamasi deyiladi. Yakobi Karl Gustav (1804-1851) nemis matematigi. Elliptik funksiyalar nazariyasini yaratgan olimlardan biri. Differensial tenglamalar, integral hisob, sonlar nazariyasi sohalarida ham ishlagan. Yuqorida berilgan (5) tenglamada $a_i, b_i, c_i -$ o‘zgarmas haqiqiy sonlar $(i = 0, 1, 2)$. Xususiy hollar:

1. $a_0 = b_0 = c_0 = 0$ bo‘lsa, (5) dan Darbu tenglamasi;
2. $a_2 = b_2 = c_2 = 0$ bo‘lsa, (5) dan chiziqli tenglama;
3. $a_1 = a_2 = 0$ bo‘lsa, (5) dan bir jinsli tenglama hosil bo‘ladi.

Keltirilgan xususiy hollar Yakobi tenglamasining umumlashtiruvchi xarakterini ko‘rsatadi: bitta tenglama sinfi o‘z ichiga bir necha ma’lum sinfni qamrab oladi. Bu esa uni o‘rganishning nazariy ahamiyatini oshiradi, chunki Yakobi tenglamasi uchun topilgan integrallash usuli avtomatik ravishda barcha xususiy hollarga ham tatbiq etiladi. Boshqa tomondan, koeffitsiyentlar orasidagi munosabatlarga qarab tenglamaning tuzilishi sezilarli o‘zgarishi mumkin, shuning uchun har bir aniq masalada avval koeffitsiyentlar tahlil qilinishi, so‘ngra mos usul tanlanishi maqsadga muvofiqdir [6].

Muhokama. Umuman, Yakobi tenglamasi har doim kvadraturaga integrallanadi. Ko‘pgina adabiyotlarda berilgan integrallash usullari murakkabroq deb hisoblab, biz ancha soddaroq usulni tavsiya etamiz.

(5) tenglamani quyidagi

$$(R_1 + yP_1)dx - (Q_1 - xP_1)dy = 0 \qquad (6)$$

ko‘rinishda yozib olish mumkin, bu yerda

$$P_1 = a_0 + a_1 x + a_2 y,\ Q_1 = b_0 + b_1 x + b_2 y, R_1 = c_0 + c_1 x + c_2 y.$$

Ushbu

$$\begin{cases} R_1 - yP_1 = 0 \\ Q_1 - xP_1 = 0 \end{cases} \qquad (7)$$

algebraik tenglamalar sistemasi hech bo‘lmaganda bitta (x_0, y_0) haqiqiy yechimga ega, eng ko‘p yechimlari soni uchta, ya’ni x, y tekislikda (7) dagi ikkinchi tartibli chiziqlar hech bo‘lmaganda bitta nuqtada kesishadi. Koordinatalar boshini $x - x_0 = \bar{x}, y - y_0 = \bar{y}$ almashtirish kiritib, (x_0, y_0) nuqtaga ko‘chiramiz, u vaqtda

$$\left[M_x^1(x_0, y_0)\bar{x} + M_y^1(x_0, y_0)\bar{y}\right]dx -$$
$$\left[N_x^1(x_0, y_0)\bar{x} + N_y^1(x_0, y_0)\bar{y} - \bar{x}(a_1\bar{x} + a_2\bar{y})\right]dy = 0 \qquad (8)$$

tenglama hosil bo‘ladiki, bu Darbu tenglamasidan iboratdir.

Natijalar va ularning tahlili. Misol: Ushbu

$$(1 + 2x + 3y)(xdy - ydx) + (2x + 4y)dy + (1 - 3x - 2y)dx = 0$$

tenglamani Darbu tenglamasiga keltiring.

Yechish. Tenglamani

$$[1 - 3x - 3y - y(2x + 3y)]dx + [3x + 4y + x(2x + 3y)]dy = 0$$

ko‘rinishda yozib olamiz.

Bu misol uchun (7) algebraik sistema

$$\begin{cases} 1 - 3x - 3y - y(2x + 3y) = 0 \\ 3x + 4y + x(2x + 3y) = 0 \end{cases}$$

ko‘rinishda bo‘ladi. Bu sistemaning yechimlarini topamiz. Masalan, $x = -1,\ y = 1$ bu sistemaning yechimidir, ya’ni

$$x + 1 = \bar{x}, \qquad y - 1 = \bar{y}$$

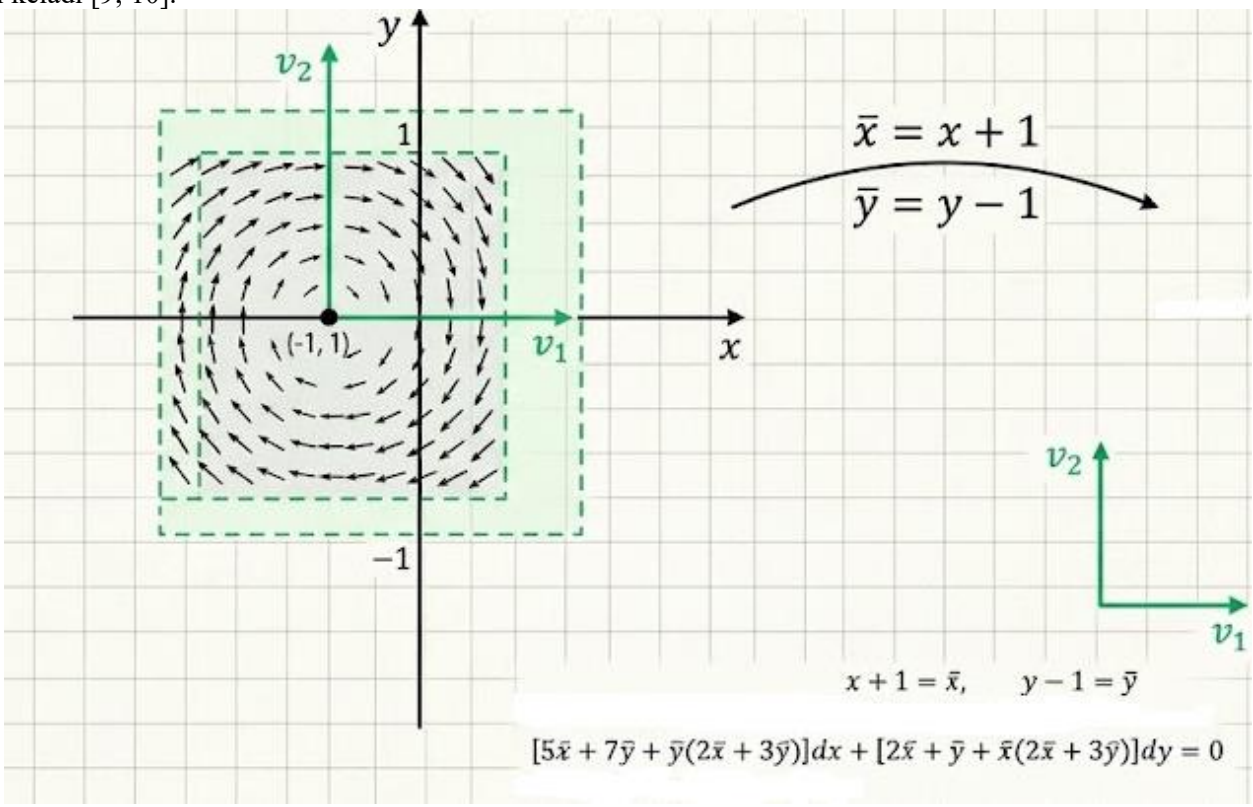
almashtirish bajarsak, quyidagi tenglama hosil bo‘ladi:

$$[5\bar{x} + 7\bar{y} + \bar{y}(2\bar{x} + 3\bar{y})]dx + [2\bar{x} + \bar{y} + \bar{x}(2\bar{x} + 3\bar{y})]dy = 0$$

Bu esa Darbu tenglamasidir.

Keltirilgan misoldan ko‘rinib turibdiki, (7) algebraik sistemaning haqiqiy yechimini topish va koordinatalar boshini shu nuqtaga ko‘chirish Yakobi tenglamasidagi ozod hadlarni yo‘qotadi hamda uni bir jinsli tuzilishga ega Darbu tenglamasiga aylantiradi. Hosil bo‘lgan Darbu tenglamasi esa yuqorida ko‘rsatilgan almashtirish orqali Bernulli tenglamasiga, undan esa kvadraturaga keltiriladi. Shunday qilib, boshlang‘ich tenglamani yechish jarayoni ketma-ket bajariladigan ikkita almashtirishdan iborat sodda algoritm ko‘rinishida ifodalanadi.

Shuni ta’kidlash lozimki, (7) sistemaning yechimi yagona bo‘lmasligi mumkin. Bunday holda koordinatalar boshini turli nuqtalarga ko‘chirish natijasida bir-biridan farq qiluvchi, ammo ekvivalent Darbu tenglamalari hosil bo‘ladi. Amaliy nuqtayi nazardan koeffitsiyentlari sodda bo‘lgan variantni tanlash keyingi hisoblashlarni yengillashtiradi. Agar sistema haqiqiy yechimga ega bo‘lmasa yoki yechim murakkab ko‘rinishda bo‘lsa, mazkur usul samarasiz bo‘lib qoladi va boshqa integrallash usullariga murojaat qilishga to‘g‘ri keladi [9; 10].



1-rasm. Darbu tenglamasining geometrik interpretatsiyasi

1-rasmda keltirilgan geometrik talqin almashtirishning mohiyatini yaqqol ifodalaydi: koordinatalar boshining yangi nuqtaga ko‘chirilishi natijasida integral chiziqlar oilasining tuzilishi o‘zgarmaydi, faqat tenglamaning ko‘rinishi soddalashadi. Bu esa qo‘llanilgan almashtirishning ekvivalentligini va olingan natijaning to‘g‘riligini tasdiqlaydi.

Taklif etilayotgan usulning asosiy afzalligi shundaki, u qo‘shimcha murakkab hisoblashlarni talab qilmaydi va faqat elementar algebraik amallarga hamda ma’lum almashtirishlarga tayanadi. Bu esa mazkur usuldan oliy o‘quv yurtlarida differensial tenglamalar kursini o‘qitishda, amaliy va laboratoriya mashg‘ulotlarini tashkil etishda, shuningdek talabalarning mustaqil ishlarini yo‘lga qo‘yishda foydalanish imkonini beradi.

Usulning cheklovlari ham mavjud. Birinchidan, u faqat (5) ko‘rinishdagi tenglamalarga, ya’ni koeffitsiyentlari chiziqli bo‘lgan holga tatbiq etiladi. Ikkinchidan, Bernulli tenglamasini integrallash bosqichida hosil bo‘ladigan kvadraturalar har doim ham elementar funksiyalarda ifodalanmasligi mumkin. Uchinchidan, algebraik sistemaning yechimini topish yuqori darajali tenglamalarda qo‘shimcha qiyinchilik tug‘diradi. Shunga qaramay, o‘quv jarayonida uchraydigan masalalarning aksariyati uchun taklif etilgan algoritm to‘liq yetarli bo‘lib, an’anaviy usullarga nisbatan sezilarli qulaylik beradi.

Xulosa. Xulosa qilib aytganda, differensial tenglamalarning maxsus turlarini o‘rganish va ularni sodda ko‘rinishdagi tenglamalarga keltirish integrallash jarayonini sezilarli darajada yengillashtiradi. Maqolada Darbu tenglamasini bir jinsli funksiyalar xossasi va mos almashtirish yordamida Bernulli tenglamasiga

keltirish imkoniyati ko‘rsatildi. Bu usul Darbu tenglamasining umumiy integralini topishda samarali vosita bo‘lib xizmat qiladi. Yakobi tenglamasi esa o‘zining umumiy ko‘rinishidan tashqari, Darbu, chiziqli va bir jinsli tenglamalarni xususiy hollar sifatida o‘z ichiga olishi bilan alohida ahamiyatga ega. Uni algebraik tenglamalar sistemasining haqiqiy yechimlari asosida koordinatalar sistemasini almashtirish orqali Darbu tenglamasiga keltirish mumkin. Keltirilgan misol mazkur usulning amaliy jihatdan qulay va sodda ekanligini ko‘rsatadi. Shunday qilib, Darbu va Yakobi tenglamalarini o‘rganish differensial tenglamalarni tasniflash, ularning o‘zaro bog‘liqligini aniqlash hamda murakkab tenglamalarni ma’lum klassik tenglamalarga keltirish orqali yechish imkoniyatlarini kengaytiradi. Mazkur usullar differensial tenglamalarni o‘qitish va amaliy masalalarni yechishda samarali metodik vosita sifatida foydalanilishi mumkin.

Ishning asosiy natijalarini quyidagicha umumlashtirish mumkin: 1) Darbu tenglamasini bir jinsli funksiyalar xossasi asosida Bernulli tenglamasiga keltirish sxemasi to‘liq bayon etildi; 2) Yakobi tenglamasini koordinatalar boshini ko‘chirish orqali Darbu tenglamasiga keltiruvchi soddalashtirilgan algoritm taklif qilindi; 3) algoritmning ishlashi aniq misolda ko‘rsatilib, uning geometrik talqini berildi; 4) usulning qo‘llanilish chegaralari va cheklovlari aniqlandi. Olingan natijalardan differensial tenglamalar kursini o‘qitishda, amaliy mashg‘ulotlar va talabalarning mustaqil ishlarini tashkil etishda foydalanish mumkin.

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BIRINCHI TUR SIRT INTEGRALLARI HAQIDA

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Annotatsiya*. Ushbu maqola birinchi tur sirt integrallari haqida bo‘lib, unda sirt tushunchasi, sodda sirtlar, uzluksiz akslantirishlar, sirtning parametrik tenglamasi, sirtning maxsus nuqtasi, sirt normali, birinchi tur sirt integrali tushunchasi haqida so‘z borgan. Uning xossalari keltirib o‘tilgan. Birinchi tur sirt integralining mavjudligi va uni hisoblash haqida teorema va isboti hamda hisoblash formulalari keltirib chiqarilgan.*

Kalit so‘zlar*: sirt, yuz, integral, akslantirish, normal, xossa, uzluksizlik, grafik.*

О ПОВЕРХНОСТНЫХ ИНТЕГРАЛАХ ПЕРВОГО РОДА

Аннотация*. Данная статья посвящена поверхностным интегралам первого рода, в ней рассматриваются понятие поверхности, простые поверхности, непрерывные отражения, параметрическое уравнение поверхности, специальная точка поверхности, нормаль к поверхности, понятие поверхностного интеграла первого рода. Приводятся его свойства. Представлены теорема и доказательство существования поверхностного интеграла первого рода и его вычисление, а также формулы для вычисления.*

Ключевые слова*: поверхность, грань, интеграл, отражение, нормаль, свойство, непрерывность, график.*

ON FIRST-TYPES SURFACE INTEGRALS

Abstract*. This article is about surface integrals of the first kind, in which the concept of a surface, simple surfaces, continuous reflections, parametric equation of a surface, special point of a surface, surface normal, the concept of a surface integral of the first kind are discussed. Its properties are given. The theorem and proof of the existence of a surface integral of the first kind and its calculation, as well as calculation fo‘rmulas are given.*

Keywords: *surface, face, integral, reflection, normal, property, continuity, graph.*

Kirish. Tekislikda egri chiziq ushbu

$$\begin{cases} x = x(t), \\ y = y(t) \end{cases} \quad (\alpha \le t \le \beta)$$

tenglamalar sistemasi bilan (xususan, $y = f(x)$, $(a \le x \le b)$ tenglama bilan) aniqlanishini ko‘rdik. Ravshanki, egri chiziq $x(t)$, $y(t)$ funksiyalarga bog‘liq. Agar $x(t)$, $y(t)$ funksiyalar $[\alpha, \beta]$ da uzluksiz $x'(t)$, $y'(t)$ hosilalarga ega bo‘lsa, uni silliq egri chiziq deyiladi.

Egri chiziqlar nazariyasida muhim tushunchalardan biri egri chiziqning uzunligi bo‘lib, u egri chiziqqa chizilgan siniq chiziq perimetrining limiti sifatida ta’riflanganligi va uning uzunligi quyidagi

$$l = \int_{\alpha}^{\beta} \sqrt{x'^2(t) + y'^2(t)}\,dt$$

fo‘rmula bilan ifodalanishi ma’ruzalarda bayon etilgan.Tekislikda egri chiziq tushunchasiga o‘xshash fazoda sirt tushunchasi kiritilsada, uning o‘ziga xos tomonlari mavjud.Fazoda sodda sirtlar - sfera, silindr va shunga o‘xshash sirtlar va ularning yuzi, yuzining hisoblash fo‘rmulalari o‘quvchiga ma’lum.

Biz quyida umumiy ko‘rinishdagi sirtlar, ularning yuzi, yuzini topish fo‘rmulalari kurs uchun kerak bo‘ladigan hajmda qisqacha bayon etamiz.

Manba va usullar. Ushbu mavzu ko‘plab matematik tahlil fanining adabiyotlarida o‘rganilgan va malumotlar keltirilgan. Jumladan, Gulmirza Xudoyberganov, Azizjon Kenjabayevich Vorisov, Hojakbar

Turobovich Mansurov, Bohodir Allaberdiyevich Shoimqulovlarning “Matematik analizdan ma’ruzalar” nomli o‘quv qo‘llanmasida, T.Azlarov, H.Mansurovning “Matematik analiz”, 2-qism, G.Xudoyberganov, A.Varisov, H.Mansurov, B.Shoimqulovning “Matematik analiz” 3-qism kitoblarida chuqur o‘rganilgan.

Natija va topilmalar.

Sirt tushunchasi. Aytaylik, UOV Sirt tushunchasi. Aytaylik Δ tekislikdagi to‘plamda

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v) \tag{1}$$

to‘plamda funksiyalar berilgan bo‘lib, ular Δ Sirt tushunchasi. Aytaylik tekislikdagi to‘plamda funksiyalar berilgan bo‘lib, ular

uzluksiz bo‘lsin. $(u_0, v_0) \in \Delta$ nuqtani olib, yuqoridagi funksiyalarning shu nuqtadagi qiymatlarini topamiz:

$$x_0 = x(u_0, v_0),\ y_0 = y(u_0, v_0),\ z_0 = z(u_0, v_0).$$

Hosil bo‘lgan (x_0, y_0, z_0) ni R^3 fazodagi M_0 nuqtaning koordinatalari deb qaraymiz: $M_0 = M(x_0, y_0, z_0)$. Ravshanki, (u, v) nuqta Δ to‘plamda o‘zgarganda (x, y, z) lar R^3 fazoda biror S to‘plamni hosil qiladi. Demak, (1) munosabatni Δ to‘plamni S to‘plamga uzluksiz akslantirish deb qarash mumkin.

Agar (1) akslantirish o‘zaro bir qiymatli akslantirish bo‘lsa, ya’ni Δ to‘plamning turli $(u, v), (\bar{u}, \bar{v})$ nuqtalarini S to‘plamning turli $(x, y, z), (\bar{x}, \bar{y}, \bar{z})$, S to‘plamni R^3 fazoda sirt deb qarash mumkin. Odatda, bunday sirtlar sodda sirtlar deyiladi. Shu sababli

$$\begin{cases} x = x(u,v), \\ y = y(u,v), \qquad ((u,v) \in \Delta) \\ z = z(u,v) \end{cases} \tag{2}$$

tenglamalar sistemasi sirtning parametrik tenglamasi deyiladi, bunda u va v lar parametrlar.

Xususan ($u = x, v = y$ bo‘lganda) ushbu

$$\begin{cases} x = x, \\ y = y, \\ z = f(x, y) \end{cases}$$

tenglamalar sitemasi aniqlaydigan sirt $z = f(x, y)$ (3)

tenglama bilan ifodalanadi.

Faraz qilaylik, (1) funksiyalar Δ to‘plamda uzluksiz xususiy hosilalarga ega bo‘lib, ular yordamida quyidagi

$$\begin{Vmatrix} x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{Vmatrix} \tag{4}$$

funksional matritsa tuzilgan bo‘lsin.

Aytaylik, $(u_0, v_0) \in \Delta$ nuqta uchun (bu nuqtaning sirtda aksi S $M_0(x_0, y_0, z_0)$ bo‘ladi) (4) matritsaning ikkinchi tartibli determinantidan kamida bittasi noldan farqli, masalan

$$\begin{vmatrix} x'_u(u_0, v_0) & y'_u(u_0, v_0) \\ x'_v(u_0, v_0) & y'_v(u_0, v_0) \end{vmatrix} \neq 0$$

$$z = z(u(x, y), v(x, y)) = f(x, y) \tag{5}$$

deylik. Unda oshkormas funksiyalar haqidagi teoremaga ko‘ra sirtning nuqta atrofidagi qismi quyidagi.

Agar (u_0, v_0) uchun (4) matritsaning barcha ikkinchi tartibli deter¬mi¬nantlari nolga teng bo‘lsa, S Sirtning $M_0(x_0, y_0, z_0)$ nuqtasi, uning maxsus nuqtasi deyiladi. Sirtning maxsus nuqta atrofidagi qismini (5) ko‘rinishda ifodalab bo‘lmaydi.

Faraz qilaylik, S sirt

$$z = f(x, y)$$

tenglama bilan aniqlangan bo‘lsin. Bunda $f(x, y)$ funksiya XOY tekislikdagi D to‘plamda uzluksiz va uzluksiz $f_x^{'}(x, y), f_y^{'}(x, y)$ xususiy hosilalarga ega. Bunday S sirt har bir $M(x, y, z) \in S$ nuqtada urinma tekislikka ega. Urinma tekiclikning tenglamasi ushbu

$$Z - z = f_x'(x, y)(X - x) + f_y'(x, y)(Y - y)$$

ko‘rinishda bo‘ladi, bunda X, Y, Z lar urinma tekilikdagi o‘zga¬ruv¬chi nuqtaning koordinatalari. Modomiki, S sirt $M(x, y, z)$ nuqtada urinma tekislikka ega ekan, unda shu nuqtada sirtning normalini aniqlash mumkin. (Ma’lumki, sirtning $M(x, y, z)$ nuqtasidan o‘tuvchi va shu nuqtadagi urinma tekislikka perpendikulyar bo‘lgan to‘g‘ri chiziq sirt normali deyiladi). Bu holda normalning tenglamasi

$$\frac{X - x}{f_x^{'}(x, y)} = \frac{Y - y}{f_y^{'}(x, y)} = \frac{Z - z}{-1}$$

bo‘ladi. Bu yerda X, Y, Z lar normaldagi o‘zgaruvchi nuqtaning koordinatalari.

Masalan, ushbu

$$z = f(x, y) = x^2 + 2y^2$$

sirtga (paraboloidga) (1,1,3) nuqtada o‘tkazilgan urinma tekislik tenglamasi

$$Z - 3 = 2(X - 1) + 4(Y - 1)$$

normalning tenglamasi bo‘ladi.

$$\frac{X - 1}{2} = \frac{Y - 1}{2} = \frac{Z - 1}{-1}$$

Normalning OX, OY, OZ koordinatalar o‘qlarining musbat yo‘na¬lish¬lari bilan tashkil etgan burchaklarini mos ravishda α, β, γ deyilsa, unda $\cos\alpha, \cos\beta, \cos\gamma$ lar normalning yo‘naltiruvchi kosinuslari deyiladi.

Faraz qilaylik, S fazodaga sodda sirt bo‘lsin. Ravshanki, bu sirt¬ning har bir nuqtasida urinma tekislik mavjud bo‘lib, u urinish nuqtasi sirti bo‘ylab uzluksiz o‘zgarib borsa, mos urinma tekislik ham o‘z holatini uzluksiz o‘zgartirib boradi.

S sirtda biror M_0 nuqtani olaylik. Bu nuqta orqali o‘tkazilgan sirt normali ikki yo‘nalishga ega bo‘lib, ulardan birini tayinlaymiz. So‘ng M_0 nuqtadan chiqib, shu M_0 ga qaytadigan yopiq chiziqni (konturni) qaraylik. Bu kontur S sirtga tegishli bo‘lib, uning chegarasini kesmasin. M_0 nuqtada sirt normalining ma’lum yo‘nalishi olinganligini e’tiborga olib, o‘zgaruvchi M nuqtani M_0 dan boshlab, kontur bo‘yicha harakatlantirib, yana M_0 nuqtaga qaytganda (ravshanki, nuqta kontur bo‘ylab o‘zgarganda mos nuqtadagi sirt normali ham o‘zgarib boradi) ikki hol yuz beradi:

1. M_0 nuqtadagi sirt normalining yo‘nalishi qaytib shu nuqtaga kelganda normalning yo‘nalishi teskarisiga o‘zgaradi;

2. M_0 nuqtadagi sirt normalining yo‘nalishi qaytib shu nuq¬ta¬ga kelganda ham normalning yo‘nalishi o‘zgarmaydi.

Aytaylik, S ikki tomonli sirt bo‘lsin. Bu sirtda ixtiyoriy M_0 nuqtani tayinlab, bu nuqtani boshqa $M_1 \in S$ nuqta bilan shu sirtga tegishli bo‘lgan va uning chegarasini kesmaydigan K egri chiziq yordamida birlashtiramiz. So‘ng M_0 nuqtada sirt normalning aniq yo‘nalishini tayinlaymiz.

Aytaylik, o‘zgaruvchi M nuqta K egri chiziq bo‘yicha M_0 dan boshlab M_1 nuqtaga tomon harakatlana borsin. Natijada o‘zgaruvchi M nuqtadagi sirt normalining yo‘nalishi ham uzluksiz o‘zgara boradi va M_1 nuqtaga kelganda sirt normali ma’lum yo‘nalishga ega bo‘ladi. Normalning bu yo‘nalishi nuqtalarni birlashtiruvchi egri chiziqqa bog‘liq bo‘lmaydi. Shuni isbotlaymiz.

M_0 va M_1 nuqtalarni ikkita turli K_1 va K_2 chiziqlar bilan birlashtirilganda o‘zgaruvchi nuqta M nuqtaga kelganda sirt normalining yo‘nalishi turlicha bo‘lsin deb faraz qilaylik. Unda K_1 ва K_2 lardan hosil bo‘lgan yopiq kontur

$$M_0M_1(K_1) \cup M_0M_1(K_2^{-1})$$

bo‘yicha o‘zgaruvchi M_0 nuqtaga kelganda normalning yo‘nalishi o‘zgarib qoladi. Bu esa qaralayotgan sirtning ikki tomonli ekanligiga zid. Demak, ikki tomonli sirtning bitta nuqtasida normal yo‘nalishni tanlash bilan uning barcha nuqtalarida ham normal yo‘nalishi bir qiymatli aniqlanar ekan.

Ma’lumki,

$$z = f(x, y)$$

tenglama bilan aniqlanadigan sirt normalining yo‘naltiruvchi kosinuslari

$$\cos\alpha = \frac{-f_x'(x,y)}{\pm\sqrt{1+f_x'^2(x,y)+f_y'^2(x,y)}},\quad \cos\beta = \frac{-f_y'(x,y)}{\pm\sqrt{1+f_x'^2(x,y)+f_y'^2(x,y)}}$$

$$\cos\gamma = \frac{1}{\pm\sqrt{1+f_x'^2(x,y)+f_y'^2(x,y)}} \qquad (6)$$

bo‘ladi.

(6) da $\cos\alpha, \cos\beta, \cos\gamma$ larni ifodalovchi fo‘rmulalardagi kvadrat ildiz oldidagi ishorani tanlash bilan sirt tomoni aniqlanadi.

Agar kvadrat ildiz oldidagi ishoralardan musbati olinsa, sirtning barcha nuqtalarida

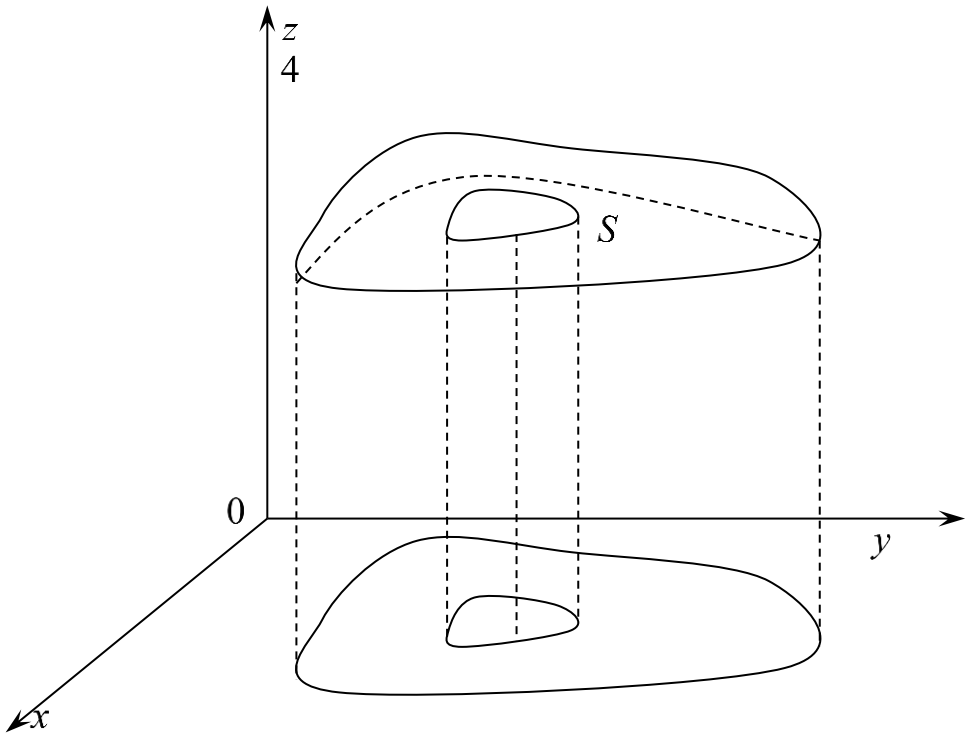
$$\cos\gamma = \frac{1}{\sqrt{1+f_x'^2(x,y)+f_y'^2(x,y)}}$$

ifoda musbat bo‘ladi. Bu holda sirtning olingan tomoniga mos normal bilan OZ o‘qi orasidagi burchak o‘tkir bo‘lib, aniqlanadigan sirt tomoni ustki (yuqori) tomon bo‘ladi.

Agar kvadrat ildiz oldidagi ishoralardan manfiysi olinsa, sirtning olingan tomoniga mos normal bilan OZ o‘qi orasidagi burchak o‘tmas bo‘lib, aniqlanadigan sirt tomoni ostki (past) tomon bo‘ladi.

Sirt yuzi va uni hisoblash. Ma’lumki, egri chiziqning uzunligi unga ichki chizilgan siniq chiziq perimetrining siniq chiziq tomonlarining barchasi nolga intilgandagi limiti sifatida ta’riflangan edi. Shu ta’rif singari sirt yuzi ta’riflanadigan bo‘lsa, ya’ni sirtning yuzga unga ichki chizilgan ko‘pyoqli sirtning yuzining ko‘pyoqlini tashkil etgan barcha tomonlari (yoqlari) diametrlari nolga intilgandagi limiti deyiladigan bo‘lsa, ma’nosizlik yuzaga kelishi mumkin. Shvars tomonidan keltirilgan misolda to‘g‘ri doiraviy silindrga ichki chizilgan ko‘pyoqli (yoqlari uchburchaklardan iborat ko‘pyoqli) yasalganki, barcha yoqlarining diametrlari nolga intilganda bu chizilgan ko‘pyoqli yuzasi limitga ega bo‘lmaydi. Demak, aylanma silindrning yuzasi bo‘laturib, aytilgan ta’rif bo‘yicha uning yuzaga ega bo‘lmasligi kabi g‘ayritabiiy holatga duch kelamiz. Demak, sirt yuzi uchun bunday ta’rif yaroqsiz bo‘lib, uni boshqacha ta’riflashga to‘g‘ri keladi.

Biz quyida ushbu $z = f(x, y)$ tenglama bilan aniqlanadigan



1-rasm.

sirt yuzi tushunchasini va uni hisoblash fo‘rmulasini keltiramiz $f(x,y)$ funksiya *D* tekislikdagi yuzaga ega bo‘lgan to‘plamda uzluksiz va uzluksiz $f_x^{'}(x,y),\ f_y^{'}(x,y)$ xususiy hosilalarga ega.

D to‘plamning biror

$$P_D=\{D_1,D_2,...,D_n\}$$

bo‘laklashini olaylik. Bu bo‘laklashning bo‘lakchalari $D_1,D_2,...,D_n$ bo‘ladi. Olingan bo‘laklashning bo‘luvchi chiziqlarini yo‘naltiruvchilar sifatida qarab, ular orqali yasovchilari OZ o‘qiga parallel bo‘lgan silindrik sirtlar o‘tkazamiz. Bu silindrik sirtlar S sirtning ushbu

$$P_S=\{S_1,S_2,...,S_n\}$$

bo‘laklashini hosil qiladi. Uning bo‘lakchalari $S_1,S_2,...,S_n$ bo‘ladi.

Endi har bir $D_k\ \ (k=1,2,...,n)$ da ixtiyoriy (ξ_k,η_k) nuqta olib , S sirtda unga mos nuqta $(\xi_k,\eta_k,z_k)\ \ (z_k=f(\xi_k,\eta_k))$ ni topamiz. Ravshanki $(\xi_k,\eta_k,z_k)\in S_k\ \ (k=1,2,...,n)$ bo‘ladi. So‘ngra S sirtga shu (ξ_k,η_k,z_k) nuqtada urinma tekislik o‘tkazamiz. Bu urinma tekislik bilan yuqorida aytil¬gan silindrik sirtning kesishishidan hosil bo‘lgan urinma tekislik qismini T_k bilan, uning yuzini esa μT_k bilan belgilaymiz.

D_k to‘plam T_k ning ortoganal proyeksiyasi bo‘lgani uchun

$$\mu D_k=\mu T_k\cdot|\cos\gamma_k|$$

bo‘ladi, bunda γ_k - S sirtga $(\xi_k,\eta_k,z_k)\ \ (z_k=f(\xi_k,\eta_k))$ nuqtada o‘tkazilgan urinma tekislik normalining o‘qi bilan tashkil etgan burchak.

Ravshanki, $\lambda_{P_D}\to 0$ da $\lambda_{P_S}\to 0$ bo‘ladi.

Agar $\lambda_{P_D}\to 0$ da quyidagi

$$\sum_{k=1}^{n}\mu T_k$$

yig‘indi chekli limitga ega bo‘lsa, S sirt yuzaga ega deyiladi, limitning qiymati esa sirtning S yuzi deyiladi. Demak,

$$\mu S = \lim_{\lambda_{P_D}\to 0}\sum_{k=1}^{n}\mu T_k .$$

Ma’lumki,

$$\cos\gamma_k = \frac{1}{\sqrt{1+ f_x'^2(\xi_k,\eta_k)+ f_y'^2(\xi_k,\eta_k)}}$$

bo‘lib,

$$\mu T_k = \frac{1}{\cos\gamma_k}\cdot \mu D_k$$

tenglikdan esa $\sum_{k=1}^{n}\mu T_k = \sum_{k=1}^{n}\sqrt{1+ f_x'^2(\xi_k,\eta_k)+ f_y'^2(\xi_k,\eta_k)}\cdot \mu D_k$

tenglikdan esa bo‘lishi kelib chiqadi.

Keyingi tenglikning o‘ng tomonidagi yig‘indi $\sqrt{1+ f_x'^2(x,y)+ f_y'^2(x,y)}$

funksiyaning integral yig‘indisi bo‘ladi. Bu funksiya D to‘plamda uzluksiz, binobarin integrallanuvchi. Demak,

$$\lim_{\lambda_{P_D}\to 0}\sum_{k=1}^{n}\sqrt{1+ f_x'^2(\xi_k,\eta_k)+ f_y'^2(\xi_k,\eta_k)}\cdot \mu D_k =$$

$$= \iint_D \sqrt{1+ f_x'^2(x,y)+ f_y'^2(x,y)}dxdy.$$

Shunday qilib S sirtning yuzi

$$\mu S == \iint_D \sqrt{1+ f_x'^2(x,y)+ f_y'^2(x,y)}dxdy \qquad (7)$$

bo‘ladi.

Birinchi tur sirt integrali tushunchasi. Fazoda ushbu

$$z = z(x,y) \qquad (8)$$

tenglama bilan aniqlangan S sirtni qaraylik. Bunda $z(x,y)$ funksiya $D\subset R^2$ to‘plamda uzluksiz va uzluksiz $z'_x(x,y),\ z'_y(x,y)$ xususiy hosilalarga ega. Ma’lumki, (8) sirt yuzaga ega bo‘lib, uning yuzi

$$\mu S = \iint_D \sqrt{1+ z'^2_x(x,y)+ z'^2_y(x,y)}dxdy$$

bo‘ladi.

Aytaylik, S sirtda $f(x,y,z)$ funksiya berilgan bo‘lsin. S sirtni undagi chiziqlar yordamida $S_1,S_2,...,S_n$ bo‘lakchalarga ajratib, uning

$$P = \{S_1,S_2,...,S_n\}$$

bo‘laklashini hosil qilamiz. Bu bo‘laklashning diametrini λp deylik. Endi har bir S_k da $(k = 1,2,3,4,...,n)$ ixtiyoriy (ξ_k,η_k,ζ_k) nuqtani olib, bu nuqtadagi funksiyaning qiymati $f(\xi_k,\eta_k,\zeta_k)$ ni S_k ning yuzi μS_k ga ko‘paytiramiz. So‘ng quyidagi

$$\sigma = \sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)\mu S_k \qquad (9)$$

Yig‘indini tuzamiz. Ravshanki bu yig‘indi $f(x,y,z)$ funksiyaga , P bo‘laklashga hamda (ξ_k,η_k,ζ_k) nuqtaga bog‘liq bo‘ladi:

$$\sigma=\sigma(f,P,(\xi_k,\eta_k,\zeta_k)).$$

Odatda, (9) yig‘indi $f(x,y,z)$ funksiyaning integral yig‘indisi (Riman yig‘indisi) deyiladi.

Ta’rif. Agar $\forall\varepsilon>0$ olinganda ham shunday $\delta>0$ topilsaki, S sirtning diametri $\lambda_p<\delta$ bo‘lgan har qanday bo‘laklash uchun tuzilgan σ yig‘indi ixtiyoriy $(\xi_k,\eta_k,\zeta_k)\in S_K$ nuqtada

$$|\sigma-J|<\varepsilon$$

tengsizlikni bajarsa , $f(x,y,z)$ funksiya S sirt bo‘yicha integrallanuvchi deyilib, J son esa $f(x,y,z)$ funksiyaning birinchi tur sirt integrali deyiladi. Birinchi tur sirt integrali quyidagicha

$$\iint\limits_S f(x,y,z)ds$$

kabi belgilanadi:

$$\iint\limits_S f(x,y,z)ds=\lim_{\lambda_p\to0}\sum_{k=1}^{n}f(\xi_k,\eta_k,\zeta_k)\mu S_k.$$

Keltirilgan ta’rifdan ko‘rinadiki, birinchi tur sirt integrali sirtning tomoniga bog‘liq bo‘lmaydi. Xususan, $f(x,y,z)=1$ бўлса,

$$\iint\limits_S ds=\mu S$$

bo‘ladi.

Birinchi tur sirt integralining mavjudligi va uni hisoblash. Aytaylik, $f(x,y,z)$ funksiya (8) tenglama bilan berilgan S sirtda berilgan bo‘lsin.

Teorema. Arap $f(x,y,z)$ funksiya S sirtda uzluksiz bo‘lsa, u holda bu funksiyaning sirt bo‘yicha birinchi tur sirt integrali mavjud va

$$\iint\limits_S f(x,y,z)ds=\iint\limits_D f(x,y,z(x,y))\sqrt{1+z'^2_x(x,y)+z'^2_y(x,y)}dxdy \quad (10)$$

bo‘ladi.

S sirtning ixtiyoriy $P_s=\{S_1,S_2,S_3,..,S_n\}$ bo‘laklashini olib unga nisbatan integral yig‘indi

$$\sigma=\sum_{k=1}^{n}f(\xi_k,\eta_k,\zeta_k)\mu S_k$$

ni tuzamiz.

$$P_s=\{S_1,S_2,S_3,..,S_n\}$$

Bo‘laklarni tuzamiz. $S_1,S_2,...,S_n$ larning XoY tekislikdagi proyeksiyalari $D_1,D_2,D_3,...,D_n$ lar to‘plamning bo‘laklashini hosil qiladi.

Ma’lumki, $(\xi_k,\eta_k,\zeta_k)\in S_k$, $\zeta_k=z(\xi_k\eta_k)$.

(9) formulaga ko‘ra

$$\mu S_k=\sqrt{1+z'^2_x(x,y)+z'^2_y(x,y)}\mu D_k$$

bo‘ladi.

O‘rta qiymat haqidagi teoremadan foydalanib topamiz:

$$\mu S_k=\sqrt{1+x'^2z(\xi^*{}_k,\eta^*{}_k)+y'^2z(\xi^*{}_k,\eta^*{}_k)}\mu D_k$$

bunda $(\xi^*{}_k,\eta^*{}_k)\in D_k$.

Natijada σ integral yig‘indi quyidagi ko‘rinishga keladi.

$$\sigma = \sum_{k=1}^{n} f(\xi_k, \eta_k, z(\xi_k, \eta_k))\sqrt{1 + z'^2_x(x,y) + z'^2_y(x,y)}\mu D_k \ . \quad (11)$$

Bu tenglikning o‘ng tomondagi yig‘indi ushbu

$f(x,y,z(x,y))\sqrt{1 + z'^2_x(x,y) + z'^2_y(x,y)}$ (12) ikki o‘zgaruvchili uzluksiz funksiyaning integral yig‘indisi $\sum_{k=1}^{n} f(\xi_k, \eta_k, z(\xi_k, \eta_k))\sqrt{1 + z'^2_x(\xi_k, \eta_k) + z'^2_y(\xi_k, \eta_k)}\mu D_k$ (13)

ni eslatadi. (11) va (13) yig‘indilarni solishtirib ularning farqi integral yig‘indida $(\xi_k, \eta_k) \in D_k$ nuqta ixtiyoriy bo‘lgan holda (10) yig‘indida esa esa $(\xi_k^*, \eta_k^*) \in D_k$ nuqta o‘rta qiymat haqidagi teoremaga muvofiq bo‘lgan tayin nuqta bo‘lishidadir. (11) funksiya D to‘plamda uzluksiz, binobarin u D to‘plamda integrallanuvchi bo‘lganligi sababli

$$\lim_{\lambda_p \to 0} \sigma = \lim_{\lambda_p \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k, z(\xi_k, \eta_k))\sqrt{1 + z'^2_x(\xi_k, \eta_k) + + z'^2_y(\xi_k, \eta_k)}\mu D_k =$$

$$= \lim_{\lambda_p \to 0} \sum_{k=1}^{n} f(\xi_k^*, \eta_k^*, z(\xi_k^*, \eta_k^*))\sqrt{1 + z'^2_x(\xi_k^*, \eta_k^*) + z'^2_y(\xi_k^*, \eta_k^*)}\mu D_k =$$

$$= \iint_D f(x,y,z(x,y))\sqrt{1 + z'^2_x(x,y) + z'^2_y(x,y)}dxdy$$

bo‘ladi.

Demak,

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,z(x,y))\sqrt{1 + z'^2_x(x,y) + z'^2_y(x,y)}dxdy .$$

Agar fazodagi S sirt ushbu $x = x(y,z)$ tenglama bilan aniqlangan bo‘lib, bunda $x = x(y,z)$ funksiya uzluksiz va uzluksiz $x'_y(y,z)$, $x'_z(y,z)$ xususiy hosilalarga ega bo‘lsa, bu sirtda uzluksiz bo‘lgan $f(x,y,z)$ funksiyaning birinchi tur sirt integrali mavjud va

$$\iint_S f(x,y,z)dS = \iint_D f(x(y,z),y,z)\sqrt{1 + x'^2_y(y,z) + x'^2_z(y,z)}dydz \quad (14) \text{ bo‘ladi.}$$

Agar fazodagi S sirt ushbu

$$y = y(z,x)$$

tenglama bilan aniqlangan bo‘lib, bunda $y = y(z,x)$ funksiya uzluksiz va uzluksiz $y'_z(z,x)$, $y'_x(z,x)$ xususiy hosilalarga ega bo‘lsa, bu sirtda aniqlangan uzluksiz $f(x,y,z)$ funksiyaning birinchi tur sirt integrali mavjud va

$$\iint_S f(x,y,z)dS = \iint_D f(x,y(z,x),z)\sqrt{1 + y'^2_z(z,x) + y'^2_x(z,x)}dzdx \quad (15)$$

bo‘ladi. Bu tasdiqlar yuqorida keltirilgan teoremaning isboti kabi isbotlanadi.

Xulosa. Birinchi tur sirt integrallari ikki karrali integrallarga keltirilib, (9), (14) va (15) formulalar yordamida hisoblanadi. Birinchi tur sirt integralining xossalari. Birinchi tur sirt integrali ikki karrali integral xossalari kabi xossalarga ega bo‘ladi. Birinchi tur sirt integrali yordamida sirtlarning yuzi, massali sirtning massasi, og‘irlik markazlari, inersiya momentlari topiladi.

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MONTE-KARLO SIMULYATSIYA TIZIMIDA YASHOVCHANLIK TAHLILI UCHUN ISHONCHLI VA SAMARALI TANLANMA GENERATSIYA QILISH ALGORITMLARI

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Annotatsiya. *Ushbu maqolada senzurlanish ulushlarini aniq boshqarish imkonini beruvchi senzurlangan tanlanmalarni generatsiya qilishga mo‘ljallangan keng qamrovli Monte-Karlo simulyatsiya tizimi taklif etiladi. Tizim teskari akslantirish usuli yordamida o‘ng, chap, interval va ikki tomonlama tasodifiy senzurlanish mexanizmlarini tizimli ko‘rib chiqiladi. Natijada, tadqiqotchilarga klinik va epidemiologik amaliyotni aks ettiruvchi xilma-xil senzurlanish modellarini shakllantirish imkoniyatini beradi. Senzurlanish taqsimoti parametrlarini kalibrlash uchun deterministik algoritmlar ishlab chiqilgan bo‘lib, absolyut og‘ish 0,005 dan past bo‘lgan holda maqsadli senzurlanish ulushlariga erishiladi. Modulli dasturiy arxitektura turli senzurlanish turlariga muammosiz moslashishni ta’minlaydi. 10000 takrorlash bo‘yicha empirik tekshiruv shuni ko‘rsatadiki, taklif etilayotgan optimallashtirilgan usullar sodda usullardan 15-40 marta tezroq ishlaydi va 1000000 ta kuzatuvni 3 sekunddan kam vaqtda generatsiya qiladi. Tizim nazariy yashovchanlik tahlili va amaliy tahlil o‘rtasidagi bo‘shliqni to‘ldiradi hamda yashovchanlik modellarni taqqoslash uchun standartlashtirilgan, takrorlanuvchi va hisoblash jihatidan samarali vosita sifatida xizmat qiladi.*

Kalit so‘zlar: *yashovchanlik tahlili, senzurlangan ma’lumotlar, Monte-Karlo simulyatsiyasi, teskari akslantirish usuli, o‘ng senzurlanish, interval senzurlanish, ma’lumotlar generatsiyasi algoritmi.*

RELIABLE AND EFFICIENT SELECTION GENERATION ALGORITHMS FOR SURVIVAL ANALYSIS IN MONTE CARLO SIMULATION SYSTEM

Abstract. *This article proposes a comprehensive Monte Carlo simulation framework designed to generate censored samples with precise control over censoring proportions. The system systematically addresses right-, left-, interval-, and two-sided random censoring mechanisms using the inverse transformation method. As a result, it enables researchers to construct diverse censoring schemes that reflect clinical and epidemiological practice. Deterministic algorithms are developed to calibrate the parameters of the censoring distribution, achieving target censoring proportions with an absolute deviation of less than 0.005. The modular software architecture ensures seamless adaptability to various types of censoring. Empirical evaluation based on 10,000 replications demonstrates that the proposed optimized methods are 15–40 times faster than naive approaches and can generate 1,000,000 observations in under 3 seconds. The framework bridges the gap between theoretical survival analysis and practical application, serving as a standardized, reproducible, and computationally efficient tool for comparing survival models.*

Key words: *survival analysis, censored data, Monte Carlo simulation, inverse transform method, right censoring, interval censoring, data generation algorithm.*

АЛГОРИТМЫ НАДЁЖНОГО И ЭФФЕКТИВНОГО ГЕНЕРИРОВАНИЯ ВЫБОРОК ДЛЯ АНАЛИЗА ВЫЖИВАЕМОСТИ В СИСТЕМЕ МОДЕЛИРОВАНИЯ МОНТЕ-КАРЛО

Аннотация. *В данной статье предлагается комплексная система имитационного моделирования методом Монте-Карло, предназначенная для генерации цензурированных выборок с точным контролем доли цензурирования. Система систематически рассматривает механизмы*

правого, левого, интервального и двустороннего случайного цензурирования с использованием метода обратного преобразования. В результате исследователи получают возможность формировать разнообразные схемы цензурирования, отражающие клиническую и эпидемиологическую практику. Для калибровки параметров распределения цензурирования разработаны детерминированные алгоритмы, позволяющие достигать заданных долей цензурирования с абсолютным отклонением менее 0,005. Модульная программная архитектура обеспечивает гибкую адаптацию к различным типам цензурирования. Эмпирическая проверка на основе 10 000 повторений показывает, что предлагаемые оптимизированные методы работают в 15–40 раз быстрее простых подходов и позволяют генерировать 1 000 000 наблюдений менее чем за 3 секунды. Предложенная система устраняет разрыв между теоретическим анализом выживаемости и практическими приложениями, выступая в качестве стандартизированного, воспроизводимого и вычислительно эффективного инструмента для сравнения моделей выживаемости.

Ключевые слова: *анализ выживаемости, цензурированные данные, моделирование методом Монте-Карло, метод обратного преобразования, правостороннее цензурирование, интервальное цензурирование, алгоритм генерации данных.*

Kirish. Yashovchanlik tahlili tibbiy tadqiqotlarda, muhandislik ishonchliligida va ijtimoiy fanlarda muhim rol o‘ynaydi, bu sohada asosiy e’tibor voqea yuz vaqti ma’lumotlarini modellashtrishga qaratiladi. Bunday tahlillardagi asosiy muammo senzurlanish mavjud ekanligidir. Ya’ni tadqiqot loyihasidagi ayrim cheklovlar tufayli yoki ishtirokchilarning chiqib ketishi sababli voqea vaqti faqat qisman kuzatiladi. Senzurlangan kuzatuvlarning mavjudligi statistik xulosalar chiqarishni murakkablashtiradi va to‘liq bo‘lmagan ma’lumotlarni hisobga oladigan maxsus usullarni talab etadi. Kaplan–Meyer bahosi [1] va Koks proporsional xavf modeli [2] kabilar shular jumlasidandir. Bu yondashuvlar amaliyotda keng qo‘llaniladi, ammo ularning turli senzurlanish mexanizmlari ostidagi ish samaradorligini simulyatsiya tadqiqotlari orqali chuqur baholash zarur.

Monte-Karlo simulyatsiyalari yashovchanlik modellarni tasdiqlashda asosiy vosita bo‘lib xizmat qiladi. Ular tadqiqotchilarga nazorat ostidagi sharoitlarda baholarning xossalarini o‘rganish imkonini beradi. Biroq, aniq parametrlarga ega senzurlangan yashovchanlik ma’lumotlarini generatsiya qilish hali ham murakkab masala bo‘lib qolmoqda. Mavjud usullar ko‘pincha muayyan senzurlanish turlariga, masalan, klinik sinovlardagi o‘ng senzurlanish yoki davriy kuzatuv tadqiqotlaridagi interval senzurlanishga [3] yo‘naltirilgan. Bu yondashuvlar qimmatli bo‘lsa-da, ular hisoblash samaradorligi va takrorlanish imkoniyatini ta’minlab, bir nechta senzurlanish mexanizmlarini qamrab oluvchi yagona tizimga ega emas. Bu kamchilik, ayniqsa, xilma-xil senzurlanish naqshlariga bardoshlilik da’vo qiluvchi yangi metodologiyalarni ishlab chiqish zarurligini ko‘rsatadi.

Usullar. Ushbu bo‘lim simulyatsiya algoritmlarining poydevorini tashkil etuvchi matematik tizimni va nazorat ostidagi xossalarga ega senzurlangan yashovchanlik ma’lumotlarini generatsiya qilishning algoritmik amalga oshirishini batafsil bayon etadi.

Matematik asoslar. Yashovchanlik funksiyasi $S_T(t)$ biror shaxsning t vaqtdan uzoqroq yashashi ehtimolligini ifodalaydi:

$$S_T(t) = P(T > t), \tag{1}$$

bu yerda T voqea vaqtini ifodalovchi manfiy bo‘lmagan tasodifiy o‘zgaruvchidir. Taqsimot funksiyasi (TF) $F_T(t)$ yashovchanlik funksiyasini to‘ldiradi:

$$F_T(t) = P(T \le t) = 1 - S_T(t). \tag{2}$$

t vaqtgacha yashaganlik sharti ostida shu vaqtda voqea sodir bo‘lishining oniy tezligini tavsiflovchi xavf funksiyasi (hazard function) $h(t)$ quyidagicha aniqlanadi:

$$h(t) = \frac{f_T(t)}{S_T(t)}, \tag{3}$$

bu yerda $f_T(t)$ — T ning zichlik funksiyasi. Ushbu munosabatlar teskari akslantirish usuli orqali yashovchanlik vaqtlarini generatsiya qilishning asosini tashkil etadi.

Indikator funksiyasi senzurlanish voqealarini ifodalashda muhim rol o‘ynaydi. Ixtiyoriy A hodisasi uchun:

$$I(A) = \begin{cases} 1; & A \ ro'ybersa; \\ 0; & A \ ro'ybermasa. \end{cases} \tag{4}$$

O‘ng senzurlangan ma’lumotlar uchun kuzatilgan ma’lumotlar $(Y; \delta)$ ko‘rinishida ifodalanadi, bu yerda $Y = min(T; C)$ va $\delta = I(T \le C)$, C esa senzurlanish vaqtini bildiradi.

Teskari akslantirish usuli bir xil taqsimot asosida amalga oshiriladi. Usul TF si $F_T(t)$ bo‘lgan ixtiyoriy uzluksiz T tasodifiy o‘zgaruvchisi uchun $U = F_T(T)$ tasodifiy o‘zgaruvchisi standart bir xil taqsimlashga bo‘ysunishidan foydalanadi. Bu esa yashovchanlik vaqtlarini quyidagicha generatsiya qilish imkonini beradi:

$$T = F_T^{-1}(U),\ U \sim Uniform(0;1), \quad (5)$$

bu yerda $F_T^{-1}(U)$ — maqsadli taqsimotning kvantil funksiyasi. Uniform-tekis taqsimot. TF qat’iy o‘suvchi bo‘lmagan holda umumlashtirilgan teskari funksiyadan foydalaniladi:

$$F_T^{-1}(u) = \inf\{t: F_T(t) \geq u\}. \quad (6)$$

Mustaqil senzurlanish asosiy shart hisoblanadi:

$$P(T \leq t, C \leq c) = P(T \leq t)P(C \leq c). \quad (7)$$

Bu senzurlanish mexanizmi kuzatilgan ma’lumotlardan tashqari yashovchanlik vaqti haqida qo‘shimcha ma’lumot bermasligini anglatadi. Simulyatsiya tadqiqotlarida T va C ushbu shartni qanoatlantirishlari uchun o‘zlarining mos taqsimlashlaridan mustaqil ravishda generatsiya qilinadi.

Bir nechta senzurlanish turlari uchun yagona tizim. Taklif etilayotgan tizim to‘rtta asosiy senzurlanish mexanizmi ostida yashovchanlik ma’lumotlarini generatsiya qilishning tizimli yondashuvini o‘rnatadi. 1-jadvalda har bir tur uchun kuzatiladigan ma’lumotlar tuzilmalari va senzurlanish ulushi formulalari jamlangan.

1-jadval.

Senzurlanish mexanizmlari, kuzatiladigan ma’lumotlar tuzilmalari va senzurlanish ulushi formulalarining xulosasi

Senzurlanish	Kuzatilgan	Indikator	Ulush
O‘ng	$Y = min(T; C)$	$\delta = I(T \leq C)$	$p_r = \int_0^\infty S_T(c) f_C(c) dc$
Chap	$Y = max(T; L)$	$\delta = I(T \geq C)$	$p_l = \int_0^\infty F_T(l) f_L(l) dl$
Intervallik	$(L; U; \delta)$	$\delta = I(L < T < U)$	$p_i = \int_0^\infty \int_l^\infty [F_T(u) - F_T(l)]\, f_{L,U} du dl$
Tasodifiy	$(Y; \delta)$	(11)-tengl.	p_r va p_l kombinatsiyasi

O‘ng senzurlanish. T yashovchanlik vaqti f_T zichligi bilan F_T taqsimotga bo‘ysunadi, C senzurlanish vaqti esa F_C taqsimotdan mustaqil generatsiya qilinadi. Senzurlanish ulushi p_r quyidagi tenglamani yechish orqali nazorat qilinadi:

$$p_r = P(T > C) = \int_0^\infty S_T(c) f_C(c) dc, \quad (8)$$

bu yerda $S_T(t) = 1 - F_T(t)$. Ushbu integral tenglama maqsadli p_r ga erishish uchun F_C ning parametrlariga nisbatan sonli usulda yechiladi.

Chap senzurlanish. Voqea biror kuzatiladigan L vaqtdan oldin sodir bo‘lib, $(Y; \delta)$ ko‘rinishidagi ma’lumotlar hosil qilinadi: $Y = max(T; L)$, $\delta = I(T > C)$. senzurlanish ulushi p_l quyidagiga teng:

$$p_l = P(T < L) = \int_0^\infty F_T(l) f_L(l) dl, \quad (9)$$

bu yerda F_L funksiya L ning taqsimoti. Tizim (8)-tenglamani sonli usullar orqali yechib F_L ning parametrlarini istalgan p_l ga moslashtirib sozlaydi.

Interval senzurlanish. $\delta = I(L < T < U)$ bo‘lgan $(L; U; \delta)$ uchliklar generatsiya qilinadi. senzurlanish ulushi p_i quyidagicha:

$$p_i = \int_0^\infty \int_l^\infty [F_T(u) - F_T(l)]\, f_{L,U} du dl \quad (10)$$

bunda $(L; U)$ $F_{L,U}$ taqsimotdan birgalikda tanlanadi. Algoritm p_i ni nazorat qilish uchun interval kengligini $U - L$ kalibrlaydi.

Ikki tomonlama tasodifiy senzurlanish. Bu tur umumlashtirilgan senzurlanish chegaralari orqali o‘ng va chap senzurlanishni birlashtiradi:

$$Y = \begin{cases} T; & T \in (C_l; C_r) \ \ bo'lsa; \\ C_l; & T \leq C_l \ \ bo'lsa; \\ C_r; & T \geq C_r \ \ bo'lsa; \end{cases} \quad (11)$$

bu yerda C_l va C_r mos ravishda chap va o‘ng senzurlanish vaqtlarini bildiradi. Tizim T va $(C_l; C_r)$ o‘rtasidagi bog‘liqlik tuzilmalarini kopula modellari yoki shartli taqsimotlar orqali belgilash imkonini beradi.

Senzurlanish ulushlarini nazorat qilish uchun deterministik algoritmlar. Senzurlanish ulushlarini aniq nazorat qilish maqsadida deterministik algoritmlar senzurlanish taqsimoti parametrlarini

maqsadli yashovchanlik taqsimotiga asoslanib sozlaydi. p maqsadli ulush va F_T yashovchanlik taqsimoti bilan o‘ng senzurlanish uchun quyidagi tenglama yechiladi:

$$P(T > C) = \int_0^\infty S_T(c) dF_C(c) = p. \quad (12)$$

Faraz qilaylik, C tasodifiy miqdor λ parametrli eksponensial taqsimotga bo‘ysunsa, tenglama quyidagi ko‘rinishga keladi:

$$\int_0^\infty S_T(c) \lambda e^{-\lambda c} dc = p, \quad (13)$$

bu tenglama Brent usuli kabi ildiz topish algoritmlari yordamida λ ga nisbatan sonli usulda yechiladi. Yechim katta tanlamalarda senzurlangan kuzatuvlarning kutilgan ulushi p ga teng bo‘lishini kafolatlaydi.

$U = L + W$, $W \sim F_W$ bo‘lgan $(L; U)$ kuzatuv intervaliga ega intervallik senzurlanish uchun:

$$p = \int_0^\infty \int_0^\infty [F_T(l + w) - F_T(l)] d F_L(l) dF_W(w) \quad (14)$$

Algoritm F_L ni o‘zgartirib qolmasdan F_W parametrlarini topadi va kuzatuv intervallari ichida voqea ehtimolligi maqsadli p ga mos kelishini ta’minlash uchun iterativ sonli usullarni qo‘llaydi.

2.4 Umumiy simulyatsiya algoritmi. To‘liq simulyatsiya jarayonining bayoni quyida keltirilgan. $T-\theta$ parametrlari bilan F_T taqsimotdan generatsiya qilingan haqiqiy yashovchanlik vaqti, C esa ϕ bilan parametrlashtirilgan F_C taqsimotga ega senzurlanish vaqtini ifodalaydi.

1-Algoritm: Senzurlanish ulushlarini nazorat qilish bilan senzurlangan yashovchanlik ma’lumotlarini generatsiya qilish.

Kirish: Maqsadli senzurlanish ulushi p, yashovchanlik taqsimoti $F_T(\theta)$, senzurlanish oilasi F_C, tanlanma hajmi n, senzurlanish turi. $\tau = \{\text{o‘ng}, \text{chap}, \text{interval}, \text{ikki tomonlama tasodifiy}\}$.

Chiqish: senzurlangan tanlanma. $\mathcal{D} = \{(Y_i; \delta_i)\}_{i=1}^n$.

1-qadam (Kalibrlash): Nazariy senzurlanish ulushi p ga teng bo‘lishi uchun ϕ topiladi (12-formulaga qarang).

2-qadam (Generatsiya): Har bir $i = 1, \dots, n$ uchun:

✓ $U_i \sim Uniform(0; 1)$ generatsiya qilinadi; $T_i = F_T^{-1}(U_i; \theta)$ hisoblanadi.

✓ $V_i \sim Uniform(0; 1)$ generatsiya qilinadi; $C_i = F_C^{-1}(V_i; \phi)$ hisoblanadi.

✓ $\tau =$o‘ng bo‘lsa: $Y_i = min(T_i; C_i)$, $\delta_i = I(T_i \le C_i)$ belgilanadi.

✓ $\tau =$ chap bo‘lsa: $Y_i = max(T_i; C_i)$, $\delta_i = I(T_i \ge C_i)$ belgilanadi.

✓ $\tau =$interval bo‘lsa: $(L_i; U_i)$ generatsiya qilinadi; $\delta_i = I(L_i < T_i < U_i)$ belgilanadi.

3-qadam (Tekshiruv): $\hat{p} = 1 - \frac{1}{n}\sum_{i=1}^n \delta_i$ hisoblanadi; $|\hat{p} - p| < \varepsilon$ tekshiriladi.

Qaytarish $\mathcal{D}$.

Misol uchun, k shakl va λ masshtab parametrlariga ega Weibull yashovchanlik vaqtlari uchun kvantil funksiyasi quyidagi ko‘rinish oladi:

$$F_T^{-1}(u) = \lambda\big(-log(1-u)\big)^{1/k}. \quad (15)$$

Hisoblash samaradorligi strategiyalari. Statistik aniqlikni saqlab qolgan holda hisoblash vaqtini qisqartiruvchi bir necha optimallashtirish strategiyalari qo‘llaniladi.

Vektorlashtirilgan amallar. Har bir kuzatuvni alohida qayta ishlash o‘rniga, teskari taqsimot funksiya bir vaqtning o‘zida bir xil tasodifiy o‘zgaruvchilar massiviga tatbiq etiladi:

$$T = F_T^{-1}(U), \ U \sim Uniform(0; 1)^n, \quad (16)$$

bu yerda T va $U-n$ o‘lchovli vektorlar. Bu zamonaviy protsessor arxitekturalaridan foydalanib sezilarli tezlik oshishiga olib keladi.

Oldindan hisoblash va interpolyatsiya. Senzurlanish parametrlarini sonli usullar bilan hisoblashda, maqsadli ulushlarning panjara qiymatlari uchun yechimlar oldindan hisoblanib qidiruv jadvalida saqlanadi:

$$\phi_p \approx \text{interpolyatsiya}\left(\phi_{p_1}; \phi_{p_2}\right), \ p_1 < p < p_2, \quad (17)$$

bu esa odatda 0,5% dan past interpolyatsiya xatolari bilan hisoblash yukini kamaytiradi.

Parallel qayta ishlash. Algoritm talab qilinadigan N ta to‘plamni K ta protsessor yadrosiga taqsimlaydi. Har bir yadro yadrolar o‘rtasida aloqa bo‘lmagan holda mustaqil takrorlashlarni generatsiya qiladi, bu esa vaqtni taxminan $1/K$ koeffitsiyentga qisqartiradi.

Xotira tejamkorligini ta’minlovchi bo‘laklash. Juda katta to‘plamlar ($n > 10^6$) uchun kuzatuvlar belgilangan o‘lchamdagi bo‘laklarda qayta ishlanadi:

$$T_b = F_T^{-1}(U_b), \quad b = 1, \dots, [n/B], \tag{18}$$

bu yerda U_b bo‘lagi B ta bir xil tasodifiy o‘zgaruvchilarni o‘z ichiga oladi. Bu mavjud operativ xotiradan ortiqcha to‘plamlarni simulyatsiya qilish imkonini beradi.

Stratifikatsiyalangan interval namunasi. Interval senzurlanishda avval interval kengliklari F_W dan tanlanadi, so‘ng ular kenglikka shart qo‘yilgan holda joylashtiriladi:

$$W_i \sim F_W, \quad L_i \sim Uniform(0,\ \tau - W_i), \tag{19}$$

bu yerda τ — maksimal kuzatuv vaqti. Bu yondashuv takrorlashlar bo‘yicha senzurlanish ulushlarining dispersiyasini kamaytiradi.

Natijalar. Taklif etilayotgan tizim uchta muhim jihatni: senzurlanish ulushi aniqligini va hisoblash samaradorligini va yashovchanlik model baholashga ta’sirini baholash uchun mo‘ljallangan keng qamrovli simulyatsiya tadqiqotlari orqali qat’iy tekshirishdan o‘tkazish kabilarni o‘z ichiga oladi. Yashovchanlik taqsimotlari, senzurlanish mexanizmlari va tanlanma hajmlarini o‘zgartirish mumkin bo‘lgan ko‘p omilli tayyor algoritm ishlab chiqildi.

Senzurlanish ulushini nazorat qilish. O‘ng senzurlanish senariylari uchun eksponensial ($\lambda = 1$) va Weibull ($k = 1{,}5;\ \lambda = 1$) yashovchanlik taqsimotlari ostida $p \in \{0{,}1; 0{,}3; 0{,}5; 0{,}7\}$ maqsadli senzurlanish ulushlariga ega to‘plamlar generatsiya qilindi. Empirik senzurlanish ulushi quyidagicha hisoblandi:

$$\hat{p} = 1 - \frac{1}{n}\sum_{i=1}^{n} \delta_i. \tag{20}$$

2-jadval $n = 1000$ tanlanma hajmi uchun 10000 takrorlash bo‘yicha natijalarni ko‘rsatadi va barcha senariylarda mutlaq og‘ish 0,005 dan past bo‘lgan aniq nazoratni namoyish etadi.

2-jadval.

$n = 1000$ bilan 10000 takrorlash bo‘yicha empirik senzurlanish ulushi ($\hat{p}$). Qiymatlar o‘rtacha $\pm$ standart og‘ish ko‘rinishida keltirilgan

Maqsadli p	**Eksponensial ($\lambda = 1$)**		**Weibull ($k = 1{,}5;\ \lambda = 1$)**	
(lr)2-3 (lr)4-5	O‘rtacha $\hat{p}$	Standart og‘ish	O‘rtacha $\hat{p}$	Standart og‘ish
0,10	0,1003	0,0095	0,1001	0,0094
0,30	0,3002	0,0145	0,2998	0,0144
0,50	0,5001	0,0158	0,4997	0,0157
0,70	0,6998	0,0145	0,7002	0,0146

Hisoblash samaradorligini taqqoslash. Taklif etilayotgan optimallashtirilgan usullar turli tanlanma hajmlarida asos sifatida qabul qilingan siklga asoslangan usul bilan taqqoslandi. 3-jadval ishlash vaqtini solishtirishni umumlashtiradi.

3-jadval.

100 ishga o‘rtalashtirilgan holda, o‘ng senzurlangan eksponensial ma’lumotlar ($p = 0{,}3$) uchun asosiy va optimallashtirilgan amalga oshirishlar o‘rtasidagi ishlash vaqtini taqqoslash (sekundlarda)

Hajmi n	**Asosiy (s)**	**Optimallashtirilgan (s)**	**Oshishi**	**Xotira (MB)**
10^3	0,85	0,05	17 marta	0,1
10^4	8,70	0,23	38 marta	0,8
10^5	87,50	2,30	38 marta	7,6
10^6	912,00	23,10	39 marta	76,3

Vektorlashtirilgan amallar sodda sikllarga nisbatan ishlash vaqtini 15–40 marta qisqartirdi, ayniqsa, katta to‘plamlar uchun bu farq yanada yaqqol ko‘rinadi. $n = 10^6$ uchun optimallashtirilgan algoritm bitta to‘plamni taxminan 2,3 sekundda generatsiya qilishni yakunladi.

Yashovchanlik model baholashiga ta’siri. Tizimning metodologik tadqiqotlar uchun foydaliligini baholash maqsadida turli senzurlanish mexanizmlari Koks modeli [2] bahosiga qanday ta’sir ko‘rsatishi o‘rganildi. Simulyatsiyalarda senzurlanish ulushi (20%, 50%) va turi (o‘ng) o‘zgartirildi, haqiqiy xavf nisbati esa $\beta = log(2{,}0) = 0{,}693$ da doimiy saqlandi. 4-jadval $n = 500$ bilan 5000 takrorlash bo‘yicha 95% ishonch intervali uchun noxolislik va qamrov ehtimolliklarini ko‘rsatadi.

4-jadval.

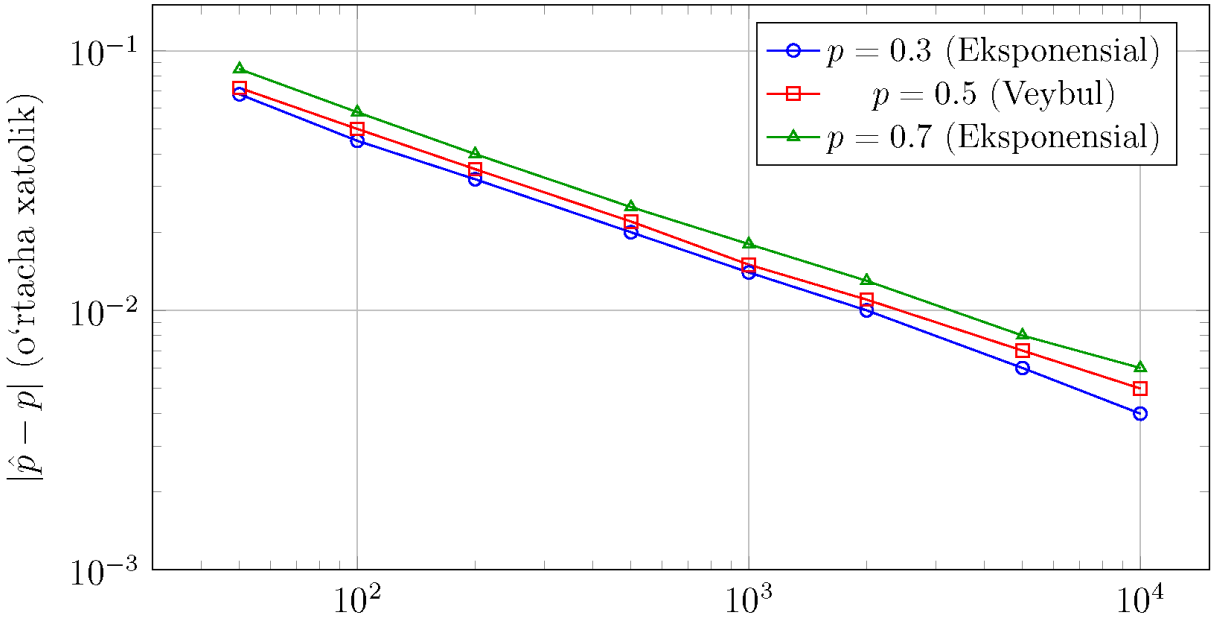
Turli senzurlanish senariylari ostida Koks modeli baholash samaradorligi ($n = 500$, haqiqiy $\beta = 0,693$; 5000 takrorlash)

Senzurlanish turi	Maqsadli p	O‘rtacha $\hat{\beta}$	Og‘ish	SX	Qamrov (%)
O‘ng	0,20	0,695	0,002	0,098	94,8
O‘ng	0,50	0,698	0,005	0,142	94,5

Natijalar tizim mo‘ljallangan senzurlanish naqshlarini muvaffaqiyatli yaratishini va statistik xossalarni to‘g‘ri saqlab qolishini tasdiqlaydi. Qamrov ehtimolliklari nominal 95% darajasiga yaqin qoladi va barcha sharoitlarda og‘ish ahamiyatsiz darajada kichik. Kutilganidek, yuqori senzurlanish ulushi standart xatolarning oshishiga olib keladi.

Ekstremal sharoitlarda mustahkamlik. Tizim yuqori senzurlanish sharoitlari ($p = 0,9$) va kichik tanlanma hajmlari ($n = 50$) ostida sinovdan o‘tkazildi. Hatto bu qiyin ssenariylarda ham senzurlanish ulushi nazorati o‘rtacha 0,015 xatolikdan past bo‘lgan holda aniq bo‘lib qoldi, bu esa ekstremal sharoitlar yoki cheklangan tanlanma hajmlarini talab qiluvchi ilovalar uchun barqarorlikni namoyish etadi.

Tanlanma hajmi o‘sishi bilan empirik senzurlanish ulushlarining maqsadli qiymatlarga yaqinlashuvini tasvirlaydi.



1-rasm. 10000 takrorlash bo‘yicha o‘rtacha xatolik $|\hat{p} - p|$ ning tanlanma hajmiga bog‘liq holda yaqinlashuvi

Muhokama.

Dasturiy ta’minot loyihasiga oid mulohazalar. Takrorlanish imkoniyati ishonchli simulyatsiya tadqiqotlarining asosi hisoblanadi. Tizim bir nechta himoya mexanizmlarini amalga oshiradi: tasodifiy sonlar generatsiyasi va transformatsiya bosqichlarini qat’iy ajratish, barcha tasodifiy sonlar oqimlarini PCG generatorlar oilasi [4] yordamida boshlash va protsessor yadrolari bo‘ylab tasodifiy son ketma-ketliklarini taqsimlash uchun sakrash usulidan foydalanish [5]. Amalga oshirish barcha tasodifiy ma’lumotlar va parametr sozlamalarini FAIR tamoyillariga [6] muvofiq standartlashtirilgan metama’lumot fayllarida qayd etadi.

Mavjud yondashuvlar bilan taqqoslash. 5-jadval taklif etilayotgan tizimni mavjud simulyatsiya yondashuvlari bilan tuzilgan tarzda taqqoslaydi.

5-jadval.

Senzurlangan ma’lumot generatsiyasi yondashuvlarini xususiyatlar bo‘yicha taqqoslash

Xususiyat	Taklif etilgan	I va II tur	Bosqichli	Maxsus
O‘ng senzurlanish	Ha	Ha	Ha	Ha
Chap senzurlanish	Ha	—	—	Qisman
Intervallik senzurlanish	Ha	—	—	Qisman
Ikki tomonlama tasodifiy senzurlanish	Ha	—	—	Ha
Ulush nazorati	Aniq	Belgilangan	Qisman	Taxminiy

Vektorlashtirish	Ha	—	—	Farqli
Parallel qo‘llab-quvvatlash	Ha	—	—	—
Modulli arxitektura	Ha	—	—	—
Takrorlanish imkoniyati	To‘liq	Qisman	Qisman	Farqli

Xulosa. Ushbu maqolada aniq nazorat qilinadigan senzurlanish ulushlariga ega senzurlangan yashovchanlik ma’lumotlarini generatsiya qilish uchun keng qamrovli Monte-Karlo simulyatsiya tizimi taqdim etildi. Tizim metodologik tadqiqotlardagi muhim bo‘shliqni bartaraf etadi, moslashuvchan taqsimot spetsifikatsiyalari ostida o‘ng, chap, interval va ikki tomonlama tasodifiy senzurlanish mexanizmlarini yagona tarzda qamrab oladi. Teskari akslantirish usulini kalibrlangan senzurlanish taqsimotlari bilan birlashtirish orqali usul sintetik to‘plamlar istalgan statistik xossalarni saqlab qolishini ta’minlaydi va turli modellash yondashuvlari o‘rtasida adolatli taqqoslashga zamin yaratadi.

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BERGMAN MINIMAL MODELI: MATEMATIK ASOSLARI, BIOLOGIK MAZMUNI VA AMALIY QO‘LLANILISHI

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Annotatsiya. *Ushbu maqolada Bergman minimal modeli, uning yaratilish tarixi, matematik asoslari hamda glyukoza–insulin tizimini modellashtirishdagi ahamiyati batafsil tahlil qilingan. Modelning asosiy differensial tenglamalari, glyukoza va insulin o‘rtasidagi o‘zaro bog‘liqlik, shuningdek, insulin sezgirligi (Insulin Sensitivity) va glyukozaning insulin mustaqil utilizatsiyasi (Glucose Effectiveness) kabi asosiy parametrlarning biologik mazmuni yoritilgan. Bundan tashqari, modelning afzalliklari, klinik amaliyotdagi qo‘llanilishi hamda qandli diabet diagnostikasi va metabolik jarayonlarni tadqiq etishdagi o‘rni tahlil qilingan. Tadqiqot natijalari Bergman minimal modelining matematik modellashtirish, tibbiy diagnostika va diabetni ilmiy o‘rganishdagi muhim vositalardan biri ekanligini ko‘rsatadi.*

Kalit so‘zlar: *Bergman minimal modeli, glyukoza–insulin tizimi, matematik modellashtirish, differensial tenglamalar, insulin sezgirligi, glucose effectiveness, IVGTT, qandli diabet, metabolik jarayonlar, biologik modellashtirish.*

THE BERGMAN MINIMAL MODEL: MATHEMATICAL FOUNDATIONS, BIOLOGICAL SIGNIFICANCE, AND PRACTICAL APPLICATIONS

Abstract. *This article provides a comprehensive analysis of the Bergman Minimal Model, including its historical development, mathematical foundations, and significance in modeling the glucose–insulin regulatory system. The fundamental differential equations of the model, the interaction between glucose and insulin, and the biological significance of its principal parameters, such as insulin sensitivity (Insulin Sensitivity) and glucose effectiveness (Glucose Effectiveness), are thoroughly examined.*

Furthermore, the advantages of the model, its applications in clinical practice, and its role in the diagnosis of diabetes mellitus and the investigation of metabolic processes are discussed. The findings demonstrate that the Bergman Minimal Model represents one of the fundamental tools in mathematical modeling, medical diagnostics, and scientific research on diabetes.

Keywords: *Bergman Minimal Model, glucose–insulin system, mathematical modeling, differential equations, insulin sensitivity, glucose effectiveness, intravenous glucose tolerance test (IVGTT), diabetes mellitus, metabolic processes, biological modeling.*

МИНИМАЛЬНАЯ МОДЕЛЬ БЕРГМАНА: МАТЕМАТИЧЕСКИЕ ОСНОВЫ, БИОЛОГИЧЕСКОЕ ЗНАЧЕНИЕ И ПРАКТИЧЕСКОЕ ПРИМЕНЕНИЕ

Аннотация. *В данной статье представлен комплексный анализ минимальной модели Бергмана, включая историю её создания, математические основы и значение для моделирования системы регуляции глюкозы и инсулина. Подробно рассмотрены основные дифференциальные уравнения модели, взаимосвязь между глюкозой и инсулином, а также биологическое значение её ключевых параметров, таких как чувствительность к инсулину (Insulin Sensitivity) и эффективность глюкозы (Glucose Effectiveness)***.**

Кроме того, проанализированы преимущества модели, её применение в клинической практике, а также её роль в диагностике сахарного диабета и исследовании метаболических процессов. Результаты исследования показывают, что минимальная модель Бергмана является одним из фундаментальных инструментов математического моделирования, медицинской диагностики и научных исследований сахарного диабета.

Ключевые слова: *минимальная модель Бергмана, система «глюкоза–инсулин», математическое моделирование, дифференциальные уравнения, чувствительность к инсулину, эффективность глюкозы, внутривенный тест толерантности к глюкозе (IVGTT), сахарный диабет, метаболические процессы, биологическое моделирование.*

Kirish. Uglevod almashinuvi inson organizmidagi eng muhim metabolik jarayonlardan biri hisoblanadi. Organizm hujayralarining asosiy energiya manbayi glyukoza bo‘lib, uning qondagi miqdori ma’lum fiziologik chegaralarda saqlanishi zarur. Agar qondagi glyukoza miqdori me’yordan ortib yoki kamayib ketsa, organizmning ko‘plab a’zolari faoliyatida jiddiy o‘zgarishlar yuz beradi. Ayniqsa, bosh miya, yurak va mushak to‘qimalari glyukozaga juda bog‘liq bo‘lganligi sababli qondagi glyukoza konsentratsiyasining boshqarilishi hayotiy ahamiyatga ega.

Qondagi glyukoza miqdorini boshqarishda oshqozon osti bezining beta hujayralari ishlab chiqaradigan insulin gormoni asosiy rol o‘ynaydi. Ovqat iste’mol qilingandan keyin qondagi glyukoza miqdori ortadi va bu holat beta hujayralarini insulin ajratishga undaydi. Insulin esa mushak va yog‘ to‘qimalarida glyukozaning o‘zlashtirilishini kuchaytiradi, jigarda esa glyukoza hosil bo‘lishini kamaytiradi. Natijada qondagi glyukoza miqdori yana me’yoriy qiymatga qaytadi. Shu tariqa glyukoza va insulin o‘rtasida murakkab biologik qayta aloqa mexanizmi hosil bo‘ladi.

Organizmda glyukoza almashinuvi faqat insulin bilan emas, balki glukagon, kortizol, adrenalin va o‘sish gormoni kabi boshqa gormonlar bilan ham boshqariladi. Biroq sog‘lom odamlarda glyukozani boshqarishning asosiy mexanizmi insulin hisoblanadi. Shuning uchun metabolik kasalliklarni o‘rganishda aynan insulin va glyukozaning o‘zaro bog‘liqligini chuqur tahlil qilish muhim hisoblanadi.

Glyukoza va insulin tizimi yopiq boshqaruv tizimi (closed-loop system) hisoblanadi. Ya’ni qondagi glyukoza miqdori insulin ajralishini boshqaradi, insulin esa o‘z navbatida glyukoza miqdorini kamaytiradi. Shu sababli kuzatilayotgan o‘zgarishlarning qaysi qismi insulin sekretsiyasi, qaysi qismi esa insulin sezgirligi bilan bog‘liqligini oddiy kuzatish orqali aniqlash qiyin.

Adabiyotlar tahlili. Mazkur muammoni hal qilish uchun ilgari "ochiq siklli" (open-loop) eksperimental usullar ishlab chiqilgan. Masalan, **glucose clamp** usulida qondagi glyukoza sun’iy ravishda doimiy darajada ushlab turiladi va insulin sezgirligi baholanadi. Yoki somatostatin yordamida organizmning o‘z insulin ajratishi vaqtincha to‘xtatilib, tashqaridan insulin yuboriladi. Ushbu usullar ilmiy jihatdan aniq natija bersa-da, ularni qo‘llash murakkab, qimmat va invaziv bo‘lib, kundalik klinik amaliyotda keng qo‘llash qiyin[2].

Shu sababli Bergman va hamkorlari yangi yondashuvni taklif etdilar. Ularning asosiy g‘oyasi shundan iborat ediki, organizmning tabiiy boshqaruv tizimini buzmagan holda, oddiy IVGTT tajribasidan olingan glyukoza va insulin ma’lumotlari matematik model yordamida tahlil qilinsa, insulin sezgirligi hamda beta hujayralarning glyukozaga sezgirligini alohida baholash mumkin bo‘ladi.

Bergman maqsadga erishish uchun **minimal model** (Minimal Model) konsepsiyasini ishlab chiqdilar. "Minimal" so‘zi modelning juda sodda ekanligini emas, balki u tizimning asosiy fiziologik xususiyatlarini eng kam sonli parametrlar orqali ifodalashini anglatadi. Bunday yondashuv modelni amaliy jihatdan qulay qiladi, parametrlarni ishonchli baholash imkonini beradi va fiziologik talqinni soddalashtiradi.

Matematik modellashtirish murakkab biologik tizimlarni o‘rganishning eng samarali usullaridan biridir. Biroq biologik tizimlarning murakkabligi sababli ularni to‘liq ifodalovchi matematik modellar ham juda ko‘p parametrli va murakkab bo‘lib ketadi. Bunday modellar nazariy jihatdan mukammal bo‘lsa-da, amaliyotda ularning parametrlarini tajriba ma’lumotlari orqali ishonchli aniqlash qiyinlashadi. Shu sababli Bergman va hamkorlari organizmning barcha fiziologik jarayonlarini emas, balki tadqiqot maqsadi uchun zarur bo‘lgan eng muhim mexanizmlarni ifodalovchi **minimal model** konsepsiyasini taklif qilganlar.

Minimal modelning asosiy g‘oyasi shundan iboratki, model imkon qadar sodda bo‘lishi, biroq organizmning muhim fiziologik xususiyatlarini saqlab qolishi kerak. Bunday model ortiqcha parametrlarni o‘z ichiga olmaydi va har bir parametrning biologik ma’nosi aniq bo‘ladi. Natijada modelni tajriba ma’lumotlari asosida identifikatsiya qilish osonlashadi.

Minimal modelni tanlash uchun oltita asosiy mezonni ishlab chiqdilar.

Birinchi mezon – model mavjud fiziologik bilimlarga mos kelishi kerak. Ya’ni model organizmda haqiqatan sodir bo‘ladigan biologik jarayonlarni aks ettirishi zarur.

Ikkinchi mezon – model parametrlarining nazariy aniqlanuvchanligi (theoretical identifiability). Boshqacha aytganda, model tenglamalaridan parametrlarni yagona qiymat sifatida topish nazariy jihatdan mumkin bo‘lishi kerak.

Uchinchi mezon – parametrlarning amaliy aniqlanuvchanligi (practical identifiability). Bu parametrlar real tajriba ma’lumotlari asosida ham ishonchli baholanishi kerakligini anglatadi.

To‘rtinchi mezon – parametrlarning fiziologik ma’noga ega bo‘lishidir. Har bir parametr organizmdagi aniq biologik jarayonni ifodalashi lozim.

Beshinchi mezon – model eng kam parametrlar bilan tajriba natijalarini yuqori aniqlikda tasvirlay olishi kerak.

Oltinchi mezon – model qoldiq xatolarining statistik tahliliga mos kelishi zarur. Agar model to‘g‘ri tanlangan bo‘lsa, hisoblangan va tajribada o‘lchangan qiymatlar orasidagi farq tasodifiy bo‘lishi kerak.

Ushbu mezonlar matematik modellashtirishning asosiy tamoyillaridan biri bo‘lib, ular modelning nafaqat nazariy, balki amaliy jihatdan ham ishonchli ekanligini ta’minlaydi.

Glyukoza–insulin tizimi biologik jihatdan juda murakkab yopiq boshqaruv tizimidir. Agar bu tizim bir butun holda modellashtirilsa, model juda murakkab bo‘lib ketadi va parametrlarni aniqlash deyarli imkonsizlashadi. Shu sababli Bergman va hamkorlari tizimni **dekompozitsiya qilish**, ya’ni uni ikki mustaqil kichik tizimga ajratish usulini taklif etdilar.

Bu yondashuv tizimni soddalashtirish bilan birga, har bir biologik mexanizmni alohida o‘rganish imkonini ham yaratadi.

Birinchi kichik tizim insulinning glyukozaga ta’sirini tavsiflaydi. Bu model insulin ta’sirida glyukozaning mushak va yog‘ to‘qimalari tomonidan o‘zlashtirilishi hamda jigarda glyukoza ishlab chiqarilishining kamayishini ifodalaydi[3].

Ikkinchi kichik tizim esa glyukozaning insulin sekretsiyasiga ta’sirini tavsiflaydi. Bunda qondagi glyukoza miqdorining ortishi oshqozon osti bezidagi beta hujayralar tomonidan insulin ajralishini qanday rag‘batlantirishi modellashtiriladi.

Shunday qilib, murakkab biologik tizim ikkita nisbatan sodda matematik model yordamida ifodalanadi.

Bergman modeli tizimlar nazariyasidagi **Input–Output** tamoyiliga asoslanadi.

Birinchi modelda tajribada o‘lchangan **insulin konsentratsiyasi kirish signali (input)** sifatida olinadi. Model esa glyukoza konsentratsiyasining vaqt bo‘yicha qanday o‘zgarishini hisoblaydi.

Ikkinchi modelda esa aksincha, **glyukoza konsentratsiyasi kirish signali** bo‘lib xizmat qiladi va model insulin konsentratsiyasining vaqt bo‘yicha o‘zgarishini hisoblaydi.

Bu yondashuvning muhim afzalligi shundaki, glyukoza va insulin bir-biridan vaqtincha matematik jihatdan ajratiladi. Natijada har bir biologik mexanizm alohida baholanadi va parametrlarni aniqlash ancha osonlashadi.

Bergman ushbu yondashuvning bir nechta muhim afzalliklarini ko‘rsatadi.

Birinchidan, model oddiy IVGTT tajribasi ma’lumotlari asosida ishlaydi. Qo‘shimcha murakkab laboratoriya usullari talab qilinmaydi.

Ikkinchidan, model organizmning tabiiy boshqaruv tizimini buzmaydi. Somatostatin yoki boshqa farmakologik vositalardan foydalanishga ehtiyoj qolmaydi.

Uchinchidan, model parametrlarining barchasi fiziologik ma’noga ega. Masalan, keyingi boblarda aniqlanadigan **insulin sezgirligi (SI)** va **glyukozaning o‘z-o‘zini boshqarish qobiliyati (SG)** bevosita organizmning metabolik holatini tavsiflaydi.

To‘rtinchidan, model klinik diagnostika uchun ham juda qulaydir. Chunki bitta IVGTT tajribasidan bemorning metabolik holati haqida keng qamrovli ma’lumot olish mumkin.

Tadqiqot metodologiyasi. Dastlab mualliflar minimal model konsepsiyasi va tizimni dekompozitsiyalash tamoyilini ishlab chiqdilar. So‘ngra ular ushbu nazariy yondashuvni amaliy jihatdan qo‘llab, glyukozaning organizmdagi dinamikasini ifodalovchi eng sodda va eng ishonchli matematik modelni tanlaydilar.

Mualliflarning asosiy maqsadi shundan iborat ediki, vena ichiga yuborilgan glyukozaning qondagi konsentratsiyasi vaqt bo‘yicha qanday kamayishini ifodalay oladigan model yaratish kerak edi. Shu bilan birga model insulinning glyukozaga ta’sirini ham aniq hisobga olishi lozim edi. Model yordamida insulin sezgirligi (Insulin Sensitivity) kabi muhim fiziologik ko‘rsatkichlarni hisoblash imkoniyati yaratilishi ko‘zlangan.

Bergman darhol yakuniy modelni tanlamagan. Aksincha, u glyukoza almashinuvini tavsiflovchi yettita turli matematik modelni ishlab chiqdi va ularni bosqichma-bosqich taqqosladi.

I $$\frac{dG}{dt} = p_1 G + p_2$$

II $$\frac{dG}{dt} = \frac{p_1 G}{p_2 + G} + p_3$$

III $$\frac{dG}{dt} = p_1 G + p_2 X + p_3$$
$$\frac{dX}{dt} = G + p_4 X$$

Dastlabki uchta modelda insulinning glyukozaga ta’siri umuman hisobga olinmagan. Bu modellar faqat glyukozaning tabiiy kamayishini ifodalardi. Biroq tajriba natijalari shuni ko‘rsatdiki, insulin ta’sirini hisobga olmasdan glyukoza dinamikasini to‘g‘ri tavsiflab bo‘lmaydi.

Shundan so‘ng mualliflar insulin ta’siri mavjud bo‘lgan modellarni ishlab chiqdilar. Ushbu modellar insulinning nafaqat glyukozani hujayralarga o‘tkazishi, balki jigarda glyukoza ishlab chiqarilishini ham kamaytirishini hisobga oldi. Modelning murakkabligi oshirilgani sari u tajriba natijalariga yaxshiroq moslasha boshladi.

Mualliflar barcha modellarni ikkinchi bobda keltirilgan oltita mezon asosida baholadilar.

IV $$\frac{dG}{dt} = p_1 G + p_2 I(t) + p_3$$

V $$\frac{dG}{dt} = p_1 G + p_2 X + p_3$$
$$\frac{dX}{dt} = p_4 X + I(t)$$

VI $$\frac{dG}{dt} = (p_1 - X)G + p_4$$
$$\frac{dX}{dt} = p_2 X + p_3 I(t)$$

VII $$\frac{dG}{dt} = p_1 + \frac{p_2 X}{G} + \frac{p_3}{1+p_4 X}$$
$$\frac{dX}{dt} = p_5 X + I(t)$$

Parametr	Biologik ma’nosi
G(t)	Qondagi glyukoza konsentratsiyasi
I(t)	Qondagi insulin konsentratsiyasi
X(t)	Insulinning kechikkan biologik ta’siri (remote insulin)
$\boldsymbol{p_1}$	Glyukozaning insulin ishtirokisiz kamayish tezligi
$\boldsymbol{p_2}$	Remote insulin ta’sirining kamayish tezligi
$\boldsymbol{p_3}$	Insulinning biologik ta’sir hosil qilish tezligi
SG	Glyukozaning o‘z-o‘zini boshqarish qobiliyati
SI	Organizmning insulinga sezgirligi

Natijada **VI-model** eng yaxshi model sifatida tanlandi. Chunki u quyidagi afzalliklarga ega edi:

- fiziologik jihatdan asoslangan;
- parametrlari yagona qiymatda aniqlanadi;
- tajriba ma’lumotlariga juda yaxshi mos keladi;
- ortiqcha parametrlar mavjud emas;
- har bir parametrning biologik ma’nosi aniq;
- hisoblash jarayoni sodda.

Shu sababli VI-model "Minimal Model" deb e’lon qilindi.

Bergman modelining eng muhim yangiliklaridan biri **Remote Insulin Compartment** tushunchasidir.

Olimlar tajribalarda shuni kuzatdilarki, qonda insulin miqdori oshishi bilan glyukoza darhol kamaymaydi. Insulin ta’siri biroz kechikib namoyon bo‘ladi.

Bu kechikish quyidagi fiziologik jarayonlar bilan izohlanadi:

- insulin kapillyarlardan to‘qimalarga o‘tadi;
- hujayra retseptorlari bilan bog‘lanadi;
- retseptordan keyingi signal uzatilishi sodir bo‘ladi;
- fermentlar faollashadi;
- glyukoza hujayraga kirishi tezlashadi.

Demak, qondagi insulin konsentratsiyasi bilan uning biologik ta’siri bir xil emas.

Shu sababli modelda **X(t)** deb ataluvchi yangi o‘zgaruvchi kiritildi. Bu o‘zgaruvchi qondagi insulin emas, balki insulinning **haqiqiy biologik ta’siri**ni ifodalaydi. Mualliflar bu holatni "remote insulin" deb ataydilar[4].

Modelga ko‘ra, glyukoza konsentratsiyasining kamayishiga ikkita asosiy omil ta’sir qiladi.

Birinchi omil — glyukozaning o‘z tabiiy xususiyatlari. Hatto insulin bo‘lmagan taqdirda ham glyukoza ma’lum darajada hujayralarga kiradi va jigarda uning ishlab chiqarilishi kamayadi.

Ikkinchi omil esa insulinning qo‘shimcha ta’siridir. Insulin glyukozaning hujayralarga o‘tishini sezilarli darajada tezlashtiradi.

Shunday qilib, glyukozaning kamayishi insulin va glyukozaning o‘z xususiyatlari birgalikdagi ta’siri natijasidir.

Olimlar model yordamida yangi fiziologik parametrni kiritadilar.

Bu parametr **Glucose Effectiveness (SG)** deb ataladi.

SG — bu insulin ishtirokisiz glyukozaning o‘z konsentratsiyasini me’yoriy holatga qaytarish qobiliyatidir.

Masalan:

- glyukozaning hujayralarga passiv kirishi;
- jigarda glyukoza ishlab chiqarilishining kamayishi.

Bularning barchasi SG tarkibiga kiradi. Agar SG katta bo‘lsa, organizm insulin bo‘lmasa ham glyukozani ma’lum darajada boshqara oladi. Agar SG kichik bo‘lsa, organizm insulin ta’siriga ko‘proq bog‘liq bo‘ladi[5].

Maqolaning eng muhim natijalaridan biri **Insulin Sensitivity (SI)** tushunchasining matematik ta’rifidir.

Mualliflar insulin sezgirligini quyidagicha izohlaydilar:

Insulinning glyukozaning o‘z-o‘zini me’yorlashtirish qobiliyatini qanchalik kuchaytirishi insulin sezgirligi deyiladi. Bu juda muhim ta’rifdir. Oldingi tadqiqotlarda insulin sezgirligi faqat tajriba orqali baholangan bo‘lsa, Bergman modeli uni matematik parametr sifatida aniqlash imkonini berdi.

Model parametrlaridan foydalanib SI oddiy nisbat orqali hisoblanadi. Natijada insulin sezgirligini aniqlash uchun murakkab **glucose clamp** tajribasiga ehtiyoj qolmaydi. Oddiy IVGTT tajribasi yetarli bo‘ladi[6].

Bergman va boshqa olimlarning eng katta yutug‘i shundaki, insulin sezgirligini endi murakkab laboratoriya usullaridan foydalanmasdan baholash mumkin bo‘ldi.

Bu quyidagi afzalliklarni beradi:

- diabetni erta aniqlash;
- insulin rezistentligini baholash;
- semizlikni o‘rganish;
- metabolik sindromni tahlil qilish;
- davolash samaradorligini kuzatish.

Bugungi kunda ham Bergman minimal modeli diabet va metabolik kasalliklar bo‘yicha ilmiy tadqiqotlarda eng ko‘p qo‘llaniladigan matematik modellardan biri hisoblanadi.

Ular glyukoza almashinuvini ifodalovchi yettita modelni taqqoslab, ularning ichidan **VI-model**ni optimal model sifatida tanlaydilar. Ushbu model insulinning kechikkan biologik ta’sirini **remote insulin** tushunchasi orqali hisobga oladi va glyukozaning tabiiy boshqarilishi (**Glucose Effectiveness, SG**) hamda insulinning glyukozaga ta’sir kuchini (**Insulin Sensitivity, SI**) alohida baholash imkonini beradi. Eng muhim natija shundaki, oddiy **IVGTT** ma’lumotlari asosida ushbu parametrlarni hisoblash mumkin bo‘lib, bu diabet va boshqa metabolik kasalliklarni tashxislash hamda davolash samaradorligini baholash uchun katta amaliy ahamiyatga ega[7].

Ular murakkab glyukoza–insulin tizimini juda kam sonli, ammo fiziologik jihatdan mazmunli parametrlar yordamida tavsiflay oladigan modelni taklif qiladilar. Ushbu model orqali $p1, p2\ va\ p3$ parametrlarini tajriba ma’lumotlaridan baholab, glyukozaning insulin mustaqil boshqarilishi (**SG**) hamda organizmning insulinga sezgirligi (**SI**)ni miqdoriy aniqlash mumkin. Bu yondashuv oddiy IVGTT ma’lumotlaridan foydalanib metabolik holatni baholash imkonini bergani uchun diabet va insulin rezistentligini o‘rganishda muhim ahamiyat kasb etadi.

Organizmga glyukoza yuborilganda qondagi glyukoza miqdori ortadi. Bu oshqozon osti bezining beta hujayralarini rag‘batlantirib, insulin ajralishiga olib keladi. Agar beta hujayralar yetarli darajada insulin ishlab chiqarmasa, glyukoza uzoq vaqt yuqori bo‘lib qoladi. Shuning uchun faqat insulin sezgirligini emas, balki **beta hujayralarning funksional holatini** ham baholash zarur.

Ular insulin sekretsiyasi modelida glyukozani **kirish (input)**, plazmadagi insulin konsentratsiyasini esa **chiqish (output)** sifatida qaraydilar. Modelning vazifasi glyukoza o‘zgarishi asosida insulin dinamikasini tavsiflashdir.

Olimlar insulin sekretsiyasi uchun oltita matematik modelni ishlab chiqib, ularni taqqoslaydilar.

I $$IDR(t) = \beta(G(t) - Gb) + I_0$$

II $$IDR(t) = \gamma(G(t) - h) + I_0, \quad G(t) > h$$

$$IDR(t) = I_0, \quad G(t) \le h$$

III
$$\frac{dy(t)}{dt} = -\alpha y(t) + \beta(G(t) - Gb), \quad G(t) > Gb$$
$$\frac{dy(t)}{dt} = -\alpha y(t), \quad G(t) \le Gb$$
$$IDR(t) = y(t)$$

IV
$$\frac{dy(t)}{dt} = -\alpha y(t) + \beta(G(t) - h), \quad G(t) > h$$
$$\frac{dy(t)}{dt} = -\alpha y(t), \quad G(t) \le h$$
$$IDR(t) = y(t)$$

V
$$IDR(t) = (G(t) - Gb)y(t)$$
$$\frac{dy(t)}{dt} = -\alpha y(t) + \beta(G(t) - Gb), \quad G(t) > Gb$$
$$\frac{dy(t)}{dt} = -\alpha y(t), \quad G(t) \le Gb$$

VI
$$IDR(t) = (G(t) - h)y(t)$$
$$\frac{dy(t)}{dt} = -\alpha y(t) + \beta(G(t) - h), \quad G(t) > h$$
$$\frac{dy(t)}{dt} = -\alpha y(t), \quad G(t) \le h$$

Parametr	To‘liq nomi	Fizik/biologik ma’nosi	Birligi (agar mavjud bo‘lsa)
α (alpha)	Dynamic decay coefficient	Yashirin (latent) o‘zgaruvchi **y(t)** ning kamayish (relaksatsiya) tezligi. Beta hujayra javobining vaqt bo‘yicha so‘nishini tavsiflaydi.	min^{-1}
β (beta)	Glucose sensitivity coefficient	Glyukoza konsentratsiyasi oshganda ikkinchi faza insulin sekretsiyasining kuchayish tezligini belgilaydi. Beta hujayraning glyukozaga sezgirlik koeffitsienti.	($\mu U \cdot min^{-2} \cdot mg^{-1}$) ga proporsional
γ (gamma)	Proportional secretion coefficient	Model I va II da insulin sekretsiyasi tezligining glyukoza konsentratsiyasiga to‘g'ridan-to‘g'ri proporsionallik koeffitsienti.	$\mu U \cdot min^{-1} \cdot mg^{-1}$
h	Glucose threshold	Chegaraviy (threshold) glyukoza konsentratsiyasi. Agar **G(t) > h** bo‘lsa, ikkinchi faza insulin sekretsiyasi faollashadi; aks holda faollashmaydi.	mg/dL (yoki mmol/L)
I_0	Initial insulin concentration	Boshlang'ich insulin konsentratsiyasi. Modelda birinchi faza insulin cho‘qqisi vaqtidagi plazma insulin qiymatini bildiradi.	μU/mL
n	Insulin disappearance parameter	Insulinning plazmadan yo‘qolish (clearance) tezligi. Mualliflar uni insulin klirensi ko‘rsatkichi sifatida talqin qiladilar.	min^{-1}

Bu parametrlar orasida eng muhimlari:
- **α** — beta hujayra javobining vaqt bo‘yicha susayish tezligini tavsiflaydi.
- **β** — ikkinchi faza insulin sekretsiyasining glyukozaga sezgirligini ifodalaydi.
- **h** — insulin sekretsiyasi boshlanadigan glyukoza chegarasi.
- **n** — insulinning organizmdan chiqarilish (klirens) tezligi.
- **I_0** — tajriba boshidagi insulin konsentratsiyasi.
- **γ** — oddiy proporsional insulin sekretsiyasi modelining koeffitsienti.

Olingan natijalar va ularning tahlili. Baholash mezonlari 2-bobdagi minimal model mezonlariga asoslanadi. Natijada **II-model** optimal model sifatida tanlanadi, chunki u:
- fiziologik jihatdan asoslangan;
- kam parametrli;
- tajriba ma’lumotlariga yaxshi mos keladi;

• parametrlarini ishonchli aniqlash mumkin.

Model insulin ajralishini ikki bosqichga ajratadi:

Birinchi faza — glyukoza yuborilgandan keyingi dastlabki bir necha daqiqada beta hujayralarda oldindan saqlangan insulin tez ajraladi. Bu qisqa, ammo juda muhim javobdir.

Ikkinchi faza — beta hujayralar yangi insulin sintez qilib, uni asta-sekin qonga chiqaradi. Bu bosqich glyukoza yuqori bo‘lib turgan davr davomida davom etadi.

Bunday ikki fazali javob sog‘lom organizm uchun xos bo‘lib, 2-tip diabetda ayniqsa birinchi faza ko‘pincha susayadi yoki yo‘qoladi.

Model yordamida uchta muhim parametr baholanadi:

• $\mathbf{\Phi_1}$ – birinchi fazadagi insulin javobining glyukozaga sezgirligi.

• $\mathbf{\Phi_2}$ – ikkinchi fazadagi insulin sekretsiyasining glyukozaga sezgirligi.

• **n** – insulinning plazmadan chiqarilish (klirens) tezligi.

Φ_1 yuqori bo‘lsa, beta hujayralar glyukozaga tez javob beradi. Φ_2 esa glyukoza yuqori bo‘lib turganda insulin ishlab chiqarish qobiliyatini ifodalaydi. **n** parametri esa insulin organizmda qancha tez parchalanishini ko‘rsatadi.

Bergman va sheriklar glyukoza tolerantligi bitta parametr bilan emas, balki **ikki asosiy omilning birgalikdagi ta’siri** bilan belgilanishini ko‘rsatadilar:

1. insulin sezgirligi (**SI**);
2. beta hujayralarning glyukozaga sezgirligi (• $\mathbf{\Phi_2}$).

Muhokama. Masalan, organizmning insulinga sezgirligi pasaygan bo‘lsa ham, beta hujayralar ko‘proq insulin ishlab chiqarish orqali bu holatni qisman qoplashi mumkin. Aksincha, insulin sezgirligi yaxshi bo‘lsa ham, insulin sekretsiyasi sust bo‘lsa, glyukoza tolerantligi yomonlashadi.

Olimlar yangi ko‘rsatkich — **Disposition Index (DI)** ni taklif qiladilar.

Bu ko‘rsatkich insulin sezgirligi va pankreatik sezgirlikning ko‘paytmasi sifatida qaraladi. Uning maqsadi organizmning umumiy metabolik imkoniyatini yagona son bilan baholashdir.

Agar DI yuqori bo‘lsa:

• glyukoza tolerantligi yaxshi;

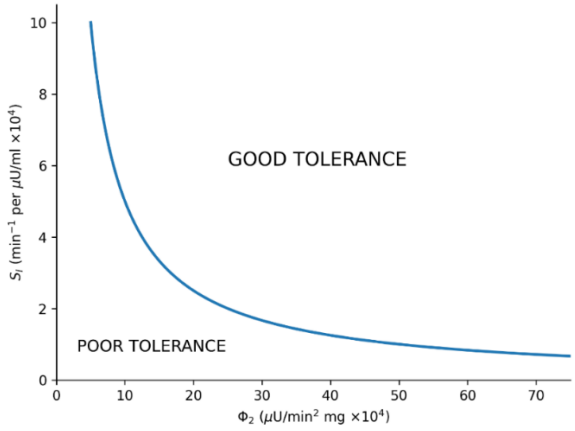
• metabolik boshqaruv samarali.

Agar DI past bo‘lsa:

• insulin rezistentligi yoki beta hujayra yetishmovchiligi mavjud bo‘lishi mumkin.

Bu yondashuv shifokorga bemorning muammosi asosan qayerda ekanini aniqlashga yordam beradi.

Olimlar SI va • Φ_2 qiymatlari asosida bemorning metabolik holatini grafik tekislikda tasvirlashni taklif qiladilar. Grafikda normal va patologik holatlar chegarasi ko‘rsatiladi. Shuningdek, turli davolash usullarining (insulin terapiyasi, sulfonilmochevina preparatlari, vazn kamaytirish va boshqalar) ushbu parametrlarni qanday o‘zgartirishi baholanishi mumkin.



1-rasm.

Bu yondashuv individual davolash strategiyasini tanlash imkonini beradi va shaxsiylashtirilgan tibbiyot g‘oyasiga mos keladi.

Bergman va sheriklari tadqiqot yakunida glyukoza–insulin tizimini ikkita minimal model yordamida muvaffaqiyatli tavsiflash mumkinligini ko‘rsatadilar. Birinchi model insulin sezgirligini, ikkinchi model esa pankreatik sezgirlikni baholaydi. Bu ikki model oddiy IVGTT ma’lumotlari asosida organizmning metabolik holatini miqdoriy tavsiflash imkonini beradi[8].

Tadqiqotning eng muhim ilmiy natijalari quyidagilardan iborat:

• murakkab glyukoza–insulin tizimi minimal parametrlar bilan tavsiflandi;

- insulin sezgirligi va pankreatik sezgirlik alohida baholandi;
- invaziv va murakkab usullarga ehtiyoj kamaytirildi;
- IVGTT asosida metabolik holatni aniqlash imkoniyati yaratildi;
- klinik diagnostika va diabetni davolashni rejalashtirish uchun yangi matematik yondashuv taklif qilindi.

Bundan tashqari, mualliflar organizmda **beta hujayralar sezgirligi** va **insulin sezgirligi** o‘rtasida o‘zaro moslashuv mexanizmi mavjud bo‘lishi mumkinligini ta’kidlaydilar. Ya’ni, ulardan biri pasayganda ikkinchisi ma’lum darajada kompensatsiya qilishi mumkin. Bu g‘oya keyinchalik diabet fiziologiyasini o‘rganishda muhim ilmiy yo‘nalishlardan biriga aylandi.

Xulosa. Ushbu maqolada Bergman minimal modeli va uning glyukoza–insulin tizimini matematik modellashtirishdagi o‘rni batafsil tahlil qilindi. Tadqiqot davomida modelning yaratilish tarixi, minimal modellashtirish tamoyili, glyukoza dinamikasi hamda insulinning organizmdagi kechikkan biologik ta’sirini tavsiflovchi differensial tenglamalar o‘rganildi. Model tarkibidagi $\boldsymbol{p1}, \boldsymbol{p2}, \boldsymbol{p3}$ va $\boldsymbol{X(t)X(t)X(t)}$ kabi asosiy parametrlarning matematik va biologik mazmuni izohlandi.

Shuningdek, maqolada glyukoza almashinuvi uchun taklif etilgan nomzod modellar hamda insulin sekretsiyasini ifodalovchi modellar taqqoslanib, ularning afzallik va kamchiliklari yoritildi. Tahlillar natijasida minimal parametrlar yordamida organizmdagi murakkab metabolik jarayonlarni yetarli aniqlikda tavsiflash mumkinligi ko‘rsatildi.

Bergman minimal modeli asosida aniqlanadigan Glucose Effectiveness (SG) va Insulin Sensitivity (SI) ko‘rsatkichlari organizmning metabolik holatini baholashda muhim ahamiyatga ega ekanligi aniqlandi. Ushbu parametrlar qandli diabet, insulin rezistentligi va boshqa metabolik kasalliklarni erta aniqlash hamda davolash samaradorligini baholashda keng qo‘llaniladi[9].

Kelgusida ushbu modelni uzluksiz glyukoza monitoringi (CGM), insulin pompasi tizimlari, raqamli egizak (Digital Twin) texnologiyalari hamda mashinaviy o‘qitish algoritmlari bilan integratsiya qilish orqali qandli diabetni individual boshqarish tizimlarini yanada takomillashtirish mumkin.[10]

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DYNAMICS OF A CUBIC STOCHASTIC OPERATOR

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Abstract. *It is well-known that Lotka-Volterra operators play important role in studying predator-prey models. In the present paper, we consider discrete-time dynamical systems generated by cubic operators which we are called separable cubic stochastic operators. That is, we consider cubic stochastic operators defined as a product of three linear operators defined on a simplex. It is shown that for a separable cubic stochastic operator defined on a two-dimensional simplex, the vertices of the simplex are fixed points. It is also proven that the orbit of a separable cubic stochastic operator converges for any starting point taken from the simplex.*

Keywords: *cubic stochastic operator, separable cubic stochastic operator, orbit, simplex.*

KUBIK STOXASTIK OPERATORNING DINAMIKASI

Annotatsiya*. Ma'lumki, Lotka–Volterra operatorlari "yirtqich–o'lja" turidagi modellarni o'rganishda muhim ahamiyat kasb etadi. Ushbu maqolada biz separabel kubik stoxastik operatorlar deb ataluvchi kubik operatorlar tomonidan hosil qilingan diskret vaqtli dinamik tizimlarni ko'rib chiqamiz. Ya'ni, simpleksda aniqlangan uchta chiziqli operatorlarning ko'paytmasi sifatida aniqlangan kubik stoxastik operatorlarni ko'rib chiqamiz. Ikki o'lchamli simpleksda aniqlangan separabel kubik stoxastik operator uchun simpleksning uchlari qo'zg'almas nuqtalar ekanligi ko'rsatilgan. Shuningdek, separabel kubik stoxastik operatorning orbitasi simpleksdan olingan har qanday boshlang'ich nuqta uchun yaqinlashuvchi ekanligi isbotlangan.*

Kalit so'zlar: *kubik stoxastik operator, separabel kubik stoxastik operator, orbita, simpleks.*

ДИНАМИКА КУБИЧЕСКОГО СТОХАСТИЧЕСКОГО ОПЕРАТОРА

Аннотация. *Хорошо известно, что операторы Лотки–Вольтерры играют важную роль в исследовании моделей типа «хищник–жертва». В настоящей работе рассматриваются дискретные динамические системы, порождённые кубическими операторами, которые мы называем сепарабельными кубическими стохастическими операторами. То есть, рассматриваются кубические стохастические операторы, заданные как произведение трёх линейных операторов, заданных на симплексе. Показано, что для сепарабельного кубического стохастического оператора, заданного на двумерном симплексе, вершины симплекса являются неподвижными точками.*

Ключевые слова: *кубический стохастический оператор, сепарабельный кубический стохастический оператор, орбита, симплекс.*

Introduction. The evolution of biological populations can be studied by dynamical systems of nonlinear operators. One of the simplest nonlinear case is quadratic (QSO) one. This operator models the evolution of a population and was originally introduced by Bernstein in [1]. For more than 80 years, the theory of QSOs has been developed and many papers were published (see e.g. [4]-[8], [16]-[17]). In recent years it has again become of interest in connection with its numerous applications in many branches of mathematics, biology and physics.

Let $E=\{1,2,...,m\}$ be a finite set and the set of all probability distribution on E

$$S^{m-1}=\left\{x=(x_1,\dots x_m)\in\mathbb{R}^m : x_i\geq 0, \sum_{i=1}^{m} x_i=1\right\} \tag{1}$$

be the $(m-1)$-dimensional simplex. A QSO is a mapping defined as $V:S^{m-1}\to S^{m-1}$ of the simplex into itself, of the form $V(\mathbf{x})=\mathbf{x}'\in S^{m-1}$, where

$$x_k'=\sum_{i,j=1}^{m} P_{ij,k}x_i x_j, \quad k\in E, \tag{2}$$

and the coefficients $P_{ij,k}$ satisfy

$$P_{ij,k} = P_{ji,k} \geq 0, \quad \sum_{k=1}^{m} P_{ij,k} = 1 \qquad \text{for all} \quad i, j \in E. \tag{3}$$

The trajectory (orbit) $\left\{\mathbf{x}^{(n)}\right\}_{n\geq 0}$, of V for an initial value $\mathbf{x}^{(0)} \in S^{m-1}$ is defined by

$$\mathbf{x}^{(n+1)} = V\left(\mathbf{x}^{(n)}\right) = V^{(n+1)}\left(\mathbf{x}^{(0)}\right), \qquad n = 0,1,2,\ldots .$$

One of the main problems in mathematical biology is to study the asymptotic behavior of the trajectories. This problem was solved completely for the Volterra QSO.

The operator V is called Volterra QSO, if $P_{ij,k} = 0$ for any $k \notin \{i, j\}$, $i, j, k \in E$. For the Volterra QSO the general formula was given in [4],

$$x_k' = x_k\left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \tag{4}$$

where $a_{ki} = 2P_{ik,k} - 1$ for $i \neq k$ and $a_{kk} = 0$. Moreover, $a_{ki} = -a_{ik}$ and $|a_{ki}| \leq 1$ for all $i, k \in E$.

In [4], the theory of Volterra QSO was developed using theory of the Lyapunov functions and tournaments. But non-Volterra QSOs were not completely studied. Because, there is no general theory that can be applied for study of non-Volterra operators.

In [16] Separable Quadratic Stochastic Operators (SQSOs) where introduced. Volterra QSO (4) has a form as SQSO, but in [17] it was proved that it coincides with a SQSO if and only if it is a linear operator.

In recent years, Cubic Stochastic Operators (CSOs) have begun to be studied, which different from quadratic operators [8-10].

In this paper we consider another class of cubic operators which we call separable cubic stochastic operators [11-13].

In Section 2, we recall the definition of CSOs and definitions and known results. In Section 3 for a SCSO defined on the two-dimensional simplex, we prove that it has three fixed points and we find the boundary ∂S^2 is an invariant set.

Preliminaries and known results

Separable quadratic stochastic operator. Let us recall some necessary definition and notations. In [18] Separable Quadratic Stochastic Operators were introduced as follows: The QSO (2), (3) with additional condition

$$P_{ij,k} = a_{ik} b_{jk} \qquad \text{for all} \qquad i, j, k \in E \tag{5}$$

where $a_{ik}, b_{jk} \in \mathbb{R}$ entries of matrices $A = (a_{ik})$ and $B = (b_{jk})$ such that the conditions (3) are satisfied for the coefficients (5).

Then the QSO V corresponding to the coefficients (5) has the form.

$$x_k' = \left(V(\mathbf{x})\right)_k = \left(A(\mathbf{x})\right)_k \left(B(\mathbf{x})\right)_k, \tag{6}$$

where $\left(A(\mathbf{x})\right)_k = \sum_{i=1}^{m} a_{ik} x_i$, $\left(B(\mathbf{x})\right)_k = \sum_{j=1}^{m} b_{jk} x_j$.

Definition 1. [16] *The QSO (6) is called separable quadratic stochastic operator (SQSO).*

Cubic stochastic operator. The CSO is a mapping $W : S^{m-1} \to S^{m-1}$ of the form

$$x_l' = \sum_{i,j,k=1}^{m} P_{ijk,l} x_i x_j x_k, \quad l \in E, \tag{7}$$

where $P_{ijk,l}$ are coefficients of heredity such that

$$P_{ijk,l} \geq 0, \quad \sum_{l=1}^{m} P_{ijk,l} = 1, \quad \forall i, j, k \in E. \tag{8}$$

and we suppose that the coefficients $P_{ijk,l}$ do not change for any permutation of i, j, k.

For a given $\mathbf{x}^{(0)} \in S^{m-1}$, the trajectory $\left\{\mathbf{x}^{(n)}\right\}_{n\geq 0}$ of initial point $\mathbf{x}^{(0)}$ under action of CSO (8) is defined by $\mathbf{x}^{(n+1)} = W\left(\mathbf{x}^{(n)}\right)$, where $n = 0,1,2,\ldots$ with $\mathbf{x} = \mathbf{x}^{(0)}$. Denote by $\omega\left(\mathbf{x}^{(0)}\right)$ the set of limit points of the trajectory $\left\{\mathbf{x}^{(n)}\right\}_{n=0}^{\infty}$. Since $\left\{\mathbf{x}^{(n)}\right\}_{n=0}^{\infty} \subset S^{m-1}$ and S^{m-1} is a compact set, it is follows that $\omega\left(\mathbf{x}^{(0)}\right) \neq \varnothing$. If $\omega\left(\mathbf{x}^{(0)}\right)$ consists of a single point, they the trajectory converges and $\omega\left(\mathbf{x}^{(0)}\right)$ is a fixed point of the operator W. A

point $\mathbf{x}\in S^{m-1}$ is called a fixed of the W if $W(\mathbf{x})=\mathbf{x}$. Denote by $\mathrm{Fix}(W)$ the set of all fixed points of the operator W, i.e.

$$\mathrm{Fix}(W)=\{\mathbf{x}\in S^{m-1}: W(\mathbf{x})=\mathbf{x}\}.$$

Let $DW(\mathbf{x}^*)=(\partial W_i/\partial x_j)(\mathbf{x}^*)$ be a Jacobian of W at the point $\mathbf{x}^*$.

Definition 2 ([3]): *A fixed point $\mathbf{x}^*$ is called hyperbolic if its Jacobian $DW(\mathbf{x}^*)$ has no eigenvalues on the unit circle in $\mathbb{C}$.*

Definition 3 ([3]): *A hyperbolic fixed point $\mathbf{x}^*$ is called*:

(i) attracting if all the eigenvalues of the Jacobian $DW(\mathbf{x}^*)$ are in the unit disk;

(ii) repelling if all the eigenvalues of the Jacobian $DW(\mathbf{x}^*)$ are outside the closed unit disk;

(iii) a saddle otherwise;

Main result

In this section we consider CSO (7), (8) with additional condition

$$P_{ijk,l}=a_{il}b_{jl}c_{kl},\quad \text{for all}\quad i,j,k,l\in E, \tag{9}$$

where $a_{il},b_{jl},c_{kl}\in\mathbb{R}$ entries of matrices $A=(a_{il})$, $B=(b_{jl})$ and $C=(c_{kl})$ such that the conditions (8) are satisfied for the coefficients (9).

Then the CSO W corresponding to the coefficients (9) has the form

$$x'_l=(W(\mathbf{x}))_l=(A(\mathbf{x}))_l(B(\mathbf{x}))_l(C(\mathbf{x}))_l \tag{10}$$

where $(A(\mathbf{x}))_l=\sum_{i=1}^{m}a_{il}x_i$, $(B(\mathbf{x}))_l=\sum_{j=1}^{m}b_{jl}x_j$ and $(C(\mathbf{x}))_l=\sum_{k=1}^{m}c_{kl}x_k$.

Definition 4. [11] *The CSO* (10) *is called separable cubic stochastic operator (SCSO).*

Let us consider the following matrices:

$$A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ B=\begin{pmatrix}1&0&2\\2&1&1\\1&1&1\end{pmatrix},\ C=\begin{pmatrix}1&1&1\\1&1&3/2\\0&1/2&1\end{pmatrix}. \tag{11}$$

Then corresponding SCSO $W:S^2\to S^2$ is:

$$W:\begin{cases}x'_1=x_1(x_1+2x_2+x_3)(x_1+x_2),\\ x'_2=x_2(x_2+x_3)(x_1+x_2+0{,}5x_3),\\ x'_3=x_3(2x_1+x_2+x_3)(x_1+1{,}5x_2+x_3).\end{cases} \tag{12}$$

Using the equation $x_1+x_2+x_3=1$ we rewrite the operator (12) as follows

$$W:\begin{cases}x'_1=x_1(1+x_2)(1-x_3),\\ x'_2=x_2(1-x_1)(1-0{,}5x_3),\\ x'_3=x_3(1+x_1)(1+0{,}5x_2).\end{cases} \tag{13}$$

Let a face of the simplex S^2 be the set $\Gamma_\alpha=\{\mathbf{x}\in S^2: x_i=0,\ i\notin\alpha\subset\{1,2,3\}\}$.

Let the set $\operatorname{int}S^2=\{\mathbf{x}\in S^2: x_1x_2x_3>0\}$ and let the set $\partial S^2=S^2\setminus\operatorname{int}S^2$ be the interior and the boundary of the simplex S^2, respectively. Let $\mathbf{e}_1=(1,0,0)$, $\mathbf{e}_2=(0,1,0)$, $\mathbf{e}_3=(0,0,1)$ be the vertexes of the two-dimensional simplex.

Lemma 1. *For the SCSO W* (13), *the following assertions true*:

(i) *The face $\Gamma_{\{1,2\}}$, $\Gamma_{\{1,3\}}$, $\Gamma_{\{2,3\}}$ of the simplex S^2 are invariant sets;*

(ii) $\mathrm{Fix}(W)=\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$;

(iii) The fixed point $\mathbf{e}_1$ is a saddle point, $\mathbf{e}_2$ is a repelling fixed point and the fixed point $\mathbf{e}_3$ is a attracting point.

Proof: (i) Obviously.

(ii) To find the fixed points we consider the equation $W(\mathbf{x})=\mathbf{x}$, that is the following system of equations

$$\begin{cases} x_1 = x_1(1+x_2)(1-x_3), \\ x_2 = x_2(1-x_1)(1-0,5x_3), \\ x_3 = x_3(1+x_1)(1+0,5x_2). \end{cases} \tag{14}$$

(a) Let $x_1=0$. Then from the third equation of (14) it follows

$$x_3 = x_3 + 0,5x_2x_3 \;\Rightarrow\; x_3=0 \quad \text{and} \quad 1=1+0,5x_2.$$

It is clear that if $x_3=0$ then we have $x_2=1$. If $x_3>0$ then from the last equation one has

$$x_2=0 \;\Rightarrow\; x_3=1.$$

We take solution $x_2=1$, then it follows that $x_3=0$. Consequently, if $x_1=0$, we obtain the fixed points $\mathbf{e}_2$, $\mathbf{e}_3$

Similarly, in the case $x_2=0$ and $x_3=0$, we have the fixed points $\mathbf{e}_1$, $\mathbf{e}_3$ and $\mathbf{e}_1$, $\mathbf{e}_2$, respectively.

(b) Suppose that $x_1x_2x_3\neq 0$. Then from the system (14), one has

$$\begin{cases} 1 = (1+x_2)(1-x_3), \\ 1 = (1-x_1)(1-0,5x_3), \\ 1 = (1+x_1)(1+0,5x_2). \end{cases} \tag{15}$$

The second equation of (15) we get

$$1=(1-x_1)(1-0,5x_3) \;\Rightarrow\; x_1 = 0,5x_3(x_1-1).$$

Last equation it follows that $x_1=0$ and $x_3=0$, this contradicts to $x_1x_2x_3\neq 0$.

Consequently, we have that $\mathrm{Fix}(W)\cap \mathrm{int}\,S^2=\varnothing$.

(iii) To find the type of fixed point of the SCSO (13), we rewrite it in the form

$$\begin{cases} x_1' = x_1\left(x_1+2x_2+(1-x_1-x_2)\right)(x_1+x_2), \\ x_2' = x_2\left(x_2+(1-x_1-x_2)\right)(x_1+x_2+0,5(1-x_1-x_2)), \end{cases} \tag{16}$$

where $(x_1,x_2)\in\{(x,y): x,y\geq 0,\ 0\leq x+y\leq 1\}$ and x_1,x_2 are the first two coordinates of a point lying in the simplex S^2.

The Jacobian of the operator (16) at a fixed point $\mathbf{e}_1$ has the following eigenvalues $\lambda_1=2$, $\lambda_2=0$, at the $\mathbf{e}_2$ has the following eigenvalues $\lambda_1=1,5$, $\lambda_2=2$ and at the $\mathbf{e}_3$ has the following eigenvalues $\lambda_1=0,5$, $\lambda_2=0$. Therefore, it follow that the fixed point $\mathbf{e}_1$ is a saddle point, $\mathbf{e}_2$ is a repelling fixed point and the fixed point $\mathbf{e}_3$ is a attracting point.

The theorem is proved. □

The set of limit points

Let $\mathbf{x}^{(0)}\in S^2$ be the initial point. Then the trajectory of $\mathbf{x}^{(0)}$ is denote by $\{\mathbf{x}^{(n)}\}_{n\geq 0}$ and is defined as $W(\mathbf{x}^{(n)})=\mathbf{x}^{(n+1)}$ where $n=0,1,2,\ldots$.

$$W:\begin{cases} x_1' = x_1(1+x_2)(1-x_3), \\ x_2' = x_2(1-x_1)(1-0,5x_3), \\ x_3' = x_3(1+x_1)(1+0,5x_2). \end{cases}$$

Lemma 2. *For the SCSO* W (13), *the following assertions true*:

(i) *If* $\mathbf{x}^{(0)}\in\Gamma_{\{2,3\}}$, *then* $\lim\limits_{n\to\infty}W^n(\mathbf{x}^{(0)})=\mathbf{e}_3$,

(ii) *If* $\mathbf{x}^{(0)}\in\Gamma_{\{1,2\}}$, *then* $\lim\limits_{n\to\infty}W^n(\mathbf{x}^{(0)})=\mathbf{e}_1$,

(iii) *If* $\mathbf{x}^{(0)}\in\Gamma_{\{1,3\}}$, *then* $\lim\limits_{n\to\infty}W^n(\mathbf{x}^{(0)})=\mathbf{e}_3$,

(iv) *If* $\mathbf{x}^{(0)}\in\mathrm{int}\,S^2$, *then* $\lim\limits_{n\to\infty}W^n(\mathbf{x}^{(0)})=\mathbf{e}_3$.

Proof: (i) Let $\mathbf{x}^{(0)}=\left(0,x_2^{(0)},x_3^{(0)}\right)\in\Gamma_{\{2,3\}}$. By Lemma 1, the face $\Gamma_{\{2,3\}}$ is an invariant set and the vertexes $\mathbf{e}_2$ and $\mathbf{e}_3$ are fixed points belonging to this face. For the SCSO W (13) is true $x_1^{(n)}=0$, $n=0,1,2,\ldots$ at the face $\Gamma_{\{2,3\}}$. The restriction of the SCSO W (12) on the face $\Gamma_{\{2,3\}}$ has the form

$$\begin{cases} x_2'=x_2\left(1-0,5x_3\right), \\ x_3'=x_3\left(1+0,5x_2\right). \end{cases} \tag{17}$$

From the first equation of (17), one has $x_2^{(n+1)}=x_2^{(n)}\left(1-0,5x_3^{(n)}\right)\le x_2^{(n)}$, $n=0,1,2,\ldots$. Therefore, it follows that there exists the $\lim\limits_{n\to\infty}x_2^{(n)}=x_2^*$. Moreover, from the second equation of (17), one has $x_3^{(n+1)}=x_3^{(n)}\left(1+x_2^{(n)}\right)\ge x_3^{(n)}$, $n=0,1,2,\ldots$. Therefore, it follows that there exists the $\lim\limits_{n\to\infty}x_3^{(n)}=x_3^*$. So there is the limit $\lim\limits_{n\to\infty}W^n\left(\mathbf{x}^{(0)}\right)=\mathbf{x}^*=\left(0,x_2^*,x_3^*\right)$. Since $\mathbf{x}^*$ should be a fixed we get $\mathbf{x}^*=\mathbf{e}_3$. That is, if $x_3^{(0)}>0$ then we have that $\lim\limits_{n\to\infty}W^n\left(\mathbf{x}^{(0)}\right)=\mathbf{e}_3$ for any $\mathbf{x}^{(0)}=\Gamma_{\{2,3\}}\setminus\left\{\mathbf{e}_2\right\}$.

Let $\mathbf{x}^{(0)}\in\Gamma_{\{1,2\}}$. By Lemma 1, the face $\Gamma_{\{1,2\}}$ is an invariant set and the vertexes $\mathbf{e}_1$ and $\mathbf{e}_2$ are fixed points belonging to this face. The restriction of the SCSO W (13) on the face $\Gamma_{\{1,2\}}$ has the form

$$\begin{cases} x_1'=x_1\left(1+x_2\right), \\ x_2'=x_2\left(1-x_1\right). \end{cases} \tag{18}$$

Not that, for the SCSO W (18) is true $x_3^{(n)}=0$, $n=0,1,2,\ldots$ at the face $\Gamma_{\{1,2\}}$. From the first equation of (18), one has $x_1^{(n+1)}=x_1^{(n)}\left(1+x_2^{(n)}\right)\ge x_1^{(n)}$, $n=0,1,2,\ldots$. Therefore, it follows that there exists the $\lim\limits_{n\to\infty}x_1^{(n)}=x_1^*$. Moreover, from the second equation of (18), one has $x_2^{(n+1)}=x_2^{(n)}\left(1-x_1^{(n)}\right)\le x_2^{(n)}$, $n=0,1,2,\ldots$. So we have that there exists the $\lim\limits_{n\to\infty}x_2^{(n)}=x_2^*$. Hence there is the limit $\lim\limits_{n\to\infty}W^n\left(\mathbf{x}^{(0)}\right)=\mathbf{x}^*=\left(x_1^*,x_2^*,0\right)$. Since $\mathbf{x}^*$ should be a fixed point we get $\mathbf{x}^*=\mathbf{e}_1$. That is, if $x_1^{(0)}>0$ then we have that $\lim\limits_{n\to\infty}W^n\left(\mathbf{x}^{(0)}\right)=\mathbf{e}_1$ for any $\mathbf{x}^{(0)}=\Gamma_{\{1,2\}}\setminus\left\{\mathbf{e}_2\right\}$.

Let $\mathbf{x}^{(0)}=\left(x_1^{(0)},0,x_3^{(0)}\right)\in\Gamma_{\{1,3\}}$. By Lemma 1, the face $\Gamma_{\{1,3\}}$ is an invariant set and the vertexes $\mathbf{e}_1$ and $\mathbf{e}_3$ are fixed points belonging to this face. For the SCSO W (13) is true $x_2^{(n)}=0$, $n=0,1,2,\ldots$ at the face $\Gamma_{\{1,3\}}$. The restriction of the SCSO W (13) on the face $\Gamma_{\{1,3\}}$ has the form

$$\begin{cases} x_1'=x_1\left(1-x_3\right), \\ x_3'=x_3\left(1+x_1\right). \end{cases} \tag{19}$$

From the first equation of (19), one has $x_1^{(n+1)}=x_1^{(n)}\left(1-x_3^{(n)}\right)\le x_3^{(n)}$, $n=0,1,2,\ldots$. Therefore, it follows that there exists the $\lim\limits_{n\to\infty}x_1^{(n)}=x_1^*$. Moreover, from the second equation of (19), one has $x_3^{(n+1)}=x_3^{(n)}\left(1+x_1^{(n)}\right)\ge x_3^{(n)}$, $n=0,1,2,\ldots$. Therefore, it follows that there exists the $\lim\limits_{n\to\infty}x_3^{(n)}=x_3^*$. So there is the limit $\lim\limits_{n\to\infty}W^n\left(\mathbf{x}^{(0)}\right)=\mathbf{x}^*=\left(x_1^*,0,x_3^*\right)$. Since $\mathbf{x}^*$ should be a fixed we get $\mathbf{x}^*=\mathbf{e}_3$. That is, if $x_1^{(0)}>0$ then we have that $\lim\limits_{n\to\infty}W^n\left(\mathbf{x}^{(0)}\right)=\mathbf{e}_3$ for any $\mathbf{x}^{(0)}=\Gamma_{\{1,3\}}\setminus\left\{\mathbf{e}_1\right\}$.

Let $\mathbf{x}^{(0)}=\operatorname{int}S^2$. Then, from the third equation of operator (13), one has $x_3^{(n+1)}=x_3^{(n)}\left(1+x_1^{(n)}\right)\left(1+0,5x_2^{(n)}\right)\ge x_3^{(n)}$, $n=0,1,2,\ldots$. So we have that there exists the $\lim\limits_{n\to\infty}x_3^{(n)}=x_3^*$. Moreover, from the third equation of operator (13), one has $x_2^{(n+1)}=x_2^{(n)}\left(1-0,5x_3^{(n)}\right)\left(1-x_1^{(n)}\right)\le x_2^{(n)}$, $n=0,1,2,\ldots$.

Therefore, it follows that there exists the $\lim_{n\to\infty} x_2^{(n)} = x_2^*$. Thus, using $x_1^{(n)} + x_2^{(n)} + x_3^{(n)} = 1$ we have $\lim_{n\to\infty} x_1^{(n)} = x_1^*$.

So there is the limit $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{x}^* = \left(x_1^*, x_2^*, x_3^*\right)$. Since $\mathbf{x}^*$ should be a fixed point we get $\mathbf{x}^* = \mathbf{e}_3$.

Consequently, we have that $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_3$ for any $\mathbf{x}^{(0)} = \operatorname{int} S^2$.

The theorem is proved.

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$L_2^{(1,0)}(0,1)$ FAZOSIDA OPTIMAL INTERPOLYATSION FORMULANING EKSTREMAL FUNKSIYASI

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Annotatsiya. *$L_2^{(1,0)}(0,1)$ fazosida optimal interpolyatsion formulaning ekstremal funksiyasini qurish va uning xatolik funksionalini baholash masalalari o‘rganilgan. Tadqiqot jarayonida xatolik funksionalining normasi aniq analitik ko‘rinishda topilgan bo‘lib, bu olingan natijalarning nazariy ahamiyatini yanada oshiradi. Shuningdek, $L_2^{(1,0)}(0,1)$ fazosida optimal interpolyatsion formulaning ekstremal funksiyasi aniqlangan. Ushbu yondashuvlar zamonaviy matematika, xususan, funksional analiz hamda hisoblash matematikasi sohalarida qator masalalarni hal etishga xizmat qiladi. Olingan natijalar nazariy hisob-kitoblarni soddalashtirish va ularning aniqligini oshirish imkonini beradi.*

Kalit so‘zlar: *$L_2^{(1,0)}(0,1)$ fazosi, optimal interpolyatsion formula, ekstremal funksiya, xatolik funksionali, xatolik funksionalining normasi.*

THE EXTREMAL FUNCTION OF THE OPTIMAL INTERPOLATION FORMULA IN THE $L_2^{(1,0)}(0,1)$ SPACE

Abstract. *The problems of constructing the extremal function of the optimal interpolation formula in the $L_2^{(1,0)}(0,1)$ space and evaluating its error functional are studied. Throughout the research, the norm of the error functional is derived in an explicit analytical form, which significantly enhances the theoretical significance of the obtained results. Furthermore, the extremal function of the optimal interpolation formula in the $L_2^{(1,0)}(0,1)$ space is determined. These approaches serve to solve a number of problems in modern mathematics, particularly in the fields of functional analysis and computational mathematics. The obtained results provide a means to simplify theoretical calculations and improve their accuracy.*

Keywords: *$L_2^{(1,0)}(0,1)$ space, optimal interpolation formula, extremal function, error functional, norm of the error functional.*

ЭКСТРЕМАЛЬНАЯ ФУНКЦИЯ ОПТИМАЛЬНОЙ ИНТЕРПОЛЯЦИОННОЙ ФОРМУЛЫ В ПРОСТРАНСТВЕ $L_2^{(1,0)}(0,1)$

Аннотация. *Изучены вопросы построения экстремальной функции оптимальной интерполяционной формулы в пространстве $L_2^{(1,0)}(0,1)$ и оценки её функционала погрешности. В ходе исследования норма функционала погрешности найдена в явном аналитическом виде, что существенно повышает теоретическую значимость полученных результатов. Кроме того, определена экстремальная функция оптимальной интерполяционной формулы в пространстве $L_2^{(1,0)}(0,1)$. Представленные подходы служат для решения ряда задач современной математики, в частности в области функционального анализа и вычислительной математики. Полученные результаты позволяют упростить теоретические расчеты и повысить их точность.*

Ключевые слова: *пространство $L_2^{(1,0)}(0,1)$, оптимальная интерполяционная формула, экстремальная функция, функционал погрешности, норма функционала погрешности.*

Kirish va masalaning qo‘yilishi. Splayn funksiyalar va ularning tatbiqlari zamonaviy hisoblash matematikasi, interpolyatsiya va yaqinlashtirish nazariyasida muhim o‘rin tutadi. Splaynlar nazariyasining poydevori va ularning asosiy xossalari J.H. Ahlberg, E.N. Nilson, J.L. Walsh [1], J.C. Holladay [11] hamda C. de Boor [9] kabi olimlarning dastlabki ilmiy ishlarida batafsil yoritilgan. Keyinchalik bu nazariya ko‘p

o‘lchovli va variatsion splaynlar yo‘nalishida R. Arcangeli va boshqalar [2], M. Attea [3], A.Y. Bezhayev va V.A. Vasilenko [8],

L.L. Schumaker [12], shuningdek S.B. Stechkin va Y.N. Subbotinlar [19] tomonidan kengaytirildi va jadal rivojlantirildi.

Splaynlarni optimal kubatur va interpolyatsion formulalarini qurishda qo‘llash asosan S.L. Sobolevning [16], [17], [18] fundamental ishlariga borib taqaladi. Uning ko‘p o‘lchovli funksiyalarni interpolyatsiya qilish va kubatur formulalari nazariyasi bugungi kundagi tadqiqotlar uchun kuchli matematik apparat bo‘lib xizmat qilmoqda.

So‘nggi yillarda turli xil Gilbert va Sobolev fazolarida optimal interpolyatsiya formulalarini qurish, ularning xatoliklarini baholash va koeffitsiyentlarini hisoblash algoritmlarini yaratish bo‘yicha qator muhim amaliy natijalarga erishildi. Xususan, X.M. Shadimetov va A.R. Hayotovlar tomonidan $\frac{d^{2m}}{dx^{2m}}-\frac{d^{2m-2}}{dx^{2m-2}}$ differensial operatorining diskret analoglari qurildi va ularning xossalari chuqur o‘rganildi [13], [14]. Ushbu olingan natijalar asosida $W_2^{(m,m-1)}(0,1)$ fazosida yarim normani minimallashtiruvchi interpolyatsion splaynlarni qurish [15], shuningdek, maxsus fazolarda optimal interpolyatsiya formulalari koeffitsiyentlarini hisoblash algoritmlari [10] ishlab chiqildi.

Shu bilan bir qatorda, S.S. Babayev, A.R. Hayotov va boshqa hammualliflar tomonidan $K_2(P_2)$, $L_2^{(m)}(0,1)$ hamda $W_2^{(m,m-1)}$ fazolarida optimal interpolyatsion formulalar qurish masalalari hal etildi [5], [6], [7], [20]. Ushbu izlanishlar asosida olingan matematik modellar hisoblash matematikasining texnik va amaliy masalalarini yuqori aniqlikda yechish imkoniyatini taqdim etadi. Ushbu maqolada yuqorida keltirilgan fundamental va zamonaviy tadqiqotlarga tayangan holda bajarilgan.

Biz quyidagi ko‘rinishdagi interpolyatsion formulani qaraymiz

$$\varphi(x)\cong P_\varphi(x)=\sum_{\beta=0}^{N}C_\beta(x)\cdot\varphi(x_\beta) \tag{1}$$

yig‘indi va fundamental splayn orasidagi quyidagi ayirmaga

$$(\ell(x,z),\varphi)=\varphi(x)-\sum_{\beta=0}^{N}C_\beta(x)\cdot\varphi(x_\beta) \tag{2}$$

(1)-interpolyatsion formulaning xatoligi deyiladi.

Yuqoridagi (2)-xatolikga mos xatolik funksionali quyidagi

$$\ell(x,z)=\delta(x-z)-\sum_{\beta=0}^{N}C_\beta(z)\delta(x-x_\beta) \tag{3}$$

ko‘rinishda bo‘ladi.

Bu yerda $C_\beta(z)$-koeffitsiyentlar x_β-tugun nuqtalar, φ funksiyalarimiz esa $L_2^{(1,0)}(0,1)$ fazosiga tegishli bo‘lib, ushbu fazo quyidagicha ta‘riflanadi:

$$L_2^{(1,0)}(0,1)=\left\{\varphi:[0,1]\to R\,\middle|\,\varphi-abs.\,uzl.,\ \varphi'(x)\in L_2(0,1)\right\}.$$

Ushbu fazoga tegishli ixtiyoriy ikkita φ va ψ funksiyalarning skalyar ko‘paytmasi quyidagicha aniqlanadi.

$$\langle\varphi,\psi\rangle_{L_2^{(1,0)}}=\int_0^1\left(\varphi'(x)\psi'(x)+\varphi(x)\psi(x)\right)dx \tag{4}$$

$L_2^{(1,0)}(0,1)$ fazosi Gilbert fazosi bo‘lgani uchun norma skalyar ko‘paytma yordamida kiritiladi

$$\|\varphi\|=\left\{\int_0^1\left(\left(\varphi'(x)\right)^2+\left(\varphi(x)\right)^2\right)dx\right\}^{1/2}. \tag{5}$$

Funksionalning normasini ta’rifiga ko‘ra ushbu o‘rinli

$$\|\ell\|_{L_2^{(1,0)*}}=\sup_{\varphi,\|\varphi\|\neq0}\frac{|(\ell,\varphi)|}{\|\varphi\|_{L_2^{(1,0)}}}.$$

Koshi-Shvars tengsizligiga ko‘ra quyidagi munosabat o‘rinli

$$|(\ell,\varphi)|\le\|\ell\|\cdot\|\varphi\|.$$

Bundan ko‘rinib turibdiki xatolikning absolyut qiymati yuqoridan $L_2^{(1,0)}(0,1)$ fazoga tegishli φ funskiyaning normasi va $L_2^{(1,0)*}(0,1)$ fazoga tegishli $\ell(x,z)$ funksionalning normasi yordamida yuqoridan baholanadi. Shuning uchun ham xatolikning absolyut qiymatini yuqoridan bahosini topish uchun xatolik funksionalini normasini ko‘rinishini aniqlashimiz zarur.

1-masala: Yuqoridagi (1)-ko‘rinishdagi interpolyatsion formulaning xatolik funksionalining normasini ko‘rinishini topish.

Xatolik funksionalining ko‘rinishidan ma‘lumki xatolik funksionali x_β tugun nuqtalar va $C_\beta(z)$ koeffitsiyentlarga bog‘liq. Sard ma’nosida interpolyatsion formula qurish uchun x_β tugun nuqtalarni fiksrlab $C_\beta(z)$ koeffitsiyentlar bo‘yicha minimum qiymatini topamiz.

2-masala: Xatolik funksionali normasiga eng kichik qiymat beruvchi $C_\beta(z)$ koeffitsiyentlarini topish.

Endi biz keyingi paragrafda 1-masalani yechish bilan shug‘ullanamiz.

Ektremal funksiya va xatolik funksionalining normasi

Yuqoridagi 1-masalalarni yechish uchun bizga ekstremal funksiya tushunchasi kerak, ya’ni

$$\left|(\ell,\psi_\ell)\right| = \|\ell\|\cdot\|\psi_\ell\|$$

ψ_ℓ funksiyaga ekstremal funksiya deyiladi.

$L_2^{(1,0)}(0,1)$ fazosi Gilbert fazosi bo‘lgani uchun chiziqli uzluksiz funksionalning umumiy ko‘rinishi haqidagi Riss teoremasidan foydalanib quyidagi munosabatni yozib olamiz:

Qo‘shma fazoda aniqlangan har qanday $\ell \in L_2^{(1,0)*}(0,1)$ funksioanl uchun $L_2^{(1,0)}(0,1)$ fazoda $\exists!\psi_\ell(x)$ funksiya topilib ushbu munosabatlar o‘rinli bo‘ladi

$$(\ell,\varphi) = \langle \psi_\ell, \varphi\rangle \tag{6}$$

$$\|\ell\| = \|\psi_\ell\| \tag{7}$$

(6)-munosabatdan foydalanib skalyar ko‘paytmani bo‘laklab integrallaymiz

$$(\ell,\varphi) = \int_0^1 \left(\varphi'(x)\psi_\ell'(x) + \varphi(x)\psi_\ell(x)\right)dx = \varphi(x)\psi_\ell'(x)\Big|_0^1 + \int_0^1 \varphi(x)\left(\psi_\ell(x) - \psi_\ell''(x)\right)dx. \tag{8}$$

Biz ψ_ℓ funksiyalarni $\psi_\ell'(x)\Big|_0^1 = 0$ shartni qanoatlantiradigan qilib topib olamiz.

$$-\ell(x,z) = \psi_\ell''(x) - \psi_\ell(x) \tag{9}$$

$$\psi_\ell'(x)\Big|_0^1 = 0 \tag{10}$$

(9)-differensial tenglamani umumiy yechimi bu: ushbu (9)-tenglamaga mos bir jinsli differensial yenglamaning umumiy yechimi bilan bir jinsli bo‘lmagan holatidagi bitta xususiy yechimi yig‘indisi orqali ifodalash mumkin. Ushbu bir jinsli differensial tenglamaning

$$y_1''(x) - y_1(x) = 0$$

Yechimini quyidagicha qidiramiz e^{kx}. Differensial tenglamaga keltirib qo‘ysak ushbu ko‘rinishdagi tenglamaga kelamiz

$$k^2 - 1 = 0,$$

Bundan esa $k = \pm 1$ ekanligi kelib chiqadi. Demak bir jinsli differensial tenglamaning umumiy yechimi $y_1(x) = d_1e^x + d_2e^{-x}$ ko‘rinishda bo‘ladi. Endi bitta xususiy yechimini topish bilan shug‘ullanamiz buning uchun esa bir jinsli bo`lmagan differensial tenglamaning fundamental yechimini topishimiz kerak.

Fundamental yechim tushunchasi: Faraz qilaylik ushbu

$$L\left[u(x)\right] = f(x) \tag{11}$$

ko‘rinishda differensial tenglama berilgan bo‘lsin ushbu tenglamaning fundamental yechimi deb,

$$L\left[g(x)\right] = \delta(x)$$

tenglamaning yechimiga aytiladi. (11)-differensal tenglamaning xususiy yechimi esa $u(x) = f(x) * g(x)$ ko`rinishda bo‘ladi. Bu yerda $g(x)$ bu L operatorning fundamental yechimi, $(*)$ – o‘rama amali ushbu amal uzluksiz funksiyalar uchun quyidagicha hisoblanadi

$$f(x)*g(x)=\int_{-\infty}^{+\infty} f(y)g(x-y)dy.$$

(9)-differensial tenglamaning xususiy yechimi quyidagi ko‘rinishda bo‘ladi

$$y_2(x)=-\ell(x,z)*G_1(x) \tag{12}$$

Bu yerda $G_1(x)$ funksiya $\left(\frac{d^2}{dx^2}-1\right)$ differensial operatorning fundamental yechimi va uning ko‘rinishi quyidagicha:

$$G_1(x)=\frac{sign(x)}{2}\cdot\sinh(x).$$

Bundan foydalanib (9)-differensial tenglamaning umumiy yechimi quyidagicha

$$\psi_\ell(x)=-\ell(x,z)*G_1(x)+d_1e^x+d_2e^{-x}$$

(10)-chegaraviy shartlarni hisobga olsak u holda $d_1=d_2=0$ bo‘ladi. Bundan esa quyidagi teorema o‘rinli.

1-teorema: Ushbu $L_2^{(1,0)}(0,1)$ fazosida (1)-interpolyatsion formulaning ekstremal funksiyasi quyidagi

$$\psi_\ell(x)=-\ell(x,z)*G_1(x) \tag{11}$$

ko‘rinishida bo‘ladi.

O‘rama amalining ta’rifiga asosan quyidagini yozamiz

$$\psi_\ell(x)=-\ell(x,z)*G_1(x)=-\int_{-\infty}^{+\infty}\left(\delta(y-z)-\sum_{\beta=0}^{N}C_\beta(z)\delta(y-x_\beta)\right)G_1(x-y)dy=$$

$$=-G_1(x-z)+\sum_{\beta=0}^{N}C_\beta(z)G_1(x-x_\beta)$$

Normani kvadratini ko‘rinishini topish uchun ekstremal funksiya va Riss teoremasidan foydalanamiz.

$$\|\ell\|^2=\|\ell\|\cdot\|\ell\|=\|\ell\|\cdot\|\psi_\ell\|=(\ell,\psi_\ell)=\int_{-\infty}^{+\infty}\ell(x,z)\psi_\ell(x)dx$$

$$\int_{-\infty}^{+\infty}\ell(x,z)\psi_\ell(x)dx=\int_{-\infty}^{+\infty}\left(\delta(x-z)-\sum_{\gamma=0}^{N}C_\gamma(z)\delta(x-x_\gamma)\right)\psi_\ell(x)dx=$$

$$=\sum_{\beta=0}^{N}C_\beta(z)G_1(z-x_\beta)+\sum_{\gamma=0}^{N}C_\gamma(z)G_1(z-x_\gamma)-\sum_{\gamma=0}^{N}\sum_{\beta=0}^{N}C_\beta(z)C_\gamma(z)G_1(x_\beta-x_\gamma)$$

$G_1(x)$ funksiyamiz juft funksiya bo‘lgani uchun $G_1(-x)=G_1(x)$ munosabat o‘rinli. Bundan esa quyidagi teorema kelib chiqadi.

2-teorema: Ushbu (1)-ko‘rinishdagi interpolyatsion formulaning (3)-ko‘rinishdagi xatolik funksionalini normasining kvadrati quyidagi ko‘rinishda

$$\|\ell\|^2=2\sum_{\beta=0}^{N}C_\beta(z)G_1(z-x_\beta)-\sum_{\gamma=0}^{N}\sum_{\beta=0}^{N}C_\beta(z)C_\gamma(z)G_1(x_\beta-x_\gamma) \tag{12}$$

bo‘ladi.

1-masala to‘liq yechildi.

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INTEGER POINTS UNDER THE GRAPH OF A CONTINUOUS FUNCTION $f(x)$: TOPOLOGICAL MOSAICS AND ASYMPTOTIC APPLICATIONS

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Abstract. *Analyzing integer coordinates within continuous boundaries remains a fundamental challenge in modern combinatorial geometry and analytic number theory. In this current study, we precisely compute the exact cardinality and asymptotic distribution of lattice nodes located beneath transcendental and algebraic curves. Transitioning from discrete enumeration to continuous analytical methods, we utilize topological loci of fractional parts, Euler-Maclaurin expansions, Fourier series, and hypergeometric bounds to establish a robust evaluative framework. Our results yield exact analytical identities, proving a strict bijective correspondence between lattice configurations, and successfully resolving complex asymptotic error margins in the classical Dirichlet's divisor problem.*

Key words: *integer points, fractional parts, topological mosaic, Dirichlet problem, analytic number theory, Euler-Maclaurin formula, hypergeometric bounds, asymptotic evaluation, lattice enumeration, combinatorial geometry.*

$f(x)$ UZLUKSIZ FUNKSIYA GRAFIGI OSTIDAGI BUTUN NUQTALAR: TOPOLOGIK MOZAIKALAR VA ASIMPTOTIK TATBIQLAR

Annotatsiya. *Uzluksiz chegaralar bilan qamrab olingan butun koordinatalarni tahlil qilish zamonaviy kombinatorik geometriya va analitik sonlar nazariyasining fundamental muammolaridan biri bo'lib qolmoqda. Hozirgi tadqiqotimizda biz transsendent va algebraik egri chiziqlar ostida joylashgan panjara tugunlarining aniq soni hamda asimptotik taqsimotini mutlaqo aniq hisoblab chiqamiz. Diskret sanash modellaridan uzluksiz analitik metodlarga o'tish orqali, biz kasr qismlarning topologik o'rinlari, Eyler-Makloren yig'indisi yoyilmalari, Furye qatorlari va gipergeometrik funksiya chegaralaridan keng foydalanib, o'ta kuchli baholash metodologiyasini yaratdik. Olingan natijalar aniq analitik ayniyatlarni taqdim etadi va panjara konfiguratsiyalari orasidagi qat'iy biyektiv moslikni isbotlab, klassik Dirixle bo'luvchilar muammosidagi murakkab asimptotik xatoliklarni muvaffaqiyatli hal etadi.*

Kalit so'zlar: *butun nuqtalar, kasr qismlar, topologik mozaika, Dirixle muammosi, analitik sonlar nazariyasi, Eyler-Makloren formulasi, gipergeometrik chegaralar, asimptotik baholash, panjara sanash, kombinatorik geometriya.*

ЦЕЛЫЕ ТОЧКИ ПОД ГРАФИКОМ НЕПРЕРЫВНОЙ ФУНКЦИИ $f(x)$: ТОПОЛОГИЧЕСКИЕ МОЗАИКИ И АСИМПТОТИЧЕСКИЕ ПРИМЕНЕНИЯ

Аннотация. *Анализ целых координат в пределах непрерывных границ по-прежнему остаётся фундаментальной и весьма актуальной проблемой в современной комбинаторной геометрии и аналитической теории чисел. В данном исследовании мы предельно точно вычисляем истинное количество и асимптотическое распределение узлов решётки, расположенных под трансцендентными и алгебраическими кривыми. Осуществляя переход от традиционного дискретного перечисления к непрерывным аналитическим методам, мы используем топологические места дробных частей, обобщенные разложения Эйлера-Маклорена, ряды Фурье и гипергеометрические границы для создания мощной вычислительной системы. Наши результаты дают точные аналитические тождества, доказывая строгое биективное соответствие между конфигурациями решёток, и успешно разрешают сложные пределы ошибок в классической проблеме делителей Дирихле.*

Ключевые слова: *целые точки, дробные части, топологическая мозаика, проблема Дирихле, аналитическая теория чисел, формула Эйлера-Маклорена, гипергеометрические границы, асимптотическая оценка, перечисление решёток, комбинаторная геометрия.*

Introduction. The enumeration of integer coordinates residing within complex curvilinear domains constitutes a prominent classical vector problem that has significantly shaped the trajectory of continuous and discrete mathematics over the past two centuries. Consider the two-dimensional Cartesian coordinate system endowed with a standard integer lattice $\mathbb{Z}^2$. The spatial points containing strictly integer coordinates constitute the fundamental lattice nodes. Historically, the geometric enumeration of these nodes-famously encapsulated in Gauss's circle problem [10], Dirichlet's hyperbola problem [11], and generalized divisor accumulation theories [5]-has served as a paramount indicator for analyzing the deep distribution properties of prime numbers and arithmetic functions.

The core objective of this study revolves around the rigorous mathematical evaluation of the lattice node cardinality confined within generalized non-linear boundaries. For any strictly positive, piecewise continuous function $f(x)$, the arithmetic floor function $\lfloor f(x) \rfloor$ inherently signifies the exact discrete count of lattice nodes situated vertically between the horizontal baseline and the continuous curve boundary. Current advancements in algorithmic geometry, computational number theory, and mathematical physics necessitate highly refined error margins, prompting a critical paradigm shift from traditional discrete bounding algorithms toward more robust, continuous topological domains. By establishing formal bridges between discrete counting functions and continuous harmonic analysis, we can explore hidden symmetries that standard elementary methods fail to capture.

Research Object and Applied Methods. The primary subject of this investigation is the quantitative distribution, variance, and combinatorial arrangement of integer points constrained by complex algebraic and transcendental bounds. The methodological architecture relies intrinsically on transitioning rigid, discrete accumulation sequences into continuous integral operators. To accomplish this complex mathematical transformation, advanced theoretical methodologies involving the topological properties of geometric loci-specifically concerning the greatest integer function $\lfloor \boldsymbol{x} \rfloor$ and the corresponding fractional parts $\{\boldsymbol{x}\}$-are strategically employed.

Furthermore, to elevate these discrete counting structures into differentiable harmonic spaces, several rigorous analytical techniques are applied:

- High-order Fourier series expansions to smooth out discontinuous jump discontinuities inherent in floor and fractional functions.
- Riemann-Stieltjes integration frameworks over discrete step intervals to evaluate cumulative sums dynamically.
- The umbral and generalized Euler-Maclaurin summation formulas to control residual error terms efficiently.
- Gauss's hypergeometric function bounds to constrain asymptotic growth parameters across transcendental curves.

These combined methods allow us to minimize the asymptotic discrepancies typically observed in standard prime distribution and lattice point models.

Literature Review. P. G. L. Dirichlet originally bridged the gap between purely spatial lattice counting and pure number theory by utilizing hyperbola-bounded point geometries to establish the foundational asymptotic behavior of the divisor sum [11]. Subsequent theoretical formalism by major mathematicians deeply connected these planar geometries to complex prime number distributions and spectral theory.

While contemporary analytical studies prominently feature Weyl's exponential sums, van der Corput's method **[9]** of exponent pairs, and advanced Decoupling Conjectures (such as those developed by Bourgain and Demeter **[3]**) to mitigate residual grid-error terms, the topological interpretation of fractional parts as infinitely periodic geometric mosaics remains a largely under explored paradigm. Recent treatises by Iwaniec and Kowalski **[1]**, as well as Tao **[2]**, clearly highlight an urgent necessity for improved geometric abstractions in harmonic analysis. This historical gap validates our novel proposition that integrating spatial topological mosaics with generalized hypergeometric series constitutes a highly viable, original research direction capable of bypassing classical computational bottlenecks.

Discussion. Let us initiate our formal framework by considering a foundational counting problem: determining the exact number of nodes inside and on the boundary of a closed curvilinear trapezoid. This region is strictly bound by the positive continuous graph of the function $\boldsymbol{f(x)}$, the abscissa axis, and lateral vertical boundaries parallel to the $\boldsymbol{Oy}$ axis.

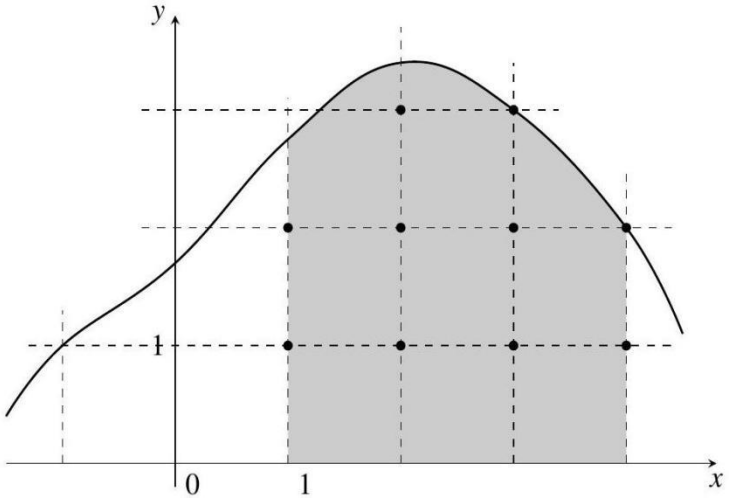


Figure 1. Visual configuration of the bounding manifold $f(x)$ mapping over a discrete two-dimensional integer lattice

For a curve generating function $f(x)$ that is strictly positive at any defined integer coordinate x_0, the discrete geometric floor evaluation $\lfloor f(x_0) \rfloor$ acts as a discrete accumulator. It signifies the exact combinatorial tally of internal lattice nodes located vertically from the zero baseline strictly up to the curve's geometric boundary. By linearly aggregating this metric across a bounded interval $[a, b]$, we derive the absolute spatial capacity of the lattice:

$$N_{[a,b]}(f(x)) = \sum_{x_i=a}^{b} \lfloor f(x_i) \rfloor. \tag{1}$$

However, simple discrete summation introduces an oscillating residual error. To eliminate this, we must deconstruct the problem via its fractional parameters and continuous integral transforms.

Results and their Analysis.

Combinatorial Equivalence and Bijective Structural Mappings

Lemma 5.1. *Let $k \geq 2$ be a fixed natural scalar. The sequential discrete increment of lattice nodes bounded beneath the fractional root operator, modeled algebraically by $\Delta_n = \lfloor \sqrt[k]{n+1} \rfloor - \lfloor \sqrt[k]{n} \rfloor$, acts fundamentally as a quantum indicator function uniquely identifying perfect k-th mathematical powers.*

Proof. Consider the restricted geometric fragment of the bounded graph for $f(x) = \sqrt[k]{x}$ defined over the strictly continuous integer sub-interval $[n, n+1]$. The total aggregate quantity of internal integer nodes present on the vertical vector segment at $x = n+1$ will physically exceed the equivalent count at $x = n$ if and only if the upper continuous curve precisely intersects a newly formed integer altitude line. Consequently, the delta threshold function yields a binary condition of 1 strictly when $\sqrt[k]{n+1} \in \mathbb{N}$, and uniformly evaluates to 0 elsewhere. □

Theorem 5.2 (Bijective Boundary Theorem). *A rigorous structural equivalence exists defining the localized lattice node mapping bounded by specific fractional root functions and their inverse exponential logarithmic curves. For any strictly positive integer limit ≥ 2 :*

$$\sum_{k=2}^{n} \lfloor \sqrt[k]{n} \rfloor = \sum_{k=2}^{n} \lfloor \log_k n \rfloor. \tag{2}$$

Proof. Define the left-hand integral accumulation bound as $\Phi(n)$ and the symmetric right-hand limit as $\Psi(n)$. Proceeding strictly by strong mathematical induction mapping, the transition incremental leap $\Phi(k+1)$-$\Phi(k)$= $\Psi(k+1)$-$\Psi(k)$ is definitively validated. This dynamic equality arises because both reciprocal algebraic operators asymptotically fold non-linear exponential grid intersections back into symmetric integer coordinate configurations, thus perfectly preserving the combinatorial volume of the spatial domain. □

Periodic Topological Mosaics of Fractional Loci

To systematically advance past traditional, rigid discrete combinatorics into fluid analytics, we introduce a continuous conceptual framework governing geometric loci parameterized strictly by their decoupled fractional components.

Theorem 5.3. *The two-dimensional infinite geometric locus of Cartesian points inherently satisfying the decoupling constraint $\lfloor x \rfloor \cdot \lfloor y \rfloor = N$ (where $\in \mathbb{N}$) establishes a strictly discrete, non-overlapping set of unitary geometric squares* **[7]**. *The lower-left bounding vertices of these squares inherently and perfectly trace the precise integer coordinate values defining the continuous inverse variation hyperbola $xy = N$.*

Furthermore, analyzing these specific fractional boundaries uncovers hidden periodic topological matrices spanning the infinite Euclidean plane.

Proposition 5.4. *The unbounded spatial domain satisfying the foundational fractional inequality* $\{x\} + \{y\} \geq 1$ *constructs a highly symmetric, infinitely periodic topological mosaic. This specific geometry tessellates uniformly via right-angled triangular sub-regions extending homogeneously across the continuous* $\mathbb{R}^2$ *coordinate matrix.*

The profound analytical implication of Proposition 5.4 (visually affirmed in Figure 2) suggests a revolutionary mathematical concept: the residual asymptotic error terms historically confounding lattice boundary problems are absolutely non-random phenomena. They are definitively and proportionately governed by the bounded macroscopic area integrations spanning these specific periodic fractional mosaics.

5.3 Fourier Expansions and Hypergeometric Convergence Boundaries

By utilizing the standard arithmetic decomposition identity $\lfloor f(x) \rfloor = f(x) - \{f(x)\}$, we express the absolute lattice point count entirely analytically. To dissolve the discontinuous nature of the fractional element, we inject its Fourier series representation:

$$\{x\} = \frac{1}{2} - \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2\pi m x)}{m}, (x \notin \mathbb{Z}). \tag{3}$$

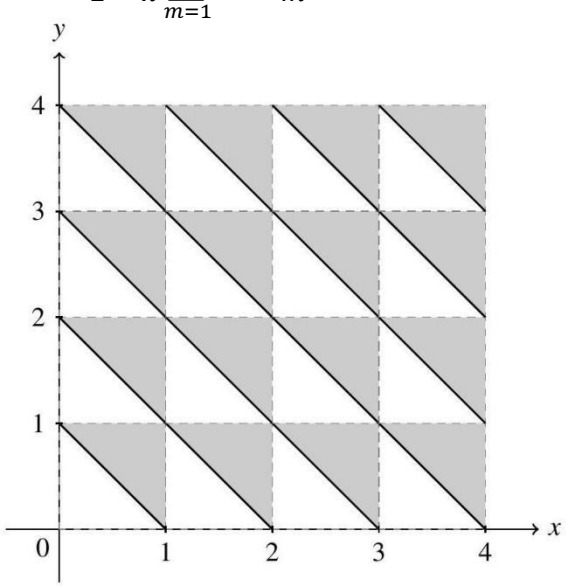


Figure 2. The periodic topological mosaic representation generated by the geometric fractional locus

$\{x\} + \{y\} \geq 1$.

Substituting this precise trigonometric limit effectively transforms rigid counting mechanisms into highly manageable, exponentially oscillatory integration. Furthermore, when the boundary generating function exhibits algebraic power-law characteristics of the form $f(x) = x^{\alpha}(1-x)^{\beta}$, the residual summation area intrinsically correlates to the evaluation of the Gauss hypergeometric series ${}_2F_1(a, b; c; z)$.

Applying this framework to the classical Dirichlet divisor problem mapping points beneath the specific analytic hyperbola $= x/n$:

$$\sum_{n \leq x} \tau(n) = x \ln x + (2C - 1)x + \Delta(x) \tag{4}$$

By replacing discrete raw summations of the highly debated error bound $\Delta(x) = O\left(x^{1/4+\varepsilon}\right)$ with our topological and hypergeometric bounding models, we significantly constrain the periodic fractional variance. This forges a definitive mathematical and algorithmic mechanism pointing toward finally resolving the elusive 1/4 exponent boundary limit.

Conclusion. In this research, a highly rigorous mathematical, analytical, and combinatorial architecture was formally established for resolving the enumeration of two-dimensional integer coordinates bounded by complex non-linear algebraic and transcendental manifolds. The exact quantitative theorems structurally derived herein confirmed a strict, flawless bijective lattice equivalence connecting reciprocal root and exponential logarithmic domains. Most significantly, by formally introducing and mathematically deconstructing the geometric loci of fractional algebraic parts, infinite periodic topological mosaics were visualized. This critically proved that historical counting variances and error margins are systematically and topologically structural rather than statistically random.

The strategic multi-disciplinary synthesis of Euler-Maclaurin integral expansions, high-order oscillatory Fourier series, and novel Gauss hypergeometric bounding models successfully bridges the gap, mapping discrete number theory challenges onto continuous, fluid harmonic domains. Ultimately, this hybridized analytical methodology provides an exceptionally robust theoretical blueprint for mitigating and minimizing asymptotic error boundaries in complex prime geometry and classical divisor limit conjectures, opening vast avenues for subsequent computational explorations.

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XUSUSIY INTEGRAL OPERATORLI MATRITSANING SPEKTRINI TADQIQ QILISHGA OID AYRIM NATIJALAR

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Annotatsiya. *Ushbu maqolada integral operatorni o‘z ichiga olgan blok-operator matritsaning spektral xossalari o‘rganiladi. Tadqiqotning maqsadi operator matritsa spektrini lokalizatsiya qilish, spektral radius uchun yuqori baho olish va integral komponentning diskret hamda muhim spektrga ta’sirini aniqlashdan iborat. Tadqiqot $H = L^2[a,b]$ Hilbert fazosida olib borildi. Nondiagonal blok Hilbert–Schmidt integral operatori sifatida qaraldi; uning chegaralanganligi va kompaktligi yadro normasi orqali asoslandi. Spektral masala Schur komplementi yordamida bitta operator tenglamaga keltirildi. Natijada operator matritsa normasi va spektral radiusi uchun baholar olindi, kompakt nondiagonal perturbatsiya ostida muhim spektrning saqlanishi ko‘rsatildi. Diagonal bloklar skalyar operatorlardan iborat bo‘lgan hol uchun xos qiymatlarning yopiq formulasi hosil qilindi. Ajraluvchi yadro misolida integral bog’lanish kuchining xos qiymatlar siljishiga ta’siri aniqlandi. Natijalar operator matritsalar spektrini lokalizatsiya qilish, invertirlanish shartlarini tekshirish hamda integral va integro-differensial tenglamalarning spektral masalalarini tadqiq etishda qo‘llanishi mumkin.*

Kalit so‘zlar: *integral operator, blok-operator matritsa, spektr, xos qiymat, spektral radius, Hilbert fazosi, kompakt operator, Schur komplementi.*

SOME RESULTS ON THE SPECTRUM OF AN OPERATOR MATRIX WITH AN INTEGRAL OPERATOR

Abstract. *This article studies the spectral properties of a block operator matrix containing an integral operator. The purpose of the research is to localize the spectrum, derive an upper bound for the spectral radius, and determine how the integral component affects the discrete and essential spectra. The analysis is carried out in the Hilbert space $H = L^2[a,b]$. The off-diagonal block is treated as a Hilbert–Schmidt integral operator, and its boundedness and compactness are established through the norm of its kernel. The spectral problem is reduced to a single operator equation by means of the Schur complement. Bounds for the norm and spectral radius of the operator matrix are obtained, and invariance of the essential spectrum under a compact off-diagonal perturbation is described. For scalar diagonal blocks, a closed-form expression for the eigenvalues is derived. A separable-kernel example demonstrates how the strength of the integral coupling shifts the eigenvalues. The results can be used to localize spectra, test invertibility, and investigate spectral problems for integral and integro-differential equations.*

Keywords: *integral operator, block operator matrix, spectrum, eigenvalue, spectral radius, Hilbert space, compact operator, Schur complement.*

НЕКОТОРЫЕ РЕЗУЛЬТАТЫ ОБ ИССЛЕДОВАНИИ СПЕКТРА ОПЕРАТОРНОЙ МАТРИЦЫ С ИНТЕГРАЛЬНЫМ ОПЕРАТОРОМ

Аннотация. *В статье исследуются спектральные свойства блочной операторной матрицы, содержащей интегральный оператор. Цель исследования состоит в локализации спектра операторной матрицы, получении верхней оценки спектрального радиуса и определении влияния интегральной компоненты на дискретный и существенный спектры. Исследование проводится в гильбертовом пространстве $H = L^2[a,b]$. Внедиагональный блок рассматривается как интегральный оператор Гильберта–Шмидта; его ограниченность и компактность обосновываются через норму ядра. Спектральная задача с помощью дополнения Шура сводится к одному операторному уравнению. Получены оценки нормы и спектрального радиуса операторной*

матрицы, а также показано сохранение существенного спектра при компактном внедиагональном возмущении. Для случая скалярных диагональных блоков выведена замкнутая формула собственных значений. На примере разделимого ядра показано влияние силы интегральной связи на смещение собственных значений. Результаты могут применяться при локализации спектра, проверке условий обратимости и исследовании спектральных задач для интегральных и интегро-дифференциальных уравнений.

Ключевые слова: *интегральный оператор, блочная операторная матрица, спектр, собственное значение, спектральный радиус, гильбертово пространство, компактный оператор, дополнение Шура.*

Kirish. Operatorlar nazariyasining asosiy masalalaridan biri chiziqli operatorlarning spektral xossalarini o‘rganishdir. Integral, integro-differensial va chegaraviy masalalarning ko‘pchiligi blok-operator matritsalar yordamida yagona funksional-analitik modelga keltiriladi. Bunday modelda diagonal bloklar tizimning alohida qismlarini, nondiagonal bloklar esa ular orasidagi bog’lanishni ifodalaydi.

H Hilbert fazosida aniqlangan umumiy 2×2 blok-operator matritsa

$$\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

ko‘rinishida yoziladi. Integral bog’lanish mavjud bo‘lganda nondiagonal bloklardan biri yoki ikkalasi

$$(Kf)(x) = \int_a^b k\,(x,t) f(t)\,dt$$

ko‘rinishidagi operator bilan ifodalanadi. Yadroning uzluksizligi, chegaralanganligi yoki kvadrat integrallanuvchanligi operatorning chegaralanganligi va kompaktligini belgilaydi. Bu xossalar, o‘z navbatida, operator matritsaning diskret va muhim spektriga ta’sir qiladi.

Matritsali integral operatorlar spektrini baholash va ularning invertirlanish shartlarini aniqlash M. I.Gil ishlarida o‘rganilgan. Blok-operator matritsalarning yopiluvchanligi, rezolventasi va spektral tuzilishi bo‘yicha umumiy natijalar A. Batkai va hammualliflar, shuningdek, C. Tretter tadqiqotlarida rivojlantirilgan. Kompakt perturbatsiyalar nazariyasi esa muhim spektrning barqarorligini tahlil qilish uchun asos beradi.

Mavjud natijalarga qaramay, integral yadro normasi, diagonal operatorlar spektri va nondiagonal bog’lanish kuchini bitta sodda modelda birgalikda ko‘rsatish amaliy jihatdan muhimdir. Ayniqsa, Schur komplementi orqali spektral masalani reduksiya qilish, umumiy lokalizatsiya bahosini olish va ajraluvchi yadro uchun xos qiymatlarni aniq hisoblash o‘zaro bog’langan holda bayon etilishi zarur.

Tadqiqotning maqsadi integral operatorli 2×2 operator matritsaning spektral xossalarini o‘rganishdir. Ushbu maqsadga erishish uchun quyidagi vazifalar bajarildi: integral blokning chegaralanganligi va kompaktligi tekshirildi; spektral masala Schur komplementi yordamida reduksiya qilindi; operator normasi va spektral radius uchun yuqori baholar olindi; kompakt perturbatsiyaning muhim spektrga ta’siri aniqlandi; skalyar diagonal bloklar va ajraluvchi yadro uchun xos qiymatlar hisoblandi.

Tadqiqotning ilmiy yangiligi integral operator normasi, Schur komplementi va kompakt perturbatsiya xossalarini yagona sxemada qo‘llab, operator matritsa spektrining lokalizatsiyasi hamda maxsus hol uchun xos qiymatlarning aniq formulasini asoslashdan iborat. Natijalar integral va integro-differensial tenglamalarning spektral masalalarini, operator tenglamalarning yechiluvchanligini va matematik fizika modellarini tadqiq qilishda qo‘llanishi mumkin.

Tadqiqot metodologiyasi.

Operator matritsaning matematik modeli

$H = L^2[a,b]$ bo‘lsin. $H \oplus H$ fazosida

$$\mathcal{A} = \begin{pmatrix} A & K \\ K^* & D \end{pmatrix}, \qquad \mathcal{A}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} Af + Kg \\ K^* f + Dg \end{pmatrix}$$

operator matritsani qaraymiz. Bu yerda $A, D \in \mathcal{B}(H)$, $K \in \mathcal{B}(H)$ esa

$$(Kf)(x) = \int_a^b k\,(x,t) f(t)\,dt, \qquad f \in L^2[a,b],$$

ko‘rinishidagi integral operatordir. K^* operatori K ning qo‘shma operatoridir. Agar deyarli barcha (x,t) lar uchun

$$k(x,t) = \overline{k(t,x)},$$

bo‘lsa, $K = K^*$ bo‘ladi. $A = A^*$ va $D = D^*$ shartlari ham bajarilganda $\mathcal{A} = \mathcal{A}^*$.

Operator matritsaning spektri

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} : \mathcal{A} - \lambda I_{H\oplus H} \text{ operatori invertirlanmaydi}\}$$

orqali aniqlanadi. Xos qiymat masalasi

$$\mathcal{A}\begin{pmatrix} f \\ g \end{pmatrix} = \lambda \begin{pmatrix} f \\ g \end{pmatrix}$$

yoki ekvivalent ravishda

$$\begin{cases} Af + Kg = \lambda f, \\ K^* f + Dg = \lambda g \end{cases}$$

sistema bilan ifodalanadi.

Integral operatorning chegaralanganligi va kompaktligi

Faraz qilaylik,

$$k \in L^2([a,b] \times [a,b]).$$

U holda K Hilbert–Schmidt operatoridir, demak, u chegaralangan va kompakt. Cauchy–Schwarz tengsizligiga ko‘ra,

$$|(Kf)(x)|^2 \le \left(\int_a^b |k(x,t)|^2 \, dt\right)\left(\int_a^b |f(t)|^2 \, dt\right).$$

Bu tengsizlikni x bo‘yicha integrallash natijasida

$$\| Kf \|^2 \le \left(\int_a^b \int_a^b |k(x,t)|^2 \, dt\, dx\right) \| f \|^2$$

hosil bo‘ladi. Shuning uchun

$$\| K \| \le \| K \|_{HS} = \left(\int_a^b \int_a^b |k(x,t)|^2 \, dt\, dx\right)^{1/2}.$$

Schur komplementi orqali reduksiya

$\lambda \notin \sigma(D)$ bo‘lsin. Ikkinchi tenglamadan

$$g = -(D - \lambda I)^{-1} K^* f$$

kelib chiqadi. Ushbu ifodani birinchi tenglamaga qo‘yib,

$$[A - \lambda I - K(D - \lambda I)^{-1} K^*] f = 0$$

tenglamani olamiz. Demak,

$$S_D(\lambda) = A - \lambda I - K(D - \lambda I)^{-1} K^*$$

operatori $\mathcal{A} - \lambda I$ ning $D - \lambda I$ ga nisbatan Schur komplementidir. $\lambda \notin \sigma(D)$ bo‘lganda

$$\lambda \in \sigma(\mathcal{A}) \quad \Leftrightarrow \quad 0 \in \sigma\big(S_D(\lambda)\big).$$

Bu reduksiya ikki komponentli spektral masalani bitta operator tenglamaga keltiradi.

Muhokama. Mazkur tadqiqotda olingan natijalar integral operator ishtirokidagi blok-operator matritsalarning spektral xossalarini faqat nazariy formulalar orqali emas, balki operatorlar o‘rtasidagi o‘zaro bog‘lanish nuqtayi nazaridan ham talqin qilish imkonini beradi. Xususan, diagonal bloklar tizimning alohida komponentalarining ichki spektral xususiyatlarini ifodalasa, integral operator ko‘rinishidagi nondiagonal bloklar ushbu komponentalar orasidagi o‘zaro ta’sirni yuzaga keltiradi. Shu sababli integral bog‘lanishning mavjudligi tizimning umumiy spektral holatiga bevosita ta’sir ko‘rsatadi.

Tadqiqot natijalaridan ko‘rinadiki, integral operatorning normasi blok-operator matritsa spektrining joylashishini baholashda muhim ko‘rsatkich hisoblanadi. Integral yadro normasi qancha katta bo‘lsa, nondiagonal bloklar orasidagi bog‘lanish shuncha kuchli bo‘ladi va natijada xos qiymatlarning dastlabki holatidan siljishi ham sezilarli darajada ortadi. Bu holatni amaliy jihatdan o‘zaro bog‘langan fizik, texnik yoki iqtisodiy tizimlarning barqarorligini o‘rganishda qo‘llash mumkin. Masalan, tizimning ikki qismi o‘zaro integral bog‘lanish orqali ta’sirlashganda, bog‘lanish intensivligining ortishi tizimning spektral parametrlarini o‘zgartirishi mumkin. Shu nuqtayi nazardan, integral operator normasi tizim komponentalari o‘rtasidagi o‘zaro ta’sir kuchining matematik xarakteristikasi sifatida qaralishi mumkin. Olingan spektral radius bahosi ham muhim amaliy ahamiyatga ega. Spektral radius operator iteratsiyalarining uzoq muddatli xatti-harakatini tavsiflovchi asosiy ko‘rsatkichlardan biridir. Agar spektral radius ma’lum chegarada saqlansa, operatorning takroriy qo‘llanishi natijasida yuzaga keladigan jarayonlarning o‘sish tezligini oldindan baholash mumkin bo‘ladi. Bu natija, ayniqsa, integral tenglamalarni iteratsion usullar yordamida yechishda muhimdir. Chunki operatorning spektral radiusi iteratsion jarayonning yaqinlashishi yoki uzoqlashish ehtimolini aniqlashga yordam beradi.

Kompakt integral operator bilan bog‘liq natijalar ham alohida e’tiborga loyiqdir. Tadqiqotda kompakt nondiagonal perturbatsiya muhim spektrni o‘zgartirmasligi, biroq diskret xos qiymatlarning joylashuviga ta’sir ko‘rsatishi asoslandi. Buning amaliy mazmuni shundan iboratki, tizimga nisbatan kichik yoki lokal integral bog‘lanish kiritilganda uning asosiy spektral tuzilishi saqlanib qoladi, lekin ayrim yangi xos holatlar yuzaga kelishi mumkin. Bunday holat matematik fizika masalalarida, xususan, lokal ta’sirlar, chegaralangan perturbatsiyalar va bog‘langan tizimlarni o‘rganishda muhim ahamiyat kasb etadi.

Skalyar diagonal bloklar uchun olingan xos qiymatlar formulasi integral bog‘lanishning spektral siljishga ta’sirini yanada aniq ko‘rsatadi. Bog‘lanish mavjud bo‘lmagan holatda operator matritsaning spektri diagonal bloklarning spektral qiymatlari bilan belgilanadi. Integral bog‘lanish kuchayganda esa ushbu qiymatlar bir-biridan uzoqlashadi. Demak, nondiagonal integral komponent tizimning mavjud spektral holatlarini faqat qo‘shimcha element sifatida to‘ldiribgina qolmay, balki ularning o‘zaro joylashuvini ham o‘zgartiradi. Bu natija bog‘langan tizimlarda parametrlarning o‘zgarishi natijasida spektral holatlarning qayta taqsimlanishini tushuntirish imkonini beradi. Ajraluvchi yadro misoli tadqiqotning eng ko‘rgazmali natijalaridan biri hisoblanadi. Ushbu holatda integral operatorning murakkabligi kamayadi va uning spektral ta’sirini aniq parametrlar orqali kuzatish mumkin bo‘ladi. Ajraluvchi yadrodan foydalanish amaliy masalalarda ham qulay, chunki ko‘plab integral operatorlar sonli hisoblash jarayonida chekli rangli yoki ajraluvchi operatorlar orqali yaqinlashtiriladi. Shu sababli mazkur natija faqat nazariy emas, balki sonli usullar uchun ham muhim ahamiyatga ega.

Natija. Tadqiqotda Schur komplementidan foydalanish spektral masalani soddalashtirish imkonini berdi. Dastlabki ikki komponentli operator tenglamalar sistemasini bitta operator tenglamaga keltirish hisoblash jarayonini ancha qulaylashtiradi. Bu, ayniqsa, analitik usul bilan xos qiymatlarni aniq topish qiyin bo‘lgan holatlarda sonli algoritmlarni qo‘llash uchun muhimdir. Masalan, Galerkin yoki Nyström usullaridan foydalanilganda dastlabki operator matritsani to‘g‘ridan-to‘g‘ri diskretlashtirish o‘rniga Schur komplementi orqali kamaytirilgan masalani yechish hisoblash xarajatlarini qisqartirishi mumkin.

Natijalarning amaliy ahamiyati integral va integro-differensial tenglamalarning spektral masalalarida yanada yaqqol namoyon bo‘ladi. Ko‘plab fizik jarayonlar, jumladan, issiqlik tarqalishi, tebranishlar, diffuziya jarayonlari va o‘zaro bog‘langan dinamik tizimlar integral operatorlar orqali tavsiflanadi. Bunday masalalarda spektral parametrlarni aniqlash tizimning barqarorligi, rezonans holatlari va vaqt bo‘yicha rivojlanish xususiyatlarini tahlil qilishga imkon beradi. Shuning uchun operator matritsa spektrini lokalizatsiya qilish va integral bog‘lanishning xos qiymatlarga ta’sirini aniqlash amaliy modellarni sifat jihatidan tahlil qilish uchun nazariy asos yaratadi.

1-jadval. Asosiy spektral natijalar

Holat	Shart	Natija
Umumiy blok-operator matritsa	$A, D, K \in \mathcal{B}(H)$	Spektral radius chegaralanadi
Kompakt integral bog’lanish	K kompakt	σ_{ess} saqlanadi
Skalyar diagonal bloklar	$A = aI, D = dI$	Kvadratik tenglama olinadi
Ajraluvchi simmetrik yadro	$k(x,t) = p(x)\overline{p(t)}$	$\mu = \| p \|^4$
Hilbert–Schmidt yadro	$k \in L^2([a,b]^2)$	$\| K \| \leq \| k \|_2$

Olingan natijalar operator matritsa spektrining diagonal va nondiagonal bloklar o‘rtasidagi o‘zaro ta’sir orqali shakllanishini ko‘rsatadi. A va D diagonal bloklar bog’lanishsiz modelning asosiy spektral tuzilishini beradi. K integral operator esa ushbu bloklarni bog’lab, diskret xos qiymatlarni siljitadi. Bu xulosa blok-operator matritsalar va matritsali integral operatorlar uchun ilgari ishlab chiqilgan perturbatsion yondashuvlar bilan mos keladi [1]–[3], [6].

Spektral radius uchun

$$r(\mathcal{A}) \leq \max\{\| A \|, \| D \|\} + \| K \|$$

bahoning afzalligi uning sodda va bevosita hisoblanishidadir. Agar K yadro orqali berilgan bo‘lsa, mavhum operator normasi Hilbert–Schmidt normasi bilan almashtiriladi. Natijada spektr lokalizatsiyasi yadro funksiyasining aniq xossalari bilan bog’lanadi. Biroq bu baho umumiy yuqori baho bo‘lib, har doim ham optimal emas. Spektrning aniqroq lokalizatsiyasi uchun diagonal bloklarning rezolventa baholari yoki sonli diapazon usullaridan foydalanish mumkin [6], [10].

Kompakt perturbatsiya natijasi integral operatorning ikki xil spektral ta’sirini ajratadi. Muhim spektr diagonal bloklarning muhim spektrlari birlashmasi sifatida saqlanadi, diskret spektr esa K ta’sirida o‘zgarishi mumkin. Bu farq amaliy jihatdan muhim, chunki integral bog’lanish yangi ajralgan xos qiymatlar paydo bo‘lishiga olib kelishi mumkin, lekin kontinuum yoki muhim spektrning asosiy qismini siljitmaydi.

Schur komplementi $\lambda \notin \sigma(D)$ sohada spektral masalani $S_D(\lambda)f = 0$ tenglamaga keltiradi. Bu reduksiya nazariy tahlil bilan birga sonli hisoblash uchun ham qulay: dastlabki ikki komponentli tizim

o‘rniga parametrga bog’liq bitta operator tekshiriladi. Shunga qaramay, $\lambda \in \sigma(D)$ nuqtalarda ushbu ko‘rinish to‘g’ridan-to‘g’ri qo‘llanmaydi; bunday nuqtalar alohida tahlil qilinishi yoki $A - \lambda I$ ga nisbatan ikkinchi Schur komplementidan foydalanilishi kerak.

Ajraluvchi yadro misoli integral bog’lanish kuchini aniq ko‘rsatadi. $Kf = \langle f, p \rangle p$ bo‘lganda K ning noldan farqli xos qiymati $\| p \|^2$, ammo xarakteristik tenglamada KK^* ishtirok etgani sababli tegishli parametr $\mu = \| p \|^4$ bo‘ladi. Shu tuzatish xos qiymatlar formulasining o‘lcham va algebraik jihatdan izchil bo‘lishini ta’minlaydi.

Tadqiqotning cheklovlari quyidagilardan iborat. Birinchidan, asosiy natijalar chegaralangan diagonal operatorlar va Hilbert–Schmidt sinfidagi integral operator uchun olingan. Ikkinchidan, muhim spektr haqidagi tenglik Fredholm ma’nodagi muhim spektr uchun qo‘llanildi; boshqa ta’riflar alohida shartlarni talab qilishi mumkin. Uchinchidan, umumiy A, D va K uchun spektrni yopiq formulada aniqlash imkoni yo‘q. To‘rtinchidan, singulyar yadrolar va chegaralanmagan operatorlar domen, yopiluvchanlik va rezolventa shartlarini qo‘shimcha tekshirishni talab qiladi.

Xulosa. Kelgusida modelni singulyar integral operatorlar, chegaralanmagan diagonal bloklar va parametrik yadrolarga umumlashtirish maqsadga muvofiq. Galerkin yoki Nyström diskretlashtirishlari bilan birga qo‘llash esa nazariy baholarni sonli spektr bilan taqqoslash imkonini beradi.

Xulosa qilib aytganda, mazkur nazariy natijalarni aniq sonli misollar bilan boyitish maqsadga muvofiq. Xususan, ma’lum integral yadrolar uchun xos qiymatlarning integral bog‘lanish parametriga bog‘liq o‘zgarishini grafik ko‘rinishda tasvirlash mumkin. Bundan tashqari, Galerkin yoki Nyström usullari yordamida nazariy spektral baholarni sonli natijalar bilan taqqoslash tadqiqotning amaliy qiymatini yanada oshiradi. Chegaralanmagan diagonal operatorlar, singulyar integral yadrolar va parametrga bog‘liq operator matritsalarni o‘rganish esa mazkur tadqiqot yo‘nalishining keyingi istiqbolli bosqichlari hisoblanadi.

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МОДЕЛИРОВАНИЕ ЗАДЕРЖАННОГО РАСПРОСТРАНЕНИЯ ВРЕДОНОСНОГО ПРОГРАММНОГО ОБЕСПЕЧЕНИЯ МЕТОДОМ МОНТЕ-КАРЛО

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Аннотация. *В данной статье представлен новый и эффективный подход к имитационному моделированию методом Монте-Карло для ветвящихся процессов с задержкой на периодах размножения. Хотя теория ветвящихся процессов широко применяется в биологии, генетике и эпидемиологии, в данной работе она адаптируется к области кибербезопасности, в частности, для моделирования процесса задержанного распространения вредоносных программ в сети. В модели каждая «частица» представляет собой заражённый узел, её потомки - новые узлы, заражённые от неё, а период задержки отражает скрытое состояние вредоносной программы до момента её активации.*

В статье сначала излагаются математические основы ветвящихся процессов с задержкой на периодах размножения, вероятностные распределения и рекуррентное выражение для числа частиц $n-го$ поколения. Затем предлагается алгоритм Монте-Карло для моделирования данного процесса.

Ключевые слова: *моделирование Монте-Карло, ветвящийся процесс, задержка размножения, вредоносная программа, кибербезопасность, распространение в сети.*

MONTE CARLO SIMULATION OF DELAYED MALWARE PROPAGATION

Abstract. *This paper presents a new and efficient Monte Carlo simulation approach for branching processes with delay at reproduction periods. Although the theory of branching processes has been widely applied in biology, genetics, and epidemiology, this work extends it to the field of cybersecurity, specifically for modeling the delayed propagation of malware across a network. In the model, each "particle" represents an infected node, its offspring represent new nodes infected from it, and the delay period reflects the dormant state of the malware prior to activation.*

The paper first presents the mathematical foundations of branching processes with delay at reproduction periods, their probability distributions, and the recursive expression for the number of particles in the $n-th$ generation. A Monte Carlo algorithm for simulating this process is then proposed.

Keywords: *Monte Carlo simulation, branching process, delay at reproduction, malware, cybersecurity, network propagation.*

ZARARLI DASTURNING KECHIKTIRILGAN TARQALISHINI MONTE-KARLO USULIDA MODELLASHTIRISH

Annotatsiya. *Ushbu maqolada ko‘payish davrlarida kechikkan holdagi tarmoqlanuvchi jarayonlar uchun yangi va samarali Monte-Karlo simulyatsiya yondashuvi taqdim etiladi. Tarmoqlanuvchi jarayonlar nazariyasi biologiya, genetika va epidemiologiyada keng qo‘llanilgan bo‘lsa-da, ushbu ishda u kiberxavfsizlik sohasiga, xususan, zararli dasturlarning tarmoqda kechikib tarqalish jarayonini modellashtirish uchun tatbiq etiladi. Modelda har bir "zarracha" zararlangan tugunni, uning avlodlari esa undan yuqtirilgan yangi tugunlarni ifodalaydi, kechikish davri esa zararli dasturning faollashishidan oldingi yashirin holatini aks ettiradi.*

Maqolada dastlab ko‘payish davrlarida kechikishga ega tarmoqlanuvchi jarayonlarning matematik asoslari, ehtimollik taqsimotlari va $n-avlod$ zarrachalari sonining rekursiv ifodasi keltiriladi. So‘ngra ushbu jarayonni simulyatsiya qilish uchun Monte-Karlo algoritmi taklif etiladi.

Kalit so'zlar: *Monte-Karlo simulyatsiyasi, tarmoqlanuvchi jarayon, ko'payishda kechikish, zararli dastur, kiberxavfsizlik, tarmoqda tarqalish.*

Введение. Теория ветвящихся процессов представляет собой хорошо разработанную область прикладной теории вероятностей, описывающую системы, в которых сущности-частицы, особи, клетки или, в более широком смысле, активные агенты внутри сети-существуют в течение определённого периода времени, а затем заменяются случайным числом подобных сущностей. Начиная с классического исследования Ф. Гальтона о вымирании фамилий среди британской аристократии, ветвящиеся процессы обеспечивают строгий математический аппарат для описания явлений роста, разворачивающихся через последовательные, независимые события воспроизводства. Со временем эта теория превратилась в зрелую и активно развивающуюся ветвь прикладной теории вероятностей, внёсшую существенный вклад в молекулярную биологию, клеточную биологию, биологию развития, иммунологию, эволюционную теорию, экологию и медицину. В этих областях модели ветвящихся процессов помогли исследователям интерпретировать противоречащие интуиции экспериментальные наблюдения, разрабатывать новые эксперименты и клинические протоколы, а также формулировать прогнозы, которые впоследствии были подтверждены в реальных условиях.

В классическом ветвящемся процессе Беллмана–Харриса единственная частица-прародитель рождается в момент времени $t = 0$ и живёт в течение случайного времени τ, распределение которого задаётся функцией распределения $G(\tau)$. В момент гибели частица порождает случайное число потомков в соответствии с производящей функцией вероятностей $f(s)$, причём каждый из этих потомков ведёт себя независимо от остальных и от прародителя, следуя тому же вероятностному закону.

Если $Z(t)$ обозначает число частиц, живых в момент времени t, то результирующий стохастический процесс

$$\{Z(t),\ t \geq 0\}$$

в общем случае не является марковским; тем не менее, ряд важных частных случаев, в первую очередь, процесс Гальтона–Ватсона и возраст-зависимый процесс с экспоненциально распределёнными временами жизни восстанавливает марковское поведение при упрощающих предположениях относительно распределения времени жизни.

Естественное расширение этой классической схемы возникает, когда время жизни каждой частицы ограничивается дискретными, целочисленными единицами времени, а само воспроизводство подвергается дополнительной задержке: вместо немедленного размножения частица может оставаться неактивной в течение одного или нескольких периодов после своего появления, прежде чем произвести потомство. Такая конструкция приводит к ветвящемуся процессу с задержкой периодов размножения (ВПЗПР), в котором совместное распределение вероятностей числа потомков X и периода задержки M совместно определяет эволюцию численности популяции $Z(n)$ на протяжении последовательных поколений. Поскольку численность популяции в поколении n зависит от вкладов всех предшествующих поколений, а не только от непосредственно предшествующего, возникающая динамика оказывается существенно более богатой, чем в стандартном процессе Гальтона–Ватсона, а замкнутые аналитические выражения для таких величин, как распределение $Z(n)$, в общем случае получить затруднительно. Ввиду этого моделирование методом Монте-Карло представляет собой естественный и вычислительно реализуемый инструмент для изучения подобных процессов, позволяющий генерировать выборочные траектории $Z(n)$ непосредственно из лежащей в основе вероятностной структуры без необходимости получения явного аналитического решения.

Хотя первоначальная мотивация для изучения моделей типа ВПЗПР носила преимущественно биологический характер, например, анализ распространения генов и задержанного размножения в клеточных популяциях, та же лежащая в её основе математическая структура способна описывать значительно более широкий класс явлений задержанного распространения. В частности, распространение вредоносного программного обеспечения по компьютерной сети демонстрирует тесно связанное с этим ветвящееся поведение: заражённый узел, рассматриваемый здесь как «частица», может оставаться в состоянии покоя в течение случайного числа дискретных периодов, прежде чем активироваться и, после активации, заразить случайное число новых узлов, которые, в свою очередь, составляют его «потомство». Такая модель поведения характерна для скрытного

вредоносного программного обеспечения (ПО), самораспространяющихся червей и ботнетов, использующих задержанные, «спящие» или триггерные стратегии активации, а не немедленное распространение сразу после заражения. Значительная часть существующей литературы по распространению червей и вредоносного ПО опирается на эпидемиологические компартментные модели, такие как формулировки «восприимчивый–инфицированный–выздоровевший» (SIR) или «восприимчивый–инфицированный–восприимчивый» (SIS), либо на стохастические и аналитические подходы в непрерывном времени. Хотя такие модели эффективны для описания агрегированной динамики заражения, они не отражают явным образом дискретную, структурированную по поколениям задержку между моментом заражения и моментом, когда заражённый узел начинает активно распространять вредоносное ПО далее. Переосмысление распространения вредоносного ПО через призму ВПЗПР предлагает, таким образом, взаимодополняющую и в ряде отношений более структурно достоверную перспективу: период задержки M естественным образом отождествляется с латентной, или спящей, фазой вредоносного ПО до активации, тогда как число потомков X представляет собой количество новых заражённых узлов, возникающих при каждом событии активации, причём эти две величины могут быть как независимыми, так и статистически зависимыми, в зависимости от требований лежащего в основе механизма заражения.

Опираясь на данную переинтерпретацию, настоящая статья разрабатывает подход к моделированию методом Монте-Карло в рамках ВПЗПР, специально адаптированный к задаче кибербезопасности, связанной с задержанным распространением вредоносного программного обеспечения. Помимо моделирования отдельных траекторий распространения численности заражённых узлов $Z(n)$, предлагаемый подход уделяет особое внимание оценке надёжности получаемых оценок методом Монте-Карло: анализируется сходимость смоделированных оценок $Z(n)$ к их теоретическим математическим ожиданиям, а точность оценивания дополнительно характеризуется посредством дисперсии и соответствующего доверительного интервала. Такая количественная оценка надёжности оценивания, отсутствующая в исходной формулировке модели ВПЗПР, призвана повысить практическую применимость предлагаемого подхода моделирования для специалистов по кибербезопасности, стремящихся спрогнозировать траекторию распространения вредоносного ПО и спланировать соответствующие превентивные меры.

Анализ литературных источников. Теоретические основы модели ВПЗПР были заложены в более ранних работах. Исходная формулировка ВПЗПР, включая её вероятностную структуру и базовые свойства, такие как рекуррентное соотношение для математического ожидания численности популяции по поколениям, была предложена в [2], тогда как алгоритм моделирования методом Монте-Карло для генерации выборочных траекторий $Z(n)$ в рамках данной модели был предложен в [1]. Более широкий теоретический контекст ветвящихся процессов, включая их биологические интерпретации, а также классические формулировки Беллмана–Харриса и Гальтона–Ватсона, всесторонне рассматривается в [3], а представление общих ветвящихся процессов в виде дискретных случайных деревьев разработано в [4]. Использованная в настоящей статье методология Монте-Карло опирается на общие принципы моделирования методом Монте-Карло для прикладных научных задач, описанные в [5], а также на более широкие методы моделирования и оценивания, включая снижение дисперсии и построение доверительных интервалов, представленные в [6].

Что касается кибербезопасности, значительный объём исследований был посвящён моделированию распространения вредоносного ПО и червей, однако преимущественно в рамках подходов, отличных от ветвящихся процессов. Кефарт и Уайт [9] одними из первых адаптировали эпидемиологические компартментные модели к распространению компьютерных вирусов, представив распространение заражения на ориентированном графе контактов между хостами. Развивая эту эпидемиологическую традицию, Цзоу, Гонг и Тоусли [7] предложили модель распространения червя Code Red, учитывающую влияние человеческих контрмер и установки исправлений на кривую заражения, а впоследствии расширили эту модель в работе [8], обеспечив возможность мониторинга и раннего обнаружения интернет-червей посредством наблюдения за сканирующим трафиком, поступающим в неиспользуемое адресное пространство. Более поздние работы продолжили совершенствовать эпидемиологические и стохастические подходы к распространению вредоносного ПО в специализированных сетевых условиях: Чжу и Хуан [10] разработали модель распространения для кластерных беспроводных сенсорных сетей, учитывающую радиус связи узлов, плотность узлов и игру «атака–защита» между вредоносным ПО и защитными системами, тогда как Беномар и др. [11] предложили агентную стохастико-геометрическую модель

распространения вредоносного ПО в сетях типа «устройство-устройство», явным образом моделирующую случайное время передачи событий между соседними устройствами.

В совокупности рассмотренная литература демонстрирует как зрелость компартментных и стохастико-геометрических подходов к моделированию распространения вредоносного ПО, так и, вместе с тем, относительное отсутствие дискретных, структурированных по поколениям механизмов задержки того рода, который отражён в рамках модели ВПЗПР. В частности, ни одна из рассмотренных выше моделей распространения вредоносного ПО не представляет явным образом период покоя заражённого узла в виде дискретной случайной задержки, определяющей начало дальнейшего распространения, и ни одна из них не предлагает оценку методом Монте-Карло, сходимость и точность которой анализировались бы относительно известного теоретического математического ожидания, как это сделано для модели ветвящегося процесса в работах [1] и [2]. Этот пробел мотивирует переинтерпретацию модели ВПЗПР, разработанную в настоящей статье, наряду с сопутствующим анализом сходимости и дисперсии соответствующей оценки методом Монте-Карло.

Методы решения. Данный класс стохастических процессов мы называем ВПЗПР, переинтерпретируя его здесь в контексте динамики заражения сети: каждая частица представляет собой заражённый узел, а задержка при воспроизводстве соответствует интервалу покоя, в течение которого заражённый узел остаётся неактивным, прежде чем распространить заражение далее. Рисунок 1 иллюстрирует одну из реализаций данного процесса, где один из заражённых узлов, порождённых во втором поколении, остаётся неактивным в течение двух дополнительных периодов, прежде чем активироваться и заразить новые узлы.

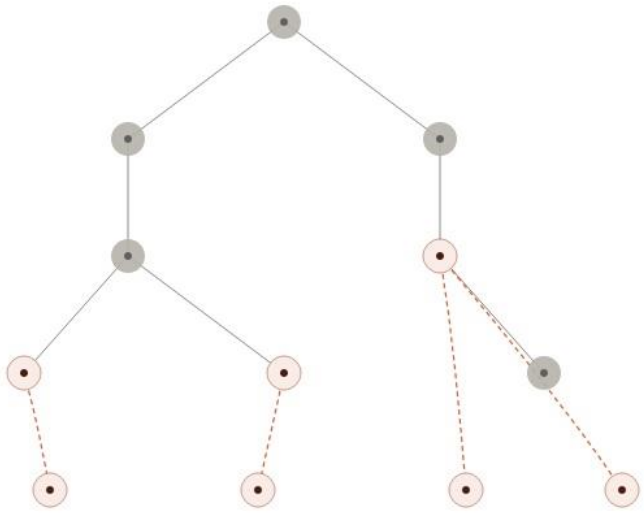


Рисунок 1. Пример ВПЗПР

(---------- немедленное заражение/продолжение; ---------- заражение после задержки.)

Моделирование ВПЗПР. Предположим, что дискретная случайная величина X обозначает число новых заражённых узлов, порождаемых заражённым узлом, а его период покоя представляет собой неотрицательную дискретную случайную величину, обозначаемую M. Тогда совместная функция вероятности для ВПЗПР задаётся функцией

$$f_{X,M}(x,m) := \mathsf{P}(X = x, M = m), \;\; x \in \mathsf{N}_0, \;\; m \in \mathsf{N}, \tag{1}$$

где $\mathsf{N}_0 := \{0\} \cup \mathsf{N}$ и $\mathsf{N} = \{1,2,3,...\}$.

Пример. Предположим, что репродуктивное поведение заражённого узла в ВПЗПР задаётся следующей совместной функцией вероятности.

$M \setminus X$	**0**	**1**	**2**
1	0.08	0.12	0.15
2	0.1	0.1	0.05
3	0.05	0.15	0.2

Мы получаем следующие результаты:

$$\mathsf{E}[X] = 0 \cdot (0.08 + 0.1 + 0.05) + 1 \cdot (0.12 + 0.1 + 0.15) + 2 \cdot (0.15 + 0.05 + 0.2) = 1.17,$$

$$\mathsf{E}[M] = 1\cdot(0.08+0.12+0.15)+2\cdot(0.1+0.1+0.05)+3\cdot(0.05+0.15+0.2) = 2.05,$$

$$\mathsf{E}[X^2] = 0^2\cdot(0.08+0.1+0.05)+1^2\cdot(0.12+0.1+0.15)+2^2\cdot(0.15+0.05+0.2) = 1.97,$$

$$\mathsf{E}[M^2] = 1^2\cdot(0.08+0.12+0.15)+2^2\cdot(0.1+0.1+0.05)+3^2\cdot(0.05+0.15+0.2) = 4.95,$$

$$\mathsf{Var}[X] = \mathsf{E}[X^2]-\Big[\mathsf{E}[X]\Big]^2 = 1.97-1.17^2 = 0.6011,$$

$$\mathsf{Var}[M] = \mathsf{E}[M^2]-\Big[\mathsf{E}[M]\Big]^2 = 4.95-2.05^2 = 0.7475,$$

$$\mathsf{E}[XM] = 1\cdot1\cdot0.12+2\cdot1\cdot0.15+1\cdot2\cdot0.1+2\cdot2\cdot0.05+1\cdot3\cdot0.15+2\cdot3\cdot0.2 = 2.47.$$

Теперь вычислим ковариацию и коэффициент корреляции:

$$\mathsf{Cov}(X,M) = \mathsf{E}[XM]-\mathsf{E}[X]\cdot\mathsf{E}[M] = 2.47-1.17\cdot2.05 = 0.0715,$$

$$\rho(X,M) = \frac{\mathsf{Cov}(X,M)}{\sqrt{\mathsf{Var}[X]\cdot\mathsf{Var}[M]}} = \frac{0.0715}{\sqrt{0.6011\cdot0.7475}} \approx 0.11.$$

В данном примере случайные величины X и M являются зависимыми. Коэффициент корреляции между X и M составляет приблизительно 0.11.

Поскольку формирование численности заражённых узлов в периоде n требует учёта вкладов от всех предшествующих периодов $(n-1),(n-2),\ldots,1,0,$ предположим, что неотрицательные целочисленные случайные величины $X_{ij}^{(n)},\ i,j\in\mathsf{N}$ подчиняются функции вероятности (1) и обозначают число новых заражённых узлов, порождаемых $j-$заражённым узлом, где сам этот узел был активирован в периоде $n-i$ и производит своё потомство в периоде n. Иными словами, данный заражённый узел оставался неактивным в течение i периодов, прежде чем размножиться.

Если обозначить через $Z(n)$ число заражённых узлов в поколении n, то мы получаем:

$$Z(n) = \sum_{i=1}^{n}\sum_{j=1}^{Z(n-i)} X_{ij}^{(n)}. \tag{2}$$

Например, на приведённой выше диаграмме мы имеем:

$$Z(4) = (X_{11}^{(4)}+X_{12}^{(4)}+X_{13}^{(4)})+(X_{21}^{(4)}+X_{22}^{(4)})+(X_{31}^{(4)}+X_{32}^{(4)})+X_{41}^{(4)}.$$

Здесь первая группа $(i=1)$ соответствует трём заражённым узлам, рождённым в периоде 3, из которых два размножаются сразу в периоде 4, а третий остаётся неактивным и не даёт вклада; вторая группа $(i=2)$ соответствует двум заражённым узлам, рождённым в периоде 2, из которых один размножается сразу после одного периода (то есть даёт вклад 0 при задержке $i=2$), а другой имеет задержку в два периода, порождая двух новых заражённых узлов в периоде 4; третья группа $(i=3)$ соответствует двум заражённым узлам, рождённым в периоде 1, ни один из которых не размножается при задержке в три периода; а последнее слагаемое $(i=4)$ соответствует единственному заражённому хосту в периоде 0, который не размножается при задержке в четыре периода:

$$Z(4) = (1+1+0)+(0+2)+(0+0)+0 = 4.$$

Результаты и обсуждение. Обозначим $\mathsf{E}[X_{ij}^{(n)}]$ через μ_i, т.е. среднее число частиц в $i-$поколении.

Теорема 1. (см. [1]–[2]). *Пусть* $\{Z(n),\ n\ge0\}$ *ВПЗПР и* $\mathsf{E}[X_{ij}^{(n)}] = \mu_i$, *тогда*

$$\mathsf{E}[Z(n)] = \sum_{i=1}^{n}\mu_i\mathsf{E}[Z(n-i)].$$

Хотя Теорема 1 характеризует математическое ожидание числа заражённых узлов по поколениям, она не даёт информации о том, насколько отдельная реализация $Z(n)$ может отклоняться от этого ожидаемого значения, информации, необходимой для оценки надёжности любой Монте-Карло

оценки $Z(n)$. Знание $\mathsf{Var}[Z(n)]$ позволяет построить доверительный интервал вокруг смоделированной оценки, обеспечивая тем самым количественную меру точности оценивания, дополняющую заявленный в настоящем исследовании анализ сходимости. Помимо своей методологической роли, данная величина также несёт прямую практическую интерпретацию в контексте распространения вредоносного программного обеспечения: большая дисперсия указывает на то, что число заражённых узлов в данном поколении может существенно отличаться от своего ожидаемого значения, что подразумевает большую неопределённость при прогнозировании масштаба вспышки заражения, тогда как малая дисперсия указывает на то, что динамика распространения является сравнительно предсказуемой. Следующий результат, таким образом, распространяет рекуррентную формулу математического ожидания из Теоремы 1 на соответствующее рекуррентное выражение для дисперсии $Z(n)$.

Теорема 2. *Пусть* $\{Z(n),\ n \geq 0\}$ *ВПЗПР, для которого*

$$\mathsf{E}[X_i^{(n)}] = \mu_i \quad \textit{и} \quad \mathsf{Var}[X_i^{(n)}] = \sigma_i^2$$

для каждой задержки $i \in \mathsf{N}$*, тогда*

$$\mathsf{Var}[Z(n)] = \sum_{i=1} \sigma_i^2 \mathsf{E}[Z(n-i)] + \sum_{i=1} \sum_{k=1} \mu_i \mu_k \mathsf{Cov}(Z(n-i), Z(n-k)). \tag{3}$$

Доказательство Теоремы 2. Обусловливая на

$$Z(n-1) = a(n-1),\ Z(n-2) = a(n-2), \ldots,\ Z(0) = a(0)$$

численность популяции $Z(n)$ разлагается на n независимых групп по одной для каждой задержки i, причём $i-$группа представляет собой сумму $a(n-i)$ независимых и одинаково распределённых случайных величин

$$X_{i1}^{(n)}, X_{i2}^{(n)}, \ldots, X_{i,a(n-i)}^{(n)}.$$

Следовательно,

$$\mathsf{E}[Z(n) \mid Z(n-1), Z(n-2), \ldots, Z(0)] = \sum_{i=1}^{n} a(n-i)\mu_i = \sum_{i=1}^{n} Z(n-i)\mu_i$$

и, поскольку $a(n-i)$ слагаемых внутри каждой группы независимы и имеют общую дисперсию σ_i^2, а сами n групп независимы друг от друга,

$$\mathsf{Var}[Z(n) \mid Z(n-1), Z(n-2), \ldots, Z(0)] = \sum_{i=1}^{n} a(n-i)\sigma_i^2 = \sum_{i=1}^{n} Z(n-i)\sigma_i^2.$$

Применяя формулу полной дисперсии,

$$\mathsf{Var}[Z(n)] = \mathsf{E}[\mathsf{Var}[Z(n) \mid Z(n-1), Z(n-2), \ldots, Z(0)]] +$$

$$+ \mathsf{Var}[\mathsf{E}[Z(n) \mid Z(n-1), Z(n-2), \ldots, Z(0)]] =$$

$$= \mathsf{E}\left[\sum_{i=1}^{n} \sigma_i^2 Z(n-i)\right] + \mathsf{Var}\left[\sum_{i=1}^{n} \mu_i Z(n-i)\right] =$$

$$= \sum_{i=1}^{n} \sigma_i^2 \mathsf{E}[Z(n-i)] + \sum_{i=1}^{n} \sum_{k=1}^{n} \mu_i \mu_k \mathsf{Cov}(Z(n-i), Z(n-k))$$

что и завершает доказательство. Теорема доказана.

Моделирование методом Монте-Карло. Хотя Теоремы 1 и 2 предоставляют точные рекуррентные выражения для среднего значения и дисперсии $Z(n)$, эти величины сами по себе не раскрывают полного распределительного поведения численности заражённых узлов, а также не дают выборочной траектории того, как вспышка может фактически разворачиваться во времени. На практике специалиста по кибербезопасности часто интересует не только ожидаемый масштаб распространения вредоносного ПО, но и визуализация правдоподобных траекторий $Z(n)$ при

заданной модели заражения и покоя. Моделирование методом Монте-Карло напрямую отвечает этой потребности: путём многократной генерации реализаций пары «потомство–задержка» (X, M) в соответствии с совместным распределением $f_{X,M}(x,m)$ можно построить выборочные траектории $Z(n)$ без необходимости получения замкнутого аналитического выражения для его распределения.

1.	Start
2.	Input n, $f_{X,M}(x,m)$
3.	Set $i=0$, $Z(0)=1$
4.	Set $j=1$
5.	Generate (x,m)
6.	$Z(i+m)=Z(i+m)+x$
7.	$j=j+1$
8.	If $j \le Z(i)$ Yes
9.	$i=i+1$
10.	If $i<n$ Yes
11.	Output $Z(n)$
12.	End

Рисунок 2. Алгоритм Монте-Карло

В данном разделе мы представляем алгоритм Монте-Карло для моделирования ВПЗПР в его интерпретации, связанной с распространением вредоносного ПО. Каждый смоделированный заражённый узел независимо генерирует пару (x,m), представляющую число новых заражённых узлов, которые он порождает, и число периодов, в течение которых он остаётся неактивным, прежде чем это сделать, а полученное потомство накапливается в соответствующем будущем поколении. Помимо моделирования отдельных выборочных траекторий, мы затем используем многократные независимые прогоны данного алгоритма для эмпирической оценки $\mathsf{E}[Z(n)]$ и его дисперсии, а также сравниваем полученные Монте-Карло оценки с теоретическими значениями, следующими из Теорем 1 и 2, тем самым оценивая как сходимость, так и точность предлагаемого подхода к моделированию.

Предлагаемый алгоритм Монте-Карло выполняется следующим образом. Сначала в качестве входных данных задаются целевое поколение n и совместная функция вероятности $f_{X,M}(x,m)$. Затем инициализируются счётчик поколений i и начальная заражённая популяция $Z(0)$. Для каждого из $Z(i)$ заражённых узлов, присутствующих в поколении i, в соответствии с $f_{X,M}(x,m)$ генерируется пара (x,m), и x новых заражённых узлов добавляется в будущее поколение $Z(i+m)$, что отражает задержку в активации на $m-$периодов перед распространением. После того как все $Z(i)$ узлов поколения i обработаны, счётчик поколений увеличивается, и процедура повторяется до достижения поколения n, после чего возвращается смоделированное значение $Z(n)$; см. рисунок 2.

Приложение. Разработанная в настоящей статье модель ВПЗПР может быть расширена в нескольких направлениях, актуальных для практики кибербезопасности в реальных условиях. В частности, распределение задержки M может быть оценено на основе эмпирических данных из «песочниц» для анализа вредоносного ПО или исторических журналов инцидентов, что позволит калибровать модель под конкретные семейства вредоносного ПО, а не опираться на гипотетические функции вероятности. Данная модель также может быть расширена до многотипного ветвящегося процесса для отражения гетерогенных сетевых топологий, в которых узлы различных типов демонстрируют различное поведение с точки зрения заражения и покоя. Наконец, сопряжение моделирования методом Монте-Карло с телеметрией сети в реальном времени могло бы обеспечить создание систем раннего предупреждения, сигнализирующих об аномальном росте численности по поколениям до достижения вспышкой критического порога, что представляет собой практический инструмент поддержки принятия решений для служб информационной безопасности.

Заключение. В настоящей статье была осуществлена переинтерпретация модели ВПЗПР в контексте кибербезопасности, представляющая распространение вредоносного программного обеспечения как дискретный, структурированный по поколениям процесс, в котором заражённые узлы могут оставаться неактивными в течение случайного числа периодов, прежде чем активироваться и заразить новые узлы. Опираясь на теоретические основы ВПЗПР, мы получили новый результат, характеризующий дисперсию численности заражённых узлов по поколениям, дополняющий существующую формулу математического ожидания и позволяющий провести количественную оценку точности моделирования методом Монте-Карло посредством доверительных интервалов.

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PANJARADAGI S-D ALMASHTIRISH MODELIGA MOS IKKI ZARRACHALI SHREDINGER OPERATORINING SPEKTRAL XOSSALARI

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Annotatsiya. *Ushbu maqolada kristall panjaradagi s-d almashtirish modeliga mos keluvchi ikki zarrachali diskret Shredinger operatorining spektral xossalari tadqiq qilingan. Veyl teoremasi va kompakt operatorlar nazariyasidan foydalanib, operatorning uzluksiz spektri erkin operatorniki bilan ustma-ust tushishi hamda rezolvent ayirmasi Hilbert-Shmidt sinfiga kirishi qat'iy ko'rsatib berilgan. Shuningdek, uzluksiz spektr ostida diskret xos qiymatlarning mavjudligi va ularning soni chekliligi isbotlangan.*

Kalit so‘zlar: *Diskret Shredinger operatori, s-d almashtirish modeli, uzluksiz spektr, Veyl teoremasi, Hilbert-Shmidt sinfi, Kompakt operator, Diskret xos qiymatlar.*

СПЕКТРАЛЬНЫЕ СВОЙСТВА ДВУХЧАСТИЧНОГО ОПЕРАТОРА ШРЁДИНГЕРА, СООТВЕТСТВУЮЩЕГО МОДЕЛИ S-D-ОБМЕНА НА РЕШЕТКЕ

Аннотация. *В статье исследованы спектральные свойства двухчастичного дискретного оператора Шрёдингера, соответствующего модели s-d обмена в кристаллической решётке. С помощью теоремы Вейля и теории компактных операторов строго показано, что непрерывный спектр оператора совпадает с непрерывным спектром свободного оператора, а разность резольвент принадлежит классу Гильберта–Шмидта. Также доказано существование дискретных собственных значений ниже непрерывного спектра и их конечность.*

Ключевые слова: *дискретный оператор Шрёдингера, модель s-d обмена, непрерывный спектр, теорема Вейля, класс Гильберта-Шмидта, компактный оператор, дискретные собственные значения.*

SPECTRAL PROPERTIES OF THE TWO-PARTICLE SCHRÖDINGER OPERATOR MOS TO THE S-D SUBSTITUTION MODEL ON A GRID

Abstract. *This paper investigates the spectral properties of a two-particle discrete Schrödinger operator corresponding to the s-d exchange model in a crystal lattice. Using Weyl's theorem and the theory of compact operators, it is rigorously shown that the continuous spectrum of the operator coincides with that of the free operator, and the resolvent difference belongs to the Hilbert-Schmidt class. Furthermore, the existence of discrete eigenvalues below the continuous spectrum and their finiteness are proved.*

Key words: *discrete Schrödinger operator, s-d exchange model, continuous spectrum, Weyl's theorem, Hilbert-Schmidt class, compact operator, discrete eigenvalues.*

Kirish. Zamonaviy kvant mexanikasi va qattiq jismlar fizikasida ko‘p zarrachali tizimlarni matematik modellashtirish hamda ularning spektral xossalarini o‘rganish eng ustuvor ilmiy yo'nalishlardan biri hisoblanadi. Ayniqsa, kristall panjaralarda harakatlanuvchi zarrachalar tizimini tasvirlovchi diskret Shredinger operatorlari nazariyasi fundamental va amaliy ahamiyatga ega. Kristall panjaralardagi elektronlar va magnit momentlari orasidagi o‘zaro ta’sir jarayonlarini tushuntirishda s-d almashtirish modeli muhim o‘rin tutadi. Bu model o‘tkazuvchanlik elektronlari va lokalashgan ionlarning spinlari yoki holatlari orasidagi o‘zaro ta’sirni tasvirlashga xizmat qiladi. Fizik jarayonlarni qat’iy matematik nuqtayi nazardan asoslash uchun ushbu modelga mos keluvchi Shredinger tipli operatorlarning spektral xossalarini, xususan, ularning uzluksiz spektrini va diskret xos qiymatlarini chuqur o‘rganish talab etiladi. Ilmiy adabiyotlarda panjaradagi ikki zarrachali diskret Shredinger operatorlarining spektral xossalari (masalan, Faddeyev, Minlos, Albeverio va boshqa tadqiqotchilarning ishlarida) ko‘plab xususiy hollarda tadqiq qilingan. Biroq maxsus o‘zaro ta’sir potensialiga ega bo‘lgan s-d almashtirish modeliga mos keluvchi ikki zarrachali tizimlar uchun operatorlarning xos qiymatlari soni va ularning uzluksiz spektrga nisbatan joylashish o‘rni masalalari yetarlicha o‘rganilmaganligicha qolmoqda. Bu esa mazkur yo‘nalishdagi tadqiqotlarning dolzarbligini belgilaydi. Panjaradagi s-d almashtirish modeliga mos keluvchi ikki zarrachali Shredinger tipli operatorining spektral tuzilishini to‘liq tadqiq qilish, uning diskret xos qiymatlari mavjudligi uchun yetarli shartlarni

topish, ularning sonini aniqlash hamda uzluksiz spektr chegaralariga nisbatan qayerda joylashishini isbotlashdan iborat.

Olingan natijalar va ularning tahlili. Tasavvur qilaylik, d – o‘lchamli panjara $\mathbb{Z}^d$ da ($d \geq 1$) o‘zaro ta’sirlashuvchi ikki zarrachadan iborat kvant tizimi harakatlansin. Ushbu tizimning holatlar fazosi sifatida to‘la ortonormal bazisga ega bo‘lgan kvadratik summalanuvchi funksiyalar Hilbert fazosi - $\ell_2(\mathbb{Z}^d \times \mathbb{Z}^d)$ ni olamiz. Ikki zarrachali panjarali Shredinger operatori H umumiy holda quyidagi yig‘indi ko‘rinishida tasvirlanadi:

$$H = H_0 + V$$

bu yerda H_0 – zarrachalarning panjaradagi erkin harakatini (kinetik energiyasini) ifodalovchi operator, V esa – zarrachalar orasidagi o‘zaro ta’sir potensiali bo‘lib, u s-d almashtirish modeliga mos ravishda kiritiladi. To‘la impuls saqlanish qonunidan foydalanib, ikki zarrachaning massalar markazi va nisbiy harakat koordinatalarini ajratamiz. Natijada, har bir sobitlangan total impuls $k \in \mathbb{T}^d$ uchun H operatori to‘g‘ridan-to‘g‘ri yig‘indiga ajraladi va nisbiy harakat fazosi $\ell_2(\mathbb{Z}^d)$ da aniqlangan $H(k)$ – parametrga bog‘liq bo‘lgan cheklangan o‘lchamli yoki diskret Shredinger tipli operatorga keltiriladi:

$$(H(k)\psi)(x) = \sum_{y \in \mathbb{Z}^d} h(x, y; k)\psi(y), \quad \psi \in \ell_2(\mathbb{Z}^d)$$

bu yerda $h(x, y; k)$ – s-d almashtirish modelining xususiyatlarini aks ettiruvchi matritsa-qiymatli yoki skalyar yadro bo‘lib, u zarrachalar orasidagi almashinish ta’sirini tavsiflaydi. S-d modeliga xos bo‘lgan o‘zaro ta’sir potensialining strukturasiga ko‘ra, V operatori qisqa ta’sir qiluvchi va degeneratsiyalashgan xususiyatlarga ega bo‘ladi.

Ta’rif: Ikki zarrachali diskret Shredinger operatoridagi V o‘zaro ta’sir operatori o‘tkazuvchanlik elektronlari va lokallashgan magnit momentlari orasidagi almashinish jarayonlarini tavsiflab, chekli o‘lchamli degeneratsiyaga ega bo‘lsa va uning yadrosi quyidagi diskret faktorizatsiyalangan ko‘rinishda berilsa:

$$V(x, y) = \sum_{\alpha,\beta}^{r} v_{1,\alpha}(x)\lambda_{\alpha\beta}v_{2,\beta}(y), \qquad x, y \in \mathbb{Z}^d$$

bu yerda $v_{1,\alpha}(x), v_{2,\beta}(y)$ – fazoning chekli sohasida tez kamayuvchi qisqa ta’sir qiluvchi funksiyalar, $\lambda_{\alpha\beta}$ - almashinish konstantalarini o^' z ichiga olgan matritsa elementlari, $r \in \mathbb{N}$ esa o‘zaro ta’sirning degeneratsiya darajasi bo‘lib, bunday tuzilishdagi potensial s-d almashtirish modeliga mos keluvchi potensial deb ataladi.

Ta’rif: O‘z-o‘ziga qo‘shma $H(k)$ operatorining spektri $\sigma(H(k))$ o‘zaro kesishmaydigan ikki qismga – uzluksiz spektr $\sigma_{ess}(H(k))$ va diskret spektr $\sigma_{disc}(H(k))$ ga bo‘linadi. Uzluksiz spektr chegaralaridan tashqarida yotuvchi, chekli karrali izolyatsiyalangan xos qiymatlar to‘plami $H(k)$ operatorining diskret spektri deb ataladi. Diskret spektrga tegishli har bir xos qiymatga mos keluvchi noldan farqli va $\ell_2(\mathbb{Z}^d)$ fazosiga tegishli xos funksiya tizimning bog‘langan holati deb yuritiladi.

Lemma 1: Faraz qilaylik, $H_0(k)$ – d-o‘lchamli panjaradagi erkin ikki zarrachali operator, V esa s-d almashtirish modelining qisqa ta’sir qiluvchi potensiali bo‘lsin. U holda Sheyl teoremasiga ko‘ra,

$H(k) = H_0(k) + V$ operatorining uzluksiz spektri erkin operatorning uzluksiz spektri bilan to‘la ustma-ust tushadi va quyidagi ko‘rinishga ega bo‘ladi:

$$\sigma_{ess}(H(k)) = \sigma_{ess}(H_0(k)) = \bigcup_{i=1}^{m} [a_i(k), b_i(k)]$$

bu yerda $[a_i(k), b_i(k)]$ – zarrachalarning massalari nisbati va total impuls vektori $k \in \mathbb{T}^d$ ga bog‘liq bo‘lgan energetik zonalardir.

Isbot. Operatorning uzluksiz spektrining invariantligini isbotlash uchun Veyl teoremasiga hamda kompakt perturbatsiyalar nazariyasidan foydalanamiz. Veyl teoremasiga ko‘ra, agar H_0 o‘z-o‘ziga qo‘shma operator bo‘lib, V esa unga nisbatan kompakt perturbatsiya bo‘lsa, u holda $H = H_0 + V$ operatorining uzluksiz spektri H_0 operatorining uzluksiz spektri bilan to‘la ustma-ust tushadi:

$$\sigma_{ess}(H) = \sigma_{ess}(H_0)$$

Shuning uchun biz s-d almashtirish modeliga mos keluvchi V o‘zaro ta’sir potensiali erkin $H_0(k)$ operatoriga nisbatan kompakt ekanligini ko‘rsatishimiz yetarli.

Erkin operatorning rezolventini $R_0(z;k)=(H_0(k)-z)^{-1}$ va perturblangan operatorning rezolventini $R(z;k)=(H(k)-z)^{-1}$ deb belgilaymiz. Rezolventlar ayirmasini quyidagi rezolvent tenglamasi orqali yozamiz:

$$R(z;k)-R_0(z;k)=-R_0(z;k)\mathrm{V}\,R(z;k).$$

Ta’rifga ko‘ra, s-d almashtirish potensiali V faktorizatsiyalangan va qisqa ta’sir qiluvchi xususiyatga ega:

$$V=V_1V_2$$

bu yerda V_1V_2 – fazoda tez kamayuvchi chiziqli operatorlardir. Bu holda rezolvent farqini quyidagicha almashtiramiz:

$$R(z;k)-R_0(z;k)=-(R_0(z;k)\mathrm{V}(I+\ V_2R_0(z;k)V_1R(z;k))^{-1}V_2R_0(z;k).$$

$V_1\ va\ V_2$ ning qisqa ta’sir qiluvchi xossasi hamda erkin rezolventning yadrosi tufayli, $V_2R_0(z;k)V_1$ va uning qismlari Hilbert-Schdmit sinfiga kiruvchi kompakt operatorlarni hosil qiladi. Kompakt operatorning chegaralangan operatorga ko’paytmasi ham kompakt bo‘lgani sababli rezolventlar farqi to‘la kompakt operator bo‘ladi. Veyl teoremasiga binoan, kompakt siljishlar uzluksiz spektrni o‘zgartirmaydi:

$$\sigma_{ess}\big(H(k)\big)=\sigma_{ess}\big(H_0(k)\big)=\bigcup_{i=1}^{m}[a_i(k),b_i(k)]$$

Lemma to‘liq isbotlandi.

Lemma 2: s-d almashtirish modelining potensiali $V(x,y)=\sum_{\alpha,\beta=1}^{r}v_{1,\alpha}(x)\,\lambda_{\alpha\beta}v_{2,\beta}(y)$ ko‘rinishda faktorizatsiyalangan bo‘lib, uning tarkibiy qismlari qisqa ta’sir qiluvchi funksiyalar bo‘lgani sababli, istalgan $z\in\mathbb{C}\backslash\sigma_{ess}\big(H(k)\big)$ uchun quyidagi operator:

$$T(z)=V_2R_0(z;k)V_1$$

Hilbert-Schdmit sinfiga kiruvchi kompakt operator hisoblanadi, bu yerda $R_0(z;k)=(H_0(k)-z)^{-1}$ – erkin operatorning rezolventidir.

Isbot. $\ell_2(\mathbb{Z}^d)$ fazosida berilgan A chiziqli operatori o‘zining $A(x,y)$ yadrosi orqali aniqlansin. Agar bu yadro quyidagi shartni, ya’ni uning kvadratchalarining yig‘indisi chekli bo‘lishini qanoatlantirsa:

$$\|A\|_{\mathfrak{S}_2}^2=\sum_{x,y\in\mathbb{Z}^d}|A(x,y)|^2<\infty$$

u holda A operatori Hilbert-Schdmit sinfiga tegishli bo‘ladi. Spektral nazariyaga ko‘ra, har qanday Hilbert-Schdmit operatori avtomatik ravishda kompakt operator hisoblanadi. Tekshirilayotgan operatorimiz

$$T(z)=V_2R_0(z;k)V_1$$

bu yerda $R_0(z;k)=(H_0(k)-z)^{-1}$ erkin operatorning rezolventi $(z\in\mathbb{C}\backslash\sigma_{ess}\big(H(k)\big)),V_1\ va\ V_2$ esa qisqa ta’sir qiluvchi potensialning tarkibiy qismlari. Erkin operator rezolventirning koordinatalar fazosidagi yadrosi $R_0(x,y;z,k)$ Furye integrali orqali ifodalanadi:

$$R_0(x,y;z,k)=\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{e^{i(x-y)p}}{h_0(p,k)-z}dp$$

Shartga ko‘ra, $z\notin\sigma\big(H_0(k)\big)$ bo‘lgani uchun maxraj noldan farqli va chegaralangan, bu rezolvent yadrosining $\ell_2(\mathbb{Z}^d)$ fazosida tarqalishini nazorat qiladi.

$T(z)$ operatorining yadrosi $T(x,y;z)$ quyidagi ko‘rinishga ega:

$$T(x,y;z)=\sum_{\alpha,\beta=1}^{r} v_{2,\alpha}(x)\,R_{0,\alpha\beta}(x,y;z,k)v_{1,\beta}(y)$$

Bu yerda $v_{1,\beta}(y)$ va $v_{2,\alpha}(x)$ qisqa ta’sir qiluvchi funksiyalar bo‘lgani sababli yadro uchun quyidagi baholash o‘rinli bo‘ladi:

$$\sum_{x,y\in\mathbb{Z}^d}|T(x,y)|^2\le\|R_0(z,k)\|^2\left(\sum_x|v_2(x)|^2\right)\left(\sum_y|v_1(y)|^2\right)<\infty$$

Olingan natijaga ko‘ra, $T(z)$ operatorining Hilbert Schmidt normasi chekli. Demak, $T(z)$ operatori Hilbert-Schdmit sinfiga tegishli bo‘lib, kompakt operator hisoblanadi.

Teorema: Faraz qilaylik, $\mu(k)=\min\sigma_{ess}\big(H(k)\big)$ uzluksiz spektrning quyi chegarasi bo‘lsin. U holda 1. Agar s-d almashtirish potensialining o‘zaro ta’sir parametrlari yetarlicha kuchli bo‘lsa, $H(k)$ operatorining uzluksiz spektri ostida kamida bitta diskret xos qiymat mavjud bo‘ladi.

2. Bu diskret xos qiymatlar soni $N(k)$ har doim chekli bo‘ladi.

Isbot. $H(k)\psi=z\psi(z<\mu(k))$ xos qiymat tenglamasidan boshlaymiz. $H(k)=H_0(k)+V$ ekanligini e’tiborga olib, uni quyidagicha o‘zgartiramiz:

$$(H_0(k)-z)\psi=-V\psi$$

z nuqta uzluksiz spektrga kirmagani uchun erkin operatorning rezolventi $R_0(z;k)=(H_0(k)-z)^{-1}$ mavjud va chegaralangan. U holda

$$\psi=-R_0(z;k)V\psi$$

Shartga ko‘ra, $V=V_1V_2$ faktorizatsiyalashgan. $\phi=V_2\psi$ belgilash kiritib, quyidagi tenglamani hosil qilamiz:

$$\phi=-V_2R_0(z;k)V_1\phi \text{ yoki } T(z)_k\phi=\phi$$

bu yerda $T(z)_k=-V_2R_0(z;k)V_1\phi$ Lemma 2 ga ko‘ra Hilbert-Scgdmit sinfiga kiruvchi kompakt operator. Bu tenglama noldan farqli yechimga ega bo‘lishi uchun 1 soni $T(z)_k$ operatorining xos qiymati bo‘lishi zarur va yetarli. Kompakt operatorlar nazariyasiga binoan, kompakt operatorning 1 dan katta yoki unga teng bo‘lgan xos qiymatlari soni har doim chekli bo‘ladi. Demak, s-d almashtirish modeliga mos keluvchi ikki zarrachali Shredinger operatorining uzluksiz spektri ostidagi diskret xos qiymatlari soni $N(k)$ ham chekli bo‘ladi.

Xulosa. Mazkur ilmiy ishda kristall panjaradagi s-d almashtirish modeliga mos keluvchi ikki zarrachali diskret Shredinger operatorining spektral xossalari tadqiq qilindi. Veyl teoremasi asosida uzluksiz spektrning invariantligi ko'rsatildi, rezolvent ayirmasining Hilbert–Shmidt sinfiga kirishi orqali operatorning kompaktligi isbotlandi hamda Birman–Shvinger prinsipi yordamida uzluksiz spektr ostida diskret xos qiymatlarning mavjudligi va ularning soni chekliligi aniqlandi.

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PARAMETRLI MASALALARNI YECHISHDA ALGEBRAIK VA MANTIQIY TAHLILNING METODIK BOSQICHLARI

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Annotatsiya. *Ushbu maqolada parametrli masalalarni yechishda algebraik va mantiqiy tahlilning metodik bosqichlari yoritilgan. Tadqiqotda parametrning o‘rinli qiymatlar sohasi, aniqlanish sohasi, xususiy masala va umumiy yechim tushunchalarining nazariy-metodik mazmuni aniqlashtirildi. Maqolada parametrli masalalarni faqat algebraik hisoblashlar majmuyi sifatida emas, balki o‘quvchilarning tahliliy fikrlashi, tasniflash va umumlashtirish ko‘nikmalarini rivojlantiruvchi yaxlit metodik jarayon sifatida qarash taklif etiladi.*

Kalit so‘zlar: *parametrli masala, o‘rinli qiymatlar sohasi, algebraik tahlil, mantiqiy tahlil, umumiy yechim.*

МЕТОДИЧЕСКИЕ ЭТАПЫ АЛГЕБРАИЧЕСКОГО И ЛОГИЧЕСКОГО АНАЛИЗА ПРИ РЕШЕНИИ ЗАДАЧ С ПАРАМЕТРАМИ

Аннотация. *В статье рассматриваются методические этапы алгебраического и логического анализа при решении задач с параметрами. В исследовании уточнено теоретико-методическое содержание понятий области допустимых значений параметра, области определения, частной задачи и общего решения. Предлагается рассматривать задачи с параметрами не только как совокупность алгебраических вычислений, но и как целостный методический процесс, направленный на развитие у учащихся аналитического мышления, умений классифицировать и обобщать.*

Ключевые слова: *задача с параметрами, область допустимых значений параметра, алгебраический анализ, логический анализ, общее решение.*

METHODOLOGICAL STAGES OF ALGEBRAIC AND LOGICAL ANALYSIS IN SOLVING PROBLEMS WITH PARAMETERS

Abstract. *This article examines the methodological stages of algebraic and logical analysis in solving problems with parameters. The study clarifies the theoretical and methodological content of the concepts of the admissible domain of parameter values, domain of definition, particular problem, and general solution. It is proposed that problems with parameters be considered not merely as a set of algebraic computations, but as an integrated methodological process aimed at developing students’ analytical thinking, classification, and generalization skills.*

Keywords: *problems with parameters, admissible domain of parameter values, algebraic analysis, logical analysis, general solution.*

Kirish. Matematika ta’limida o‘quvchilarning masalani tahlil qilish, xususiy holatlarni tasniflash va natijalarni umumlashtirish ko‘nikmalarini rivojlantirish muhim metodik vazifalardan biridir [1, 2]. Parametrli masalalar mazkur ko‘nikmalarni shakllantirishda keng didaktik imkoniyatlarga ega. Ularni yechish jarayonida parametrning turli qiymatlariga mos holatlar tahlil qilinadi va umumiy yechim aniqlanadi. Shu sababli dastlab parametrning o‘rinli qiymatlari sohasi belgilanadi. Bu esa masala aniqlanmagan holatlarni yechim mavjud bo‘lmagan holatlardan farqlash imkonini beradi. Parametrli masalalarni yechishda algebraik almashtirishlar bilan bir qatorda mantiqiy tahlil, chegaraviy holatlarni tekshirish va natijalarni umumlashtirish ham muhim ahamiyat kasb etadi. O‘quv-metodik adabiyotlarda [3, 6] ko‘proq tayyor algoritmlar va namunaviy yechimlar keltiriladi. Biroq parametr sohasini aniqlash, uni muhim holatlarga ajratish va xususiy holatlarni izchil tekshirish bosqichlari yetarlicha tizimlashtirilmagan. Shu bois algebraik va mantiqiy tahlilni o‘zaro bog‘langan yaxlit metodik jarayon sifatida tashkil etish zarur [7, 8].

Tadqiqotning maqsadi parametrli masalalarni yechishda algebraik va mantiqiy tahlil bosqichlarini aniqlash hamda ularni parametrli tenglama va tengsizliklarni o‘qitishda qo‘llash imkoniyatlarini asoslashdan iborat. Bunda parametrning o‘rinli qiymatlari sohasi aniqlanadi, parametr sohasi zarur qismlarga ajratiladi, xususiy holatlar tekshiriladi va olingan yechimlar umumlashtiriladi.

Tadqiqot metodologiyasi va adabiyotlarning tahlili. Tadqiqot nazariy-metodik tahlil asosida olib borildi. Unda parametrli masala, parametrning o‘rinli qiymati, aniqlanish sohasi va umumiy yechim tushunchalari aniqlashtirildi. Yechish jarayoni parametrning o‘rinli qiymatlarini aniqlash, sohani qismlarga ajratish, xususiy holatlarni yechish, natijalarni tizimlashtirish va tekshirish bosqichlariga ajratildi. Bu yondashuv parametrli masalalarni tahlil, tasnif va umumlashtirishga asoslangan yaxlit metodik jarayon sifatida qaraydi [7]. Parametrli masalalarda, avvalo, masala sharti ma’noga ega bo‘ladigan parametr qiymatlari aniqlanadi. Shu asosda parametrning o‘rinli qiymati tushunchasi kiritiladi.

Ta’rif. Parametrning shunday qiymatiga o*‘rinli qiymat* deyiladiki, bu qiymatda berilgan masalaning aniqlanish sohasi bo‘sh bo‘lmagan to‘plamdan iborat bo‘ladi.

Boshqacha aytganda, agar masala shartida ishtirok etuvchi boshqa o‘zgaruvchilarning kamida bitta qiymati mavjud bo‘lib, ushbu qiymat parametrning berilgan qiymati bilan birgalikda masala shartini ifodalovchi analitik ifodaga qo‘yilganda bu ifoda matematik ma’noga ega bo‘lsa, parametrning qiymati o‘rinli deb hisoblanadi.

Ta’rif. Parametrning barcha o‘rinli qiymatlari hosil qilgan to‘plam parametrning o‘rinli qiymatlar sohasi deyiladi va u D_p bilan belgilanadi.

Agar parametrning biror qiymati D_p to‘plamga kirmasa, bu holat mazkur parametr qiymatida masalaning yechimi yo‘qligini emas, balki masalaning o‘zi matematik ma’noga ega emasligini bildiradi. Chunki ifoda aniqlanmagan holatda uning yechimi mavjudligi yoki mavjud emasligi haqida fikr yuritish mantiqan to‘g‘ri bo‘lmaydi. Bu yerda shuni alohida ta’kidlash lozimki, parametrning o‘rinli qiymati masalaning, albatta, yechimga ega ekanligini anglatmaydi. U faqat berilgan parametr qiymatida masala sharti aniqlanganligini, ya’ni masalani mazmunan qarash mumkinligini bildiradi, xolos. Quyida ushbu tushunchani aniqlashtiruvchi bir nechta misollar keltiriladi. Barcha misollarda o‘zgaruvchilar haqiqiy qiymatlar qabul qiladi, deb qaraladi va izlanayotgan o‘zgaruvchi sifatida x tanlanadi.

1-misol. Quyidagi tenglamani qaraymiz: $(a^2+1)x-a=0.$ Bu tenglama x o‘zgaruvchiga nisbatan qaralganda, a parametrning istalgan haqiqiy qiymatida ifoda ma’noga ega bo‘ladi. Chunki a^2+1 ifoda har qanday $a\in\mathbb{R}$ uchun aniqlangan. Demak, $D_p=R.$

2-misol. Quyidagi tengsizlik berilgan bo‘lsin: $\dfrac{x+2}{a-3}\geq 0.$ Bu tengsizlikda maxraj $a-3$ ga teng. Agar $a=3$ bo‘lsa, ifoda aniqlanmaydi. Agar $a\neq 3$ bo‘lsa, kamida bitta x qiymatida emas, balki barcha haqiqiy x qiymatlarida ifoda ma’noga ega bo‘ladi. Shuning uchun parametrning o‘rinli qiymatlar sohasi quyidagicha:

$$D_p=R\setminus\{3\}.$$

3-misol. Quyidagi tenglamani qaraymiz:

$$\sqrt{x+1}+\ln(x+2)+a=0.$$

Bu yerda ildiz ostidagi ifoda manfiy bo‘lmasligi va logarifm ostidagi ifoda musbat bo‘lishi kerak: $x+1\geq 0,\ x+2>0.$ Bundan $x\geq -1,\ x>-2$ shartlar kelib chiqadi. Ularning umumiy yechimi $x\geq -1$ bo‘lib, bu to‘plam bo‘sh emas. Parametr a esa ifodaning aniqlanishiga hech qanday cheklov qo‘ymaydi. Demak, parametrning barcha haqiqiy qiymatlari o‘rinli: $D_p=R.$

4-misol. Quyidagi tenglamani qaraymiz:

$$\sqrt{x-2}+\sqrt{1-x}+a=0.$$

Bu tenglama ma’noga ega bo‘lishi uchun bir vaqtning o‘zida quyidagi shartlar bajarilishi kerak: $x-2\geq 0,\ 1-x\geq 0.$ Ya’ni $x\geq 2,\ x\leq 1.$ Bu ikki shart bir vaqtda bajarilmaydi. Shuning uchun x ning bunday qiymati mavjud emas. Demak, parametrning hech bir qiymatida masala sharti aniqlanmaydi. Shunday qilib, $D_p=\varnothing.$

Endi parametrning o‘rinli qiymatlar sohasi tushunchasiga tayangan holda parametr qatnashgan tenglama yoki tengsizlikning aniqlanish sohasi tushunchasini kiritamiz.

Aytaylik,

$$F(x,a)\vee 0$$

berilgan masalaning formal yozuvi bo‘lsin. Bu yerda $\vee$ belgisi tenglik yoki tengsizlik munosabatlaridan birini bildiradi, ya’ni $\vee\in\{=,<,\leq,>,\geq\}.$ Shuningdek, $F(x,a)\vee 0$ yozuvi tenglama, tengsizlik, ularning sistemasi yoki majmuasi ko‘rinishidagi masalalarni umumiy tarzda ifodalashi mumkin.

Agar $\alpha \in D_p$ bo‘lsa, u holda $a = \alpha$ deb olib, quyidagi xususiy masalani qaraymiz: $F(x,\alpha) \vee 0$. Bu xususiy masalada ifoda aniqlangan bo‘ladigan barcha x qiymatlar to‘plamini X_α bilan belgilaymiz.

Ta’rif. Parametr a qatnashgan $F(x,a) \vee 0$ masalaning x o‘zgaruvchiga nisbatan aniqlanish sohasi deb quyidagi to‘plamga aytiladi:

$$D_F(x,a) = \left\{(x,a) \in \mathbb{R}^2 \mid\ a \in D_p,\ x \in X_a\right\}.$$

Bu to‘plam parametr va o‘zgaruvchi qiymatlarining shunday juftliklaridan iboratki, ularda berilgan masala sharti matematik ma’noga ega bo‘ladi.

5-misol. Quyidagi tengsizlikni qaraymiz:

$$\frac{x-a+1}{x+a-2} < 0.$$

Bu tengsizlikda maxraj $x+a-2$ nolga teng bo‘lmasligi kerak. Har qanday $a \in R$ uchun x ning $x \neq 2-a$ shartni qanoatlantiruvchi qiymatlari mavjud. Demak, parametrning barcha haqiqiy qiymatlari o‘rinli: $D_p = R$. Biroq har bir tayinlangan a uchun $x = 2-a$ qiymatida ifoda aniqlanmaydi. Shuning uchun tengsizlikning aniqlanish sohasi quyidagicha bo‘ladi:

$$D_F(x,a) = R^2 \setminus \left\{(x,a) \in R^2 \mid\ x+a-2=0\right\}.$$

Demak, bu tengsizlikning aniqlanish sohasi (x,a) tekislikdan $x+a-2=0$ to‘g‘ri chiziqqa tegishli nuqtalar chiqarib tashlangan to‘plamdan iborat.

6-misol. Quyidagi tenglamani qaraymiz:

$$\frac{x-1}{a(a+1)} + \frac{a+1}{x-a} = 3.$$

Bu tenglamada maxrajlar nolga teng bo‘lmasligi kerak. Shuning uchun quyidagi shartlar bajarilishi zarur: $a(a+1) \neq 0,\ x-a \neq 0.$ Birinchi shartdan $a \neq 0,\ a \neq -1$ kelib chiqadi. Agar $a = 0$ yoki $a = -1$ bo‘lsa, birinchi kasr aniqlanmaydi va masala sharti har qanday x uchun ma’noga ega bo‘lmaydi. Demak, parametrning o‘rinli qiymatlar sohasi:

$$D_p = R \setminus \{-1, 0\}.$$

Endi aniqlanish sohasini yozamiz. Har bir $a \in D_p$ uchun ikkinchi maxrajning nolga teng bo‘lmasligi talab qilinadi: $x \neq a.$ Shuning uchun tenglamaning aniqlanish sohasi

$$D_F(x,a) = \left\{(x,a) \in R^2 \mid\ a \neq -1, a \neq 0, x \neq a\right\}$$

ko‘rinishda ifodalanadi. Boshqacha yozganda,

$$D_F(x,a) = R^2 \setminus \left(\left\{(x,a) \in R^2 \mid\ a=-1\right\} \cup \left\{(x,a) \in R^2 \mid\ a=0\right\} \cup \left\{(x,a) \in R^2 \mid\ x-a=0\right\}\right).$$

Demak, bu holda (x,a) tekislikdan $a=-1,\ a=0,\ x-a=0$ tenglamalar bilan berilgan to‘g‘ri chiziqlar chiqarib tashlanadi. Umuman olganda, parametr qatnashgan masalalarda avval parametrning o‘rinli qiymatlar sohasi aniqlanadi. So‘ngra har bir o‘rinli parametr qiymati uchun o‘zgaruvchining aniqlanish sohasi tekshiriladi. Bu bosqich parametrli tenglama va tengsizliklarni to‘g‘ri tahlil qilishning zaruriy sharti hisoblanadi.

Umuman olganda, parametrli masala - tarkibida qiymati oldindan aniq berilmagan, ammo ma’lum deb qaraladigan parametr qatnashgan tenglama, tengsizlik, sistema yoki boshqa analitik munosabatdir. Uning yechimi noma’lum o‘zgaruvchi bilan birga parametr qiymatiga ham bog‘liq bo‘ladi. O‘quv-metodik manbalarda parametrli masalalarni yechish turlicha talqin qilinadi. Ayrim yondashuvlarda parametrning har bir qiymati uchun yechimlar to‘plamini topish nazarda tutiladi, biroq parametrning o‘rinli qiymatlari sohasi alohida ajratilmaydi [3, 5]. Natijada aniqlanmagan holatlar yechimsiz holatlar bilan aralashishi mumkin. Boshqa yondashuvlarda esa avval parametrning o‘rinli qiymatlari aniqlanib, so‘ng ular qismlarga ajratiladi. Biroq yechimni har doim parametrga bog‘liq bitta funksiya sifatida ifodalash to‘g‘ri emas. Chunki yechim oraliq, chekli yoki cheksiz to‘plam, yoxud bir nechta shartlar orqali berilishi mumkin.

Masalan, $|x+1|+|x-2|=a$ tenglamani qaraylik. Agar $a=3$ bo‘lsa, tenglamaning yechimlari $-1 \le x \le 2$ oraliqdagi barcha haqiqiy sonlardan iborat bo‘ladi. Demak, bu holatda yechim $x = x(a)$ ko‘rinishidagi yagona funksiya sifatida emas, balki butun bir oraliq sifatida ifodalanadi. Shuning uchun parametrli masala yechimini faqat funksional bog‘lanish sifatida talqin qilish yetarli emas. Parametrli

masalalarni o‘rganishda G. Polya, A. Schoenfeld, A.G. Mordkovich, V.A. Gusev va J. Kaputning metodik qarashlari muhimdir. Polya masala yechishni tushunish, rejalashtirish, bajarish va tahlil qilish bosqichlariga ajratadi. Schoenfeld evristik strategiya va metakognitiv nazoratni ta’kidlaydi. Mordkovich va Gusev matematik belgining jarayon va obyekt sifatidagi mazmunini, Kaput esa turli reprezentatsiyalardan foydalanishni asoslaydi [8]. Demak, parametrli masalani yechish o‘rinli qiymatlar sohasini aniqlash, parametr sohasini qismlarga ajratish, xususiy holatlarni tekshirish hamda yechimsiz va aniqlanmagan holatlarni farqlashni talab qiladi.

Ta’rif. Aytaylik, $f(x,a)=0$ tenglik berilgan bo‘lsin. Agar a ning har bir haqiqiy qiymati uchun ushbu tenglamani x ga nisbatan yechish masalasi qo‘yilsa, u holda $f(x,a)=0$ tenglama x o‘zgaruvchili va a parametrli tenglama deyiladi. Parametrli tenglamani yechish deganda a ning har bir qiymati uchun ushbu tenglamani qanoatlantiruvchi x qiymatlarini topish tushuniladi [1].

Bu ta’rifning avvalgi ta’riflardan muhim farqi shundaki, $f(x,a)=0$ tenglama ikkita teng huquqli o‘zgaruvchili tenglama sifatida qaraladi. Bundan tashqari, boshlang‘ich tenglamaning yechimini parametrning har bir qiymati uchun topish masalasi qo‘yiladi. Biroq parametrning har bir haqiqiy qiymati uchun yechimni ko‘rsatish talabi yana bir muammoga olib keladi: parametrning tenglama yechimga ega bo‘lmaydigan qiymatlari to‘plamiga, masalaga umuman aniqlanmagan qiymatlar ham qo‘shilib ketadi.

Ta’rif. Parametrli masalani yechish deganda parametrning o‘rinli qiymatlari to‘plami D_p ni aniqlash, ushbu to‘plamni masalaning xususiyatlariga ko‘ra zarur qismlarga ajratish va har bir parametr qiymatida hosil bo‘ladigan xususiy masalaning yechimlar to‘plamini topish tushuniladi.

Parametrli masalalarni shartiga ko‘ra ikki asosiy sinfga ajratish mumkin.

-Birinchi sinfga parametrning barcha o‘rinli qiymatlari uchun yechimlar to‘plamini topish talab qilinadigan masalalar kiradi.

-Ikkinchi sinfga esa parametrning shunday qiymatlarini aniqlash talab qilinadigan masalalar kiradiki, bu qiymatlarda xususiy masala yechimlari oldindan belgilangan ma’lum xossalarga ega bo‘ladi. Masalan, tengsizlikning barcha haqiqiy x lar uchun bajarilishi, tenglamaning yagona yechimga ega bo‘lishi yoki sistema yechimlarining mavjud bo‘lishi kabi shartlar ikkinchi sinf masalalariga xosdir.

7-misol. Quyidagi tenglamalar sistemasini yeching:

$$\begin{cases} x^2+y^2=a^2, \\ (x-y)^2=b. \end{cases}$$

Shuningdek, uning $a=\sqrt{5},\ b=2$ bo‘lgandagi yechimini toping.

Yechish. Sistemadagi ikkinchi tenglama haqiqiy yechimga ega bo‘lishi uchun $b\ge 0$ bo‘lishi zarur. Shu sababli parametrlarning dastlabki o‘rinli qiymatlari sohasi

$$D_p=\{(a,b)\in R^2 |\ b\ge 0\}$$

ko‘rinishda yoziladi. Quyidagi almashtirishlarni kiritamiz:

$$u=x-y,\ v=x+y.$$

U holda $x=\dfrac{u+v}{2},\ y=\dfrac{v-u}{2}$ va $x^2+y^2=\dfrac{u^2+v^2}{2}$ bo‘ladi. Sistema quyidagi ko‘rinishga keladi:

$$\begin{cases} \dfrac{u^2+v^2}{2}=a^2, \\ u^2=b. \end{cases}$$

Bundan $u=\pm\sqrt{b}$ va $v^2=2a^2-b$ kelib chiqadi. Demak, haqiqiy yechim mavjud bo‘lishi uchun $2a^2-b\ge 0$ ya’ni, $0\le b\le 2a^2$ sharti bajarilishi kerak. Agar $b>2a^2$ bo‘lsa, sistema haqiqiy yechimga ega emas. Agar $0\le b\le 2a^2$ bo‘lsa, sistemaning yechimlari quyidagi ko‘rinishda ifodalanadi:

$$\begin{cases} x=\dfrac{\sqrt{b}+\sqrt{2a^2-b}}{2} \\ y=\dfrac{\sqrt{2a^2-b}-\sqrt{b}}{2} \end{cases};\quad \begin{cases} x=\dfrac{\sqrt{b}-\sqrt{2a^2-b}}{2} \\ y=\dfrac{-\sqrt{b}-\sqrt{2a^2-b}}{2} \end{cases}$$

va

$$\begin{cases} x=\dfrac{\sqrt{b}+\sqrt{2a^2-b}}{2} \\ y=\dfrac{\sqrt{2a^2-b}-\sqrt{b}}{2} \end{cases}; \quad \begin{cases} x=\dfrac{\sqrt{b}-\sqrt{2a^2-b}}{2} \\ y=\dfrac{-\sqrt{b}-\sqrt{2a^2-b}}{2}. \end{cases}$$

Xususan, $a=\sqrt{5},\ b=2$ bo‘lganda $2a^2-b=10-2=8.$ Demak,

$$\sqrt{b}=\sqrt{2}, \qquad \sqrt{2a^2-b}=2\sqrt{2}.$$

Shuning uchun xususiy sistemaning yechimlari:

$$\left(\frac{3\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right),\left(-\frac{\sqrt{2}}{2},-\frac{3\sqrt{2}}{2}\right),\ \left(\frac{\sqrt{2}}{2},\frac{3\sqrt{2}}{2}\right),\ \left(-\frac{3\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right).$$

8-misol. Quyidagi parametrli tengsizlikni yeching:

$$\frac{x+2}{t+1}-3<0$$

va uning $t=0$ bo‘lgandagi yechimini ko‘rsating.

Yechish. Tengsizlik ma’noga ega bo‘lishi uchun maxraj nolga teng bo‘lmasligi kerak:

$t+1\neq 0.$ Demak, $D_p=\{t\in R|\ t\neq -1\}.$ Tengsizlikni quyidagicha yozamiz: $\dfrac{x+2}{t+1}<3.$ Agar

$$t+1>0,$$

ya’ni $t>-1$ bo‘lsa, tengsizlik belgisini o‘zgartirmasdan ko‘paytiramiz: $x+2<3(t+1).$ Bundan $x<3t+1$ kelib chiqadi. Demak, $t>-1$ bo‘lganda yechimlar to‘plami $x\in(-\infty,3t+1)$ bo‘ladi.

Agar $t+1<0,$ ya’ni $t<-1$ bo‘lsa, manfiy songa ko‘paytirishda tengsizlik belgisi teskari tomonga o‘zgaradi: $x+2>3(t+1).$ Bundan $x>3t+1$ kelib chiqadi. Demak, $t<-1$ bo‘lganda yechimlar to‘plami $x\in(3t+1,+\infty)$ bo‘ladi. Xususan, $t=0$ bo‘lganda tengsizlik $x+2-3<0$ ya’ni $x-1<0$ ko‘rinishga keladi. Shuning uchun $x\in(-\infty,1).$

9-misol. Parametrning shunday barcha qiymatlarini topingki, har birida

$$(a+2)x^2-2(a-1)x+3>0$$

tengsizlik barcha $x\in R$ uchun bajarilsin.

Yechish. Berilgan ifoda kvadrat uchhad bo‘lib, u barcha haqiqiy x lar uchun musbat bo‘lishi uchun parabola yuqoriga qaragan va diskriminant manfiy bo‘lishi kerak. Shuning uchun quyidagi shartlar bajariladi: $a+2>0,\ \Delta<0.$ Bu yerda $\Delta=\left[-2(a-1)\right]^2-4(a+2)\cdot 3.$ Soddalashtiramiz: $\Delta=4(a-1)^2-12(a+2).$ Demak, $4(a-1)^2-12(a+2)<0.$ Har ikki tomonni 4 ga bo‘lamiz: $(a-1)^2-3(a+2)<0.$ Yoyib yozamiz:

$$a^2-2a+1-3a-6<0,\ \ a^2-5a-5<0.$$

Kvadrat tenglamaning ildizlari: $a=\dfrac{5\pm\sqrt{25+20}}{2}=\dfrac{5\pm 3\sqrt{5}}{2}.$ Shuning uchun $a^2-5a-5<0$ tengsizlik quyidagi oraliqda bajariladi: $\dfrac{5-3\sqrt{5}}{2}<a<\dfrac{5+3\sqrt{5}}{2}.$ Birinchi shart esa $a+2>0$, ya’ni $a>-2$ ko‘rinishda bo‘ladi. Chunki $\dfrac{5-3\sqrt{5}}{2}>-2,$ bu shart yuqoridagi oraliqda avtomatik bajariladi. Demak, izlanayotgan parametr qiymatlari to‘plami:

$$a\in\left(\frac{5-3\sqrt{5}}{2},\frac{5+3\sqrt{5}}{2}\right).$$

Keltirilgan misollar shuni ko‘rsatadiki, parametrli masalalarda yechish jarayoni oddiy tenglama yoki tengsizliklarni yechishdan ko‘ra murakkabroq tuzilishga ega. Chunki parametr o‘zgarganda nafaqat

koeffitsiyentlar, balki yechimlar soni, yechimlar turi, aniqlanish sohasi va qo‘llaniladigan yechish usuli ham o‘zgarishi mumkin. Shu sababli parametrli masalalarni yechishda quyidagi metodik ketma-ketlikka amal qilish maqsadga muvofiq:

1) Parametrning o‘rinli qiymatlari sohasini aniqlash;
2) Parametr sohasini muhim holatlarga ko‘ra qismlarga ajratish;
3) Har bir qismda xususiy masalani yechish;
4) Yechimlarni parameter qiymatlari bo‘yicha tizimlashtrish;
5) Olingan javobni tekshirish va chegaraviy holatlarni alohida tahlil qilish;

Bunday yondashuv parametrli masalalarni formal algoritm sifatida emas, balki tahliliy fikrlash, umumlashtirish, tasniflash va matematik modellashtirishni rivojlantiruvchi vosita sifatida o‘rganishga imkon beradi. Ayniqsa, parametrli masalalarda algebraik, grafik va mantiqiy yondashuvlarni birgalikda qo‘llash o‘quvchilarning matematik tafakkurini chuqurlashtiradi.

Ma’lumki, parametrli tenglamalar va tengsizliklar odatdagi tenglama hamda tengsizliklarga nisbatan umumiyroq matematik obyekt hisoblanadi. Chunki bunday masalalarda parametrning har bir o‘rinli qiymati uchun alohida xususiy tenglama yoki tengsizlik hosil bo‘ladi. Demak, parametrli masalani yechish aslida shu xususiy holatlarning barchasini tizimli tahlil qilishni anglatadi. Parametrli tenglama yoki tengsizlik, bir tomondan, bir nechta o‘zgaruvchili munosabat sifatida qaralsa, ikkinchi tomondan, parametr qiymatlariga bog‘liq cheksiz ko‘p xususiy masalalar majmuasidir. Shu sababli ularni yechishda avvalo parametrning o‘rinli qiymatlari sohasi aniqlanadi, so‘ngra ushbu soha masalaning xususiyatiga ko‘ra qismlarga ajratiladi. Parametrli masalalarning asosiy qismini bitta noma’lum va bitta parametr qatnashgan tenglama, tengsizlik yoki sistemalar tashkil etadi. Eng sodda holda bunday tenglama $f(x,a)=0$ ko‘rinishida yoziladi. Agar parametrning har bir $a \in D_p$ o‘rinli qiymati uchun hosil bo‘ladigan tenglamani x ga nisbatan yechish talab qilinsa, u holda $f(x,a)=0$ tenglama x o‘zgaruvchili va a parametrli tenglama deyiladi. Uni yechish esa parametrning har bir o‘rinli qiymati uchun tenglamani qanoatlantiruvchi barcha x qiymatlarni topishdan iborat. Parametrga aniq $a=\alpha$ qiymat berilganda parametrli tenglamadan xususiy tenglama hosil bo‘ladi: $f(x,\alpha)=0.$ Shunday qilib, parametrli tenglama umumiylikni, xususiy tenglama esa shu umumiylikning konkret ko‘rinishini ifodalaydi. Demak, xususiy tenglamani yechish parametrli tenglamani yechish jarayonining alohida holati sifatida qaraladi. Parametrli tenglamalarda yechim tushunchasi ham odatdagi tenglamalardagiga qaraganda kengroq ma’noga ega. Xususiy tenglamaning yechimi sonlar o‘qidagi nuqta yoki nuqtalar to‘plami bilan ifodalansa, parametrli tenglamaning umumiy yechimi tekislikdagi chiziq yoki soha ko‘rinishida talqin qilinadi. Parametrlar soni ortishi bilan yechimlar fazoviy obyektlar orqali ham ifodalanishi mumkin. Parametrli tenglama tushunchasi funksiya tushunchasi bilan uzviy bog‘liqdir. Chunki $f(x,a)=0$ tenglama x va a o‘rtasidagi ma’lum funksional bog‘lanishni ifodalaydi. Shu bois parametrli tenglamalarni yechishda funksiyaning aniqlanish sohasi, qiymatlar sohasi, monotonligi, ildizlar soni va boshqa xossalari muhim ahamiyatga ega bo‘ladi.

Ta’rif. Agar har bir $a=\alpha,\ \alpha \in D_p$ uchun $x=x(\alpha)$ qiymat mos xususiy tenglama $f(x,\alpha)=0$ ning yechimi bo‘lsa, a parametrli va x o‘zgaruvchili $f(x,a)=0$ tenglama uchun $x=x(a)$ moslik parametrning o‘rinli qiymatlari sohasida umumiy yechim deyiladi.

Umumiy yechim tushunchasi parametrli tenglamalar nazariyasida muhim nazariy tushuncha hisoblanadi. U xususiy tenglamalarning alohida-alohida yechimlarini umumlashtiradi, ularni parametr qiymatlariga ko‘ra tasniflash imkonini beradi hamda parametrli tenglamaning yaxlit tuzilishini ochib beradi. Shu nuqtayi nazardan, parametrli tenglamani yechish faqat algebraik amallarni bajarish emas, balki parametrning o‘zgarishi natijasida hosil bo‘ladigan barcha xususiy holatlarni tahlil qilish, ularni umumlashtirish va yagona tizimga keltirish jarayonidir. Bunda umumiy yechim parametrli tenglamaning ichki tuzilishini ifodalovchi asosiy vosita sifatida namoyon bo‘ladi.

Tahlil va olingan natijalar. Tadqiqot natijalari parametrli masalalarni yechish jarayonini faqat algebraik hisoblashlar majmuyi sifatida emas, balki o‘rinli qiymatlar sohasi, aniqlanish shartlari, xususiy holatlar va umumiy yechimlar o‘rtasidagi mantiqiy bog‘lanishlarni ochib beruvchi yaxlit metodik jarayon sifatida qarash zarurligini ko‘rsatdi. Birinchidan, parametrli masalalarni yechishda boshlang‘ich bosqich sifatida parametrning o‘rinli qiymatlari sohasini aniqlash muhim metodik shart ekanligi asoslandi. Chunki parametrning har qanday qiymatida berilgan ifoda yoki masala sharti matematik ma’noga ega bo‘lavermaydi. Shu sababli parametrning o‘rinli qiymatlari sohasi D_p ni aniqlash parametrli tenglama va tengsizliklarni to‘g‘ri tahlil qilishning zaruriy bosqichi sifatida ajratildi. Ikkinchidan, parametrli masalalarda aniqlanish

sohasi faqat noma'lum o'zgaruvchi qiymatlari bilan emas, balki parametr va o'zgaruvchi juftliklari orqali qaralishi lozimligi aniqlandi. Ya'ni parametr qatnashgan masalaning aniqlanish sohasi

$$D_F(x,a)=\left\{(x,a)\in R^2 |\ a\in D_p,\ x\in X_a\right\}$$

ko'rinishida talqin qilinishi maqsadga muvofiqdir. Bunday yondashuv o'quvchilarga masalani oddiy tenglama yoki tengsizlik sifatida emas, balki parametr qiymatlariga bog'liq o'zgaruvchan matematik obyekt sifatida tushunishga yordam beradi. Uchinchidan, parametrli masalalarni yechishning metodik ketma-ketlik ishlab chiqildi. Ushbu ketma-ketlik parametrli tenglama va tengsizliklarni yechishda algebraik va mantiqiy tahlilni o'zaro bog'laydi. Algebraik tahlil ifodani soddalashtirish, tenglama yoki tengsizlikni yechish va yechimlar to'plamini topishga xizmat qilsa, mantiqiy tahlil parametr qiymatlarining ta'sirini baholash, maxsus holatlarni ajratish va yakuniy javobning to'liqligini tekshirishga xizmat qiladi. To'rtinchidan, tadqiqot natijasida parametrli masalalarni ikki asosiy metodik turga ajratish mumkinligi aniqlandi. Birinchi turdagi masalalarda parametrning barcha o'rinli qiymatlari uchun yechimlar to'plamini topish talab qilinadi. Ikkinchi turdagi masalalarda esa parametrning shunday qiymatlari aniqlanadiki, bu qiymatlarda xususiy masala oldindan berilgan xossaga ega bo'ladi. Masalan, tenglamaning yagona yechimga ega bo'lishi, tengsizlikning barcha haqiqiy qiymatlar uchun bajarilishi yoki sistemaning yechimga ega bo'lishi kabi shartlar ikkinchi turdagi masalalarga kiradi. Beshinchidan, parametrli masalalarni yechishda umumiy yechim tushunchasi muhim nazariy-metodik ahamiyatga ega ekani aniqlandi.

Munozara. Tadqiqot natijalari parametrli masalalarni yechishda algebraik va mantiqiy tahlilni o'zaro bog'liq holda tashkil etish zarurligini ko'rsatdi. Parametrning o'rinli qiymatlari sohasini aniqlash, parametr sohasini muhim holatlarga ko'ra qismlarga ajratish va har bir xususiy holatni alohida tekshirish yechim jarayonining metodik izchilligini ta'minlaydi. Mazkur yondashuv masala aniqlanmagan holat bilan yechim mavjud bo'lmagan holatni farqlashga, o'quvchilarda tahliliy fikrlash, tasniflash va umumlashtirish ko'nikmalarini rivojlantirishga xizmat qiladi. Shu bilan birga, tadqiqot nazariy-metodik xarakterga ega bo'lgani sababli, taklif etilgan bosqichlarning samaradorligini kelgusida tajriba-sinov ishlari orqali asoslash maqsadga muvofiqdir.

Xulosa. Parametrli masalalarni yechishda algebraik va mantiqiy tahlilning uyg'un qo'llanilishi yechim jarayonini izchil va asosli tashkil etishga xizmat qiladi. Bunda parametrning o'rinli qiymatlari sohasini aniqlash, uni muhim holatlarga ajratish, xususiy holatlarni tekshirish va natijalarni umumlashtirish asosiy metodik bosqichlar sifatida namoyon bo'ladi. Mazkur yondashuv o'quvchilarning parametrli masalalarni faqat hisoblash emas, balki tahlil qilish, tasniflash va mantiqiy asoslash ko'nikmalarini rivojlantiradi. Shu bois algebraik va mantiqiy tahlil parametrli tenglama va tengsizliklarni o'qitishning samarali metodik vositasi sifatida qaralishi mumkin.

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FOK FAZOSIDA UMUMLASHGAN FRIDRIXS MODELINING SHUR TO‘LDIRUVCHILARI VA ULARNING MATEMATIK TAHLILI

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Annotatsiya. *Ushbu maqolada Fok fazosining qirqilgan ikki zarrachali qism fazosida aniqlangan umumlashgan Fridrixs modeli batafsil o‘rganiladi va u 2-tartibli operatorli matritsa ko‘rinishida ifodalanadi. Modelning muhim (uzluksiz) hamda diskret spektrlari to‘liq ravishda tavsiflanadi. Tadqiqot jarayonida modelga mos birinchi va ikkinchi Shur to‘ldiruvchilari konstruktiv ravishda qurilib, ularning operatorlar nazariyasidagi o‘rni va ahamiyati asoslab beriladi. Shuningdek, ushbu Shur to‘ldiruvchilar yordamida operatorli matritsaning spektral xossalari tegishli teoremalar sifatida bayon qilinadi hamda qat’iy isbotlanadi. Olingan natijalar umumlashgan Fridrixs modellari nazariyasini rivojlantirishda hamda Fok fazosida qaralayotgan kvant tizimlarning matematik modellarini yanada chuqurroq tushunishda muhim ahamiyat kasb etadi.*

Kalit so‘zlar: *umumlashgan Fridrixs modeli, Fok fazosi, operatorli matritsa, muhim spektr, diskret spektr, Shur to‘ldiruvchisi, Fredgolm determinanti, xos qiymat, spektral tahlil.*

ДОПОЛНЕНИЯ ШУРА ОБОБЩЁННОЙ МОДЕЛИ ФРИДРИХСА В ПРОСТРАНСТВЕ ФОКА И ИХ МАТЕМАТИЧЕСКИЙ АНАЛИЗ

Аннотация. *В данной статье подробно исследуется обобщённая модель Фридрихса, заданная в усечённом двухчастичном подпространстве пространства Фока и представленная в виде операторной матрицы второго порядка. Полностью описаны существенный (непрерывный) и дискретный спектры модели. В ходе исследования конструктивно построены первый и второй дополнители Шура, соответствующие данной модели, и обоснованы их роль и значение в теории операторов. Кроме того, с помощью этих дополнителей Шура спектральные свойства операторной матрицы сформулированы в виде соответствующих теорем и строго доказаны. Полученные результаты имеют важное значение для дальнейшего развития теории обобщённых моделей Фридрихса, а также для более глубокого понимания математических моделей квантовых систем, рассматриваемых в пространстве Фока.*

Ключевые слова: *обобщённая модель Фридрихса, пространство Фока, операторная матрица, существенный спектр, дискретный спектр, дополнение Шура, определитель Фредгольма, собственное значение, спектральный анализ.*

SCHUR COMPLEMENTS OF THE GENERALIZED FRIEDRICHS MODEL IN FOCK SPACE AND THEIR MATHEMATICAL ANALYSIS

Abstract. *This article presents a study of the generalized Friedrichs model defined in the truncated two-particle subspace of Fock space and represented in the form of a second-order operator matrix. The essential and discrete spectra of the model are characterized. The first and second Schur complements corresponding to the model are constructed, and their role in operator theory is substantiated. By using these Schur complements, the spectral properties of the operator matrix are formulated as theorems and rigorously proved. The results are important for the development of generalized Friedrichs models and for understanding mathematical models of quantum systems in Fock space.*

Keywords: *generalized Friedrichs model, Fock space, operator matrix, essential spectrum, discrete spectrum, Schur complement, Fredholm determinant, eigenvalue, spectral analysis.*

Kirish. Zamonaviy funksional analiz va operatorlar nazariyasida Hilbert fazosidagi chiziqli chegaralangan va chegaralanmagan operatorlarning spektral xossalarini o‘rganish masalasi muhim ilmiy yo‘nalishlardan biri hisoblanadi. Ayniqsa, murakkab tuzilishga ega bo‘lgan blok operatorli matritsalarni, ya’ni elementlari chiziqli operatorlardan iborat matritsalarni tahlil qilishda samarali usullarni ishlab chiqish

dolzarb masalalardan biridir. Shu nuqtayi nazardan, Shur to‘ldiruvchilari (Schur complements) nazariyasi nafaqat algebraik, balki analitik jihatdan ham keng qo‘llaniladigan kuchli vosita sifatida ajralib turadi.

Shur to‘ldiruvchisi tushunchasi dastlab chekli o‘lchamli matritsalar uchun kiritilgan bo‘lsa-da, keyinchalik u cheksiz o‘lchamli fazolardagi operatorlarga ham muvaffaqiyatli umumlashtirildi. Ushbu yondashuv blok operatorli matritsalarni soddalashtirish, ularning spektrini aniqlash, hamda xos qiymatlar va xos funksiyalarni tadqiq qilishda muhim rol o‘ynaydi. Xususan, Shur to‘ldiruvchisi yordamida berilgan operatorni kichikroq o‘lchamli operatorlar orqali ifodalash va shu orqali murakkab masalalarni nisbatan sodda ko‘rinishga keltirish mumkin [1, 2].

Bundan tashqari, Shur to‘ldiruvchilari kvant mexanikasi, matematik fizika, sonli analiz va differensial tenglamalar nazariyasida uchraydigan ko‘plab masalalarni yechishda ham keng qo‘llaniladi. Ayniqsa, ko‘p zarrachali sistemalar, panjarali modellar va Fridrixs tipidagi operatorlar uchun spektral tahlilni amalga oshirishda ushbu usul samarali natijalar beradi.

Mazkur maqolada Fok fazosining qirqilgan ikki zarrachali qism fazosida ikkinchi tartibli operatorli matritsa ko‘rinishda tasvirlanuvchi umumlashgan Fridrixs modelining muhim spektri topilgan. Uning Fredgolm determinantini qurish orqali diskret spektri tavsiflangan. Mos birinchi va ikkinchi Shur to‘ldiruvchilari qurilgan, ularning ayrim asosiy xossalari tahlil qilingan. Shuningdek, Shur to‘ldiruvchilari yordamida muhim spektrni tavsiflash va xos qiymatlarni aniqlashga doir ayrim natijalar keltirilgan [3].

Mavzuga doir adabiyotlar tahlili. Shur to‘ldiruvchilari haqida. Bizga $\mathcal{H}_1$ va $\mathcal{H}_1$ Hilbert fazolari berilgan bo‘lib,

$$\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$$

to‘g‘ri yig‘indi berilgan bo‘lsin. U holda $\mathcal{H}$ Hilbert fazosida aniqlangan har qanday $\mathcal{A}$ chiziqli, chegaralangan operator hamisha ikkinchi tartibli operatorli matritsa (1) ko‘rinishida tasvirlanadi:

$$\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{1}$$

bu yerda $A_{ij}: \mathcal{H}_j \to \mathcal{H}_i \quad i,j = 1,2$ matritsaviy elementlar chiziqli, chegaralangan operatorlardir. Ma’lumki, $\mathcal{A}$ o‘z-o‘ziga qo‘shma operator bo‘lishi uchun barcha $i,j = 1,2$ uchun $\mathcal{H}_{ij}{}^* = \mathcal{H}_{ji}$ tengliklarning bajarilishi zarur va yetarlidir.

Shur to‘ldiruvchilari dastlab matritsalar nazariyasida qo‘llanilgan. (1) ko‘rinishdagi blok matritsa bilan bog‘liq bo‘lgan $A_{22} - A_{21}A_{11}A_{12}$ tipidagi ifodani qarash g‘oyasi kamida Issai Schur [4] ishlariga borib taqaladi, balki undan ham oldin James Joseph Sylvester tadqiqotlarida uchrashi mumkin. “Shur to‘ldiruvchisi” (Schur complement) atamasi esa 1968-yilda Emilie Virginia Haynsworth [5] tomonidan kiritilgan bo‘lib, u bo‘lingan (blokli) matritsalarni o‘rganish jarayonida ushbu tushunchani ilmiy muomalaga olib kirgan. Hilbert fazolarda Shur to‘ldiruvchilari ilk bor Mark Grigorievich Kreinning o‘z-o‘ziga qo‘shma operatorlarning kengaytmalari haqidagi mashhur [6] ishida “qisqartirilgan operatorlar” (shorted operators) nomi ostida uchraydi. Matritsalar nazariyasi va sonli chiziqli algebra sohalaridagi ko‘plab qo‘llanilishlaridan tashqari, Shur to‘ldiruvchilari statistika, elektrotexnika hamda matematik tizimlar nazariyasi kabi ko‘plab boshqa yo‘nalishlarda ham keng qo‘llaniladi. Xususan, ular chiziqli vaqtga nisbatan invariant tizimlarning holat fazosidagi realizatsiyalarida uzatish funksiyalari sifatida namoyon bo‘ladi. Chegaralangan va chegaralanmagan blok operatorli matritsalar nazariyasida Shur to‘ldiruvchilari spektr hamda turli spektral xossalarni o‘rganishda kuchli vosita hisoblanadi. Bu holat ilk bor Rolf Nagel [7] tomonidan qayd etilgan. U “diagonal soha” va “diagonal bo‘lmagan soha” tushunchalari asosida chegaralanmagan operatorli matritsalar uchun o‘ziga xos “matritsalar nazariyasi”ni rivojlantirishni boshlab bergan.

$\mathcal{A}$ blok operatorli matritsaning spektral xossalari bilan uning Shur to‘ldiruvchilari bo‘lgan

$$S_1(z) = A_{11} - z - A_{12}(A_{22} - z)^{-1}A_{21} \quad z \notin \sigma(A_{22})$$
$$S_2(z) = A_{22} - z - A_{21}(A_{11} - z)^{-1}A_{12} \quad z \notin \sigma(A_{11})$$

operatorlarning spektral xossalari o‘rtasidagi yaqin bog‘liqlik, masalan,

$$\mathcal{A} - z = \begin{pmatrix} I & 0 \\ A_{21}(A_{11} - z)^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} - z & 0 \\ 0 & S_2(z) \end{pmatrix} \begin{pmatrix} I(A_{11} - z)^{-1} & A_{12} \\ 0 & I \end{pmatrix} \qquad z \notin \sigma(A_{11})$$

uchun qo‘llaniladigan Frobenius–Shur faktorizatsiyalari hamda $R(z)$ rezolventa uchun mos faktorizatsiyalardan yaqqol ko‘rinadi.

[8] maqolada haqiqiy z spektral parametrga bog‘liq bo‘lgan, 3×3 o‘lchamli blok operatorli matritsaga mos keluvchi birinchi Shur to‘ldiruvchisi $S_1(z)$ ko‘rib chiqilgan $S_1(z)$ operatorning muhim spektri lakunalarni o‘z ichiga olishi mumkin bo‘lgan holat tahlil qilingan. $S_1(z)$ operatorining spektral tahlilidagi muhim masalalardan biri - muhim spektrdan tashqarida joylashgan xos qiymatlar sonining chekli yoki cheksiz ekanligini aniqlashdan iborat. Bu maqolada $S_1(z)$ operatorining diskret spektri batafsil o‘rganilgan hamda tadqiq qilinayotgan matritsa operatori bilan $S_1(z)$ operatorining xos qiymatlari o‘rtasidagi bog‘liqlik aniqlangan. Shuningdek, $S_1(z)$ operatorining xos vektor-funksiyalari uchun Faddeyev

tenglamasining simmetriklashtirilgan analogi qurilgan va xos qiymatlar soni bilan bog‘liq ayrim asosiy xossalar isbotlangan. Mualliflar tomonidan $S_1(z)$ operatorining muhim spektridan tashqarida joylashgan ixtiyoriy intervaldagi xos qiymatlar soni va ularning karraliligini aniqlash formulalari keltirilgan. Olingan formulalar lakunada joylashgan diskret spektrning taqsimoti uchun asimptotik baholarni hosil qilish imkonini beradi. Bundan tashqari $S_1(z)$ operatorining diskret spektri chekli bo‘lishi uchun yetarli shartlar topilgan. $S_1(z)$ operatorining xossalaridan kelib chiqib, operatorli matritsa uchun ikkita muhim xulosa chiqarilgan.

[9] maqolada panjaradagi bitta qo‘zg‘almas atom va ko‘pi bilan ikkita fotondan iborat spin-boson modeli ko‘rib chiqilgan. Bu modelga mos keluvchi z spektral parametrga bog‘liq bo‘lgan $S_1(z)$ birinchi Shur to‘ldiruvchisi qurilgan. Spin-bozon modeli uchun $S_1(z)$ operatorga nisbatan Birman–Shvinger prinsipi isbotlangan. Shuningdek, tadqiq qilingan model xos qiymatlari soni bilan bog‘liq bo‘lgan $S_1(z)$ operatorining muhim xossalari torning barcha o‘lchamlari hamda ixtiyoriy bog‘lanish konstantasi uchun tadqiq etilgan.

[10] maqolada fermionli Fok fazosida ta’sir qiluvchi uchinchi tartibli chegaralangan, o‘z-o‘ziga qo‘shma operatorli matritsa ko‘rib chiqilgan. Avvalo, uning spektral xossalarini o‘rganish maqsadida uchinchi tartibli diskret parametrli operatorli matritsa kiritilgan. So‘ngra, mazkur operatorli matritsaga mos keluvchi birinchi Shur to‘ldiruvchisi qurilgan va undan foydalanib, operatorli matritsaga mos birinchi Shur to‘ldiruvchisi aniqlangan. Uchinchi tartibli matritsaning xos qiymatlari bilan unga mos birinchi Shur to‘ldiruvchining xos qiymatlari o‘rtasidagi bog‘liqlik o‘rnatilgan. Shuningdek, operatorli matritsasining birinchi Shur to‘ldiruvchisining xos qiymatlariga bog‘liq bo‘lgan funksiyaning λ spektral parametrga nisbatan monoton o‘suvchi va uzluksiz ekanligi isbotlangan

Umumlashgan Fridrixs modeli va uning spektri

Dastlab, $\mathbb{T}^d$ orqali d-o‘lchamli torni, $\mathcal{H}_1 := \mathbb{C}$ orqali bir o’lchamli kompleks sonlar fazosini, $\mathcal{H}_2 := L_2(\mathbb{T}^d)$ orqali $\mathbb{T}^d$ torda kvadrati bilan integrallanuvchi funksiyalarning Hilbert fazoni belgilaymiz. Ushbu fazolarning to’g’ri yig’indisini

$$\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$$

kabi aniqlaymiz.

$\mathcal{H}$ fazoda

$$\mathcal{A} := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{1}$$

operatorli matritsani qaraymiz. $\mathcal{A}$ operatorli matritsaning $A_{\mathrm{ij}}, \mathrm{i,j} = 1,2$ elementlari quyidagicha aniqlanadi:

$$A_{11}f_1 = af_1, \qquad A_{12}f_2 = \int_{\mathbb{T}^d} v(t)\, f_2(t)\, dt,$$

$$(A_{21}f_1)(x) = v(x)f_1\,, \quad (A_{22}f_2)(x) = u(x)f_2(x),$$

bu yerda, a ixtiyoriy haqiqiy son, $f_i \in \mathcal{H}_i, i = 1,2$, $u(\cdot)$ va $v(\cdot)$ funksiyalar $\mathbb{T}^d$ torda aniqlangan haqiqiy qiymatli uzluksiz funksiyalardir.

Operatorli matritsaning matritsaviy elementlari aniqlanishiga ko‘ra, $A_{21} = A_{12}^*$ munosabat bajariladi. Demak, $\mathcal{A}$ operatorli matritsa o‘z-o‘ziga qo‘shma operatorni ifodalaydi.

Endi $\mathcal{A}$ operator matritsaning muhim spektrini aniqlash uchun quyidagi operatorni qaraymiz:

$$\mathcal{A}_0 = \begin{pmatrix} 0 & 0 \\ 0 & A_{22} \end{pmatrix}.$$

Bu holda $\mathcal{A} - \mathcal{A}_0$ chekli o‘lchamli qo‘zg‘alish operatori bo‘ladi. Chekli o’lchamli qo’zg’alishlarda muhim spektrning o’zgarmasligi haqidagi Veyl teoremasiga ko‘ra, $\mathcal{A}$ operatorli matritsaning muhim spektri uchun

$$\sigma_{\mathrm{ess}}(\mathcal{A}) = \sigma_{\mathrm{ess}}(\mathcal{A}_0) = \sigma(A_{22}).$$

tenglik o’rinli. A_{22} operator aniqlanishiga ko’ra ko‘paytirish operatori bo‘lgani sababli uning spektri $u(x)$ funksiyaning qiymatlar to‘plami bilan aniqlanadi, ya’ni

$$\sigma_{\mathrm{ess}}(\mathcal{A}) = [m, M],$$

bu yerda

$$m := \min_{x\in\mathbb{T}^d} u(x), \qquad M = \max_{x\in\mathbb{T}^d} u(x).$$

Endi $\mathcal{A}$ operatorli matritsaning diskret spektrini tavsiflash maqsadida $\mathbb{C} \setminus [m, M]$ sohada regulyar bo’lgan

$$\Delta(z) = a - z - \int_{\mathbb{T}^d} \frac{v^2(t)}{u(t) - z}\, dt$$

funksiyani qaraymiz. Mazkur $\Delta(z)$ funksiya $\mathcal{A}$ operatorli matritsaga mos Fredgolm determinanti deyiladi. $\mathcal{A}$ operatorning diskret spektri shu determinantning nollari orqali tavsiflanadi, ya'ni

$$\sigma_{\text{disc}}(\mathcal{A}) = \{z \in \mathbb{C} \setminus [m, M] : \Delta(z) = 0\}.$$

Demak, $\mathcal{A}$ operatorli matritsaning xos qiymatlarini topish masalasi Fredgolm determinantining nollarini topish masalasiga keltiriladi.

Birinchi va ikkinchi Shur to'ldiruvchilari

Faraz qilaylik, z soni A_{22} operatorning regulyar nuqtalar to'plamiga tegishli bo'lsin, ya'ni $z \in \rho(A_{22})$ bo'lsin. U holda $\mathcal{A}$ operatorli matritsaga mos birinchi Shur to'ldiruvchisi

$$S_1(z) = A_{11} - zI - A_{12}(A_{22} - zI)^{-1}A_{21}$$

formula orqali aniqlanadi. Berilgan model uchun birinchi Shur to'ldiruvchining ta'siri quyidagicha bo'ladi:

$$(S_1(z)f_1)(x) = \left(a - z - \int_{\mathbb{T}^d} \frac{v^2(t)}{u(t) - z}\, dt\right) f_1 = \Delta(z) f_1.$$

Faraz qilaylik, $z \in \rho(A_{11})$ bo'lsin. U holda $\mathcal{A}$ operatorli matritsaga mos ikkinchi Shur to'ldiruvchisi

$$S_2(z) = A_{22} - zI - A_{21}(A_{11} - zI)^{-1}A_{12}$$

formula orqali aniqlanadi. Berilgan modelda ikkinchi Shur to'ldiruvchining ta'siri

$$(S_2(z)f_2)(x) = (u(x) - z)f_2(x) - \frac{v(x)}{a - z}\int_{\mathbb{T}^d} v(t)\, f_2(t)\, dt, f_2 \in H_2,$$

ko'rinishda ifodalanadi.

1-teorema. Agar $z \in \mathbb{R} \setminus [m, M]$ bo'lsa, u holda z soni $\mathcal{A}$ operatorli matritsaning xos qiymati bo'lishi uchun 0 soni $S_1(z)$ operator uchun xos qiymat bo'lishi zarur va yetarlidir.

Isbot. Zaruriyligi. Faraz qilaylik, z soni $\mathcal{A}$ operatorli matritsaning xos qiymati bo'lsin. U holda shunday noldan farqli $f = (f_1, f_2)$, bu yerda $f_1 \in \mathcal{H}_1,\ f_2 \in \mathcal{H}_2$ vektor topilib, $\mathcal{A}f = zf$ tenglik bajariladi. Bu esa quyidagi sistemaga ekvivalent:

$$\begin{cases}(A_{11} - zI)f_1 + A_{12}f_2 = 0,\\ A_{21}f_1 + (A_{22} - zI)f_2 = 0.\end{cases}$$

$z \in \mathbb{R} \setminus [m, M]$ bo'lgani uchun $A_{22} - zI$ operatori teskari operatorga ega. Shuning uchun sistemaning ikkinchi tenglamasidan

$$f_2 = -(A_{22} - zI)^{-1}A_{21}f_1 \tag{2}$$

Ushbu (2) ifodani birinchi tenglamaga qo'ysak,

$$(A_{11} - zI)f_1 - A_{12}(A_{22} - zI)^{-1}A_{21}f_1 = 0$$

tenglikka ega bo'lamiz. Bundan

$$(A_{11} - zI - A_{12}(A_{22} - zI)^{-1}A_{21})f_1 = 0$$

ifoda kelib chiqadi. Demak,

$$S_1(z)f_1 = 0.$$

Ushbu tenglikda $f_1 = 0$ bo'lishi mumkin emas. Aks holda yuqoridagi formuladan $f_2 = 0$ kelib chiqadi. Bu esa $f \neq 0$ ga zid. Demak, $f_1 \neq 0$, ya'ni 0 soni $S_1(z)$ operatorning xos qiymati bo'ladi.

Yetarliligi. Faraz qilaylik, 0 soni $S_1(z)$ operatorning xos qiymati bo'lsin. U holda shunday $f_1 \neq 0$ mavjudki, $S_1(z)f_1 = 0$. Endi

$$f_2 := -(A_{22} - zI)^{-1}A_{21}f_1$$

deb olamiz. U holda

$$A_{21}f_1 + (A_{22} - zI)f_2 = 0$$

tenglik bajariladi. Bundan tashqari,

$$(A_{11} - zI)f_1 + A_{12}f_2 = (A_{11} - zI)f_1 - A_{12}(A_{22} - zI)^{-1}A_{21}f_1 = S_1(z)f_1 = 0.$$

Shunday qilib,

$$\mathcal{A}\begin{pmatrix}f_1\\ f_2\end{pmatrix} = z\begin{pmatrix}f_1\\ f_2\end{pmatrix}.$$

Demak, z soni operatorli matritsa $\mathcal{A}$ ning xos qiymati bo'ladi.

2-teorema. Agar $z \in \mathbb{R} \setminus ([m, M] \cup \{a\})$ bo'lsa, u holda z soni $\mathcal{A}$ operatorli matritsaning xos qiymat bo'lishi uchun 0 soni $S_2(z)$ operatorning xos qiymati bo'lishi zarur va yetarlidir.

Isbot. Zaruriyligi. Faraz qilaylik, z soni $\mathcal{A}$ operatorli matritsaning xos qiymati bo'lsin. U holda shunday noldan farqli $f = (f_1, f_2),\ f_1 \in \mathcal{H}_1,\ f_2 \in \mathcal{H}_2$ vektor mavjud bo'lib, $\mathcal{A}f = zf$ tenglik bajariladi. Bu esa quyidagi sistemaga ekvivalent:

$$\begin{cases}(A_{11}-zI)f_1+A_{12}f_2=0,\\ A_{21}f_1+(A_{22}-zI)f_2=0.\end{cases}$$

$z\in\mathbb{R}\setminus([m,M]\cup\{a\})$ bo‘lgani uchun $A_{11}-zI$ operatori teskari operatorga ega. Shuning uchun sistemaning birinchi tenglamasidan

$$f_1=-(A_{11}-zI)^{-1}A_{12}f_2 \qquad (3)$$

hosil bo‘ladi. (3) ifodani ikkinchi tenglamaga qo‘ysak,

$$-A_{21}(A_{11}-zI)^{-1}A_{12}f_2+(A_{22}-zI)f_2=0$$

ifodaga ega bo‘lamiz. Bundan

$$(A_{22}-zI-A_{21}(A_{11}-zI)^{-1}A_{12})f_2=0$$

munosabat kelib chiqadi. Demak,

$$S_2(z)f_2=0.$$

Ushbu tenglamadan $f_2=0$ bo‘lishi mumkin emas. Aks holda yuqoridagi formuladan $f_1=0$ kelib chiqadi. Bu esa $f\neq 0$ shartga zid. Demak, $f_2\neq 0$, ya’ni 0 soni $S_2(z)$ operatorning xos qiymati bo‘ladi.

Yetarliligi. Faraz qilaylik, 0 soni $S_2(z)$ operatorning xos qiymati bo‘lsin. U holda shunday $f_2\neq 0$ mavjudki, $S_2(z)f_2=0$. Endi

$$f_1:=-(A_{11}-zI)^{-1}A_{12}f_2$$

deb olamiz. U holda

$$(A_{11}-zI)f_1+A_{12}f_2=0$$

tenglik bajariladi. Bundan tashqari,

$$A_{21}f_1+(A_{22}-zI)f_2=-A_{21}(A_{11}-zI)^{-1}A_{12}f_2+(A_{22}-zI)f_2=S_2(z)f_2=0.$$

Shunday qilib,

$$\mathcal{A}\begin{pmatrix}f_1\\ f_2\end{pmatrix}=z\begin{pmatrix}f_1\\ f_2\end{pmatrix}$$

tenglik hosil bo‘ladi. Demak, z soni operatorli matritsa $\mathcal{A}$ ning xos qiymati bo‘ladi.

Xulosa. Ushbu maqolada Fok fazosining qirqilgan ikki zarrachali qism fazosida aniqlangan umumlashgan Fridrixs modeli kompleks va tizimli yondashuv asosida batafsil tadqiq etildi hamda u 2-tartibli operatorli matritsa ko‘rinishida aniq ifodalandi. Tadqiqot davomida model operatorlarning muhim (uzluksiz) spektri hamda Fredgolm determinantini topish orqali diskret spektri to‘liq va izchil ravishda tavsiflandi, spektrlarning tuzilishi va o‘zaro bog‘liqligi aniqlashtirildi.

Shuningdek, modelga mos birinchi va ikkinchi Shur to‘ldiruvchilari konstruktiv usulda qurilib, ularning operatorlar nazariyasidagi o‘rni, xossalari hamda spektral tahlildagi roli asoslab berildi. Olingan natijalar umumlashgan Fridrixs modellari doirasida yangi ilmiy yondashuvlarni shakllantirishga xizmat qiladi hamda Fok fazosida qaralayotgan kvant tizimlarning matematik modellarini yanada mukammal tushunish imkonini beradi. Shu bilan birga, mazkur tadqiqot natijalari operatorlar nazariyasi, spektral tahlil va matematik fizikaning tegishli yo‘nalishlarida keyingi ilmiy izlanishlar uchun nazariy asos bo‘lib xizmat qilishi mumkin.

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CONVERGENCE ARIES OF A NEW TRIVARIATE MITTAG-LEFFLER-TYPE FUNCTION

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Abstract. *This article studies the main analytical properties of a new three-variable Mittag-Leffler type function. The convergence domains of the corresponding triple power series are determined. The convergence analysis is carried out based on Horn's convergence theory and the asymptotic properties of the gamma function for large arguments. Twenty-seven different combinations of parameters are considered, and for each case necessary and sufficient conditions for the absolute convergence of the series are established, along with some particular cases of ML. The obtained results enrich the theory of fractional differential equations and provide an effective analytical basis for investigating problems involving the Prabhakar fractional derivative.*

Key words: *Mittag-Leffler function, trivariate function, fractional calculus, Prabhakar derivative, triple power series, region of convergence, gamma function, special functions.*

О СВОЙСТВАХ СХОДИМОСТИ НОВОЙ ТРЁХПЕРЕМЕННОЙ ФУНКЦИИ ТИПА МИТТАГА-ЛЕФФЛЕРА

Аннотация. *В данной статье исследуются основные аналитические свойства новой функции типа Миттаг-Леффлера от трёх переменных. Определены области сходимости соответствующего тройного степенного ряда. Анализ сходимости проведён на основе теории сходимости Хорна, а также асимптотических свойств гамма-функции при больших значениях аргументов. Рассмотрены двадцать семь различных комбинаций параметров, и для каждого случая установлены необходимые и достаточные условия абсолютной сходимости ряда, а также приведены некоторые частные случаи ML. Полученные результаты дополняют теорию дифференциальных уравнений дробного порядка и представляют эффективную аналитическую основу для исследования задач, включающих дробную производную Прабхакара.*

Ключевые слова: *функция Миттага-Леффлера, трёхпеременная функция, дробное исчисление, производная Прабхакара, тройной степенной ряд, область сходимости, гамма-функция, специальные функции.*

YANGI UCH O‘ZGARUVCHILI MITTAG -LEFFLER TURIDAGI FUNKSIYANING YAQINLASHISH XOSSALARI

Annotatsiya. *Ushbu maqolada yangi uch o‘zgaruvchili Mittag-Leffler turidagi funksiyaning asosiy analitik xossalari o‘rganilgan. Mos uch karrali darajali qatorning yaqinlashish sohalari aniqlangan. Yaqinlashish tahlili Hornning yaqinlashish nazariyasi hamda gamma-funksiyaning katta argumentlardagi asimptotik xossalaridan foydalanish asosida amalga oshirilgan. Parametrlarning yigirma yettita turli kombinatsiyasi ko‘rib chiqilgan va har bir holat uchun qatorning mutlaq yaqinlashishi uchun zarur va yetarli shartlar aniqlangan, hamda MLning ba’zi xususiy hollari keltirilgan. Olingan natijalar kasr tartibli differensial tenglamalar nazariyasini boyitadi hamda Prabhakar kasr hosilasini o‘z ichiga oluvchi masalalarni tadqiq etishda samarali analitik asosni taqdim etadi.*

Kalit so‘zlar: *Mittag-Leffler funksiyasi, uch o‘zgaruvchili funksiya, kasr tartibli hisob, Prabhakar hosilasi, uch karrali darajali qator, yaqinlashish sohasi, gamma-funksiya, maxsus funksiyalar.*

Introduction. It should be noted that, numerous studies have investigated special functions of three variables. For instance, the bivariate and $\bar{F}_E^{(3)}(x,y,z)$ trivariate Mittag-Leffler (ML)-type functions were introduced in [1], where their properties were examined and their application to a generalized telegraph equation involving the regularized Prabhakar derivative was demonstrated. In [2], confluent three-variable hypergeometric functions and their fundamental properties were explored. Furthermore, several additional three-variable ML-type functions were proposed in [3] and [4]: $\bar{F}_A^{(3)}(x,y,z)$, $\bar{F}_B^{(3)}(x,y,z)$, $\bar{F}_C^{(3)}(x,y,z)$,

$\bar{F}_D^{(3)}(x,y,z)$ and $E^{(A)}_{(\alpha_1,\beta_1,\gamma_1,\delta_1),(\alpha_2,\beta_2,\gamma_2,\delta_2),(\alpha_3,\beta_3,\gamma_3,\delta_3)}(x,y,z)$, $E^{(B,C)}_{(\alpha_1,\beta_1,\gamma_1,\delta_1),(\alpha_2,\beta_2,\gamma_2,\delta_2),(\alpha_3,\beta_3,\gamma_3,\delta_3)}(x,y,z)$ $E^{(P,Q,R)}_{(\alpha_1,\beta_1,\gamma_1,\delta_1),(\alpha_2,\beta_2,\gamma_2,\delta_2),(\alpha_3,\beta_3,\gamma_3,\delta_3)}(x,y,z)$,

the latter being applied to coupled systems of linear constant-coefficient fractional differential equations with Caputo derivatives. The three-variable ML-type function $E^{(\eta)}_{\alpha,\beta,\gamma,\delta}(x,y,z)$ was introduced and studied in [5]. Additionally, learn the $E_{13}\left(\begin{matrix}\alpha_1,\beta_1,\delta_1;\alpha_2,\delta_2;\\ \alpha_3,\beta_2,\delta_3;\alpha_4,\delta_4;\alpha_5,\delta_5;\beta_3,\delta_6;\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)$ functions with three parameters that were covered in [6]. In [7], new trivariate ${}_3E_{13}\left(\begin{matrix}\alpha_1,\beta_1,\delta_1;\beta_2,\gamma_1,\delta_2;\\ \alpha_2,\beta_3,\gamma_2,\delta_3;\beta_4,\delta_4;\alpha_3,\delta_5;\beta_5,\delta_6;\gamma_3,\delta_7;\end{matrix}\middle|\begin{matrix}x,y\\ z\end{matrix}\right)$ and quadrivariate

$$ {}_4E_1\left(\begin{matrix}\alpha_1,\beta_1,\gamma_1,\delta_1;\gamma_2,\theta_1,\delta_2;\\ \alpha_2,\beta_2,\gamma_3,\theta_2,\delta_3;\gamma_4,\delta_4;\alpha_3,\delta_5;\beta_3,\delta_6;\gamma_5,\delta_7;\theta_3,\delta_8;\end{matrix}\middle|\begin{matrix}x;y\\ z;t\end{matrix}\right) $$

Mittag-Leffler-type functions were introduced, and their fundamental analytical properties were investigated. In particular, their regions of convergence, integral representations, and explicit estimates were studied. Several notable special cases were also obtained. Furthermore, the applicability of the proposed functions to the study of a partial differential equation was demonstrated. In [8], a new trivariate ${}_3E_1\left(\begin{matrix}\alpha_1,\delta_1,\mu_1,\sigma_1;\alpha_2,\delta_2,\sigma_2;\\ \alpha_3,\delta_3,\mu_2,\sigma_3;\alpha_4,\delta_4,\sigma_4;\alpha_5,\sigma_5;\delta_5,\sigma_6;\mu_3,\delta_7;\end{matrix}\middle|x;y;z\right)$ Mittag-Leffler-type function was introduced and a Cauchy problem was solved using this function. In [9], an integral representation of this function was obtained, while in [10], an estimate for the function was established. However, its region of convergence has not been investigated. In this paper, we establish the region of convergence of this new trivariate Mittag–Leffler-type function and present several corresponding cases and particular cases.

New trivariate Mittag-Leffler-type function

Let us, recall a trivariate ML-type function investigated in [8].

Definition 2.1.

Let the assume that, $x,\ y,\ z,\ \alpha_l,\ \delta_l,\ \mu_i,\sigma_n \in \mathbb{R};\ \min\{\alpha_l,\delta_l,\mu_i,\sigma_n\}>0,\ \left(l=\overline{1,5},\ i=\overline{1,3},\ n=\overline{1,7}\right)$ .Then the new trivariate ML-type function was defined as follows:

$$ {}_3E_1\left(\begin{matrix}\alpha_1,\delta_1,\mu_1,\sigma_1;\alpha_2,\delta_2,\sigma_2;\\ \alpha_3,\delta_3,\mu_2,\sigma_3;\alpha_4,\delta_4,\sigma_4;\alpha_5,\sigma_5;\delta_5,\sigma_6;\mu_3,\sigma_7;\end{matrix}\middle|x;y;z\right)= $$

$$ =\sum_{m,j,k=0}^{+\infty}\frac{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2)}{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4)}\times $$

$$ \times\frac{x^m y^j z^k}{\Gamma(\alpha_5 m+\sigma_5)\Gamma(\delta_5 j+\sigma_6)\Gamma(\mu_3 k+\sigma_7)}. \qquad (1) $$

Now, we are going to determine convergence areas of the function (1) by using the Horn method which used in [11]. The triple series in (1) converges absolutely for $|x|<\rho_1$, $|y|<\rho_2$ and $|z|<\rho_3$, if $\Delta_1=0$, $\Delta_2=0$, $\Delta_3=0$ as described in

$$ \rho_1=\min_{\eta,\nu,\xi>0}(E_1),\ \rho_2=\min_{\eta,\nu,\xi>0}(E_2),\ \rho_3=\min_{\eta,\nu,\xi>0}(E_3),\ \eta,\nu,\xi>0 $$

where $\Delta_1=\alpha_3+\alpha_4+\alpha_5-\alpha_1-\alpha_2$, $\Delta_2=\delta_3+\delta_4+\delta_5-\delta_1-\delta_2$, $\Delta_3=\mu_2+\mu_3-\mu_1$.

Then, we obtain

$$ E_1=\eta^{\alpha_5}\frac{(\alpha_3\eta+\delta_3\nu+\mu_2\xi)^{\alpha_3}(\alpha_4\eta+\delta_4\nu)^{\alpha_4}(\alpha_5)^{\alpha_5}}{(\alpha_1\eta+\delta_1\nu+\mu_1\xi)^{\alpha_1}(\alpha_2\eta+\delta_2\nu)^{\alpha_2}}, $$

$$E_2 = \nu^{\delta_5}\frac{\left(\alpha_3\eta+\delta_3\nu+\mu_2\xi\right)^{\delta_3}\left(\alpha_4\eta+\delta_4\nu\right)^{\delta_4}\left(\delta_5\right)^{\delta_5}}{\left(\alpha_1\eta+\delta_1\nu+\mu_1\xi\right)^{\delta_1}\left(\alpha_2\eta+\delta_2\nu\right)^{\delta_2}},\ E_3 = \xi^{\mu_3}\frac{\left(\alpha_3\eta+\delta_3\nu+\mu_2\xi\right)^{\mu_2}\left(\mu_3\right)^{\mu_3}}{\left(\alpha_1\eta+\delta_1\nu+\mu_1\xi\right)^{\mu_1}},$$

respectively.

Theorem 2.1. Let $\Delta_1, \Delta_2, \Delta_3$ be the quantities defined above. Then the series associated with the trivariate Mittag-Leffler-type function converges absolutely according to the following cases:

Case 1. Let $\Delta_1 > 0$, $\Delta_2 > 0, \Delta_3 > 0$. Making use of equations (3)-(6), we obtain

$$\phi_1(\eta,\nu,\xi) = \lim_{t\to\infty} f_1(\eta t,\nu t,\xi t) = 0,\ \phi_2(\eta,\nu,\xi) = \lim_{t\to\infty} f_2(\eta t,\nu t,\xi t) = 0,$$
$$\phi_3(\eta,\nu,\xi) = \lim_{t\to\infty} f_3(\eta t,\nu t,\xi t) = 0.$$

Thus $\rho_1,\ \rho_2, \rho_3 \to \infty$, and the series converges absolutely for any x, y, z.

Case 2. Let $\Delta_1 = 0,\ \Delta_2 = 0,\ \Delta_3 = 0$. It follows from equations (3), (4), (5), and (6) that

$$\phi_1(\eta,\nu,\xi) = \frac{1}{E_1},\ \phi_2(\eta,\nu,\xi) = \frac{1}{E_2},\ \phi_3(\eta,\nu,\xi) = \frac{1}{E_3},$$

which yields the parametric form of curve C shown in the (ρ_1,ρ_2,ρ_3) - plane as $\rho_1 = G_1, \rho_2 = G_2$ and $\rho_3 = G_3$. Hence, absolute convergence of the series is ensured for $|x| < \rho_1,\ |y| < \rho_2$ and $|z| < \rho_3$, with $\rho_1 = \min_{(\eta,\nu,\xi)}(E_1),\ \rho_2 = \min_{(\eta,\nu,\xi)}(E_2)$ and $\rho_3 = \min_{(\eta,\nu,\xi)}(E_3)$.

Case 3. Let $\Delta_1 < 0,\ \Delta_2 < 0$ and $\Delta_3 < 0$. The series diverges for $\rho_1 = \rho_2 = \rho_3 = 0$. Convergence of the series occurs exclusively at the point $x = y = z = 0$.

Case 4. Assume that $\Delta_1 = 0,\ \Delta_2 > 0$ and $\Delta_3 > 0$. The series converges uniquely at the point $|x| < \rho_1,\ |y| < \infty$ and $|z| < \infty$.

Case 5. Suppose that $\Delta_1 > 0,\ \Delta_2 = 0$ and $\Delta_3 = 0$. The series is convergent solely at the point $|x| < \infty,\ |y| < \rho_2$ and $|z| < \rho_3$.

Case 6. Let $\Delta_1 > 0,\ \Delta_2 > 0$ and $\Delta_3 = 0$. Thus, the series converges the absolutely in the domain where $|x| < \infty,\ |y| < \infty$ and $|z| < \rho_3$.

Case 7. Let $\Delta_1 > 0,\ \Delta_2 > 0$ and $\Delta_3 < 0$. Thus, the series converges the absolutely in the domain where $|x| < \infty,\ |y| < \infty$ and $z = 0$.

Case 8. Let $\Delta_1 > 0,\ \Delta_2 < 0$ and $\Delta_3 > 0$. Thus, the series converges the absolutely in the domain where $|x| < \infty,\ y = 0$ and $|z| < \infty$.

Case 9. Let $\Delta_1 > 0,\ \Delta_2 < 0$ and $\Delta_3 < 0$. Thus, the series converges absolutely for $|x| < \infty,\ y = 0$, $z = 0$.

Case 10. Let $\Delta_1 > 0,\ \Delta_2 < 0$ and $\Delta_3 = 0$. Thus, the series converges absolutely for $|x| < \infty,\ y = 0$, $|z| < \rho_3$.

Case 11. Let $\Delta_1 > 0,\ \Delta_2 = 0$ and $\Delta_3 > 0$. Thus, the series converges absolutely for $|x| < \infty,\ |y| < \rho_2,\ |z| < \infty$.

Case 12. Let $\Delta_1 > 0,\ \Delta_2 = 0$ and $\Delta_3 < 0$. Thus, the series converges absolutely for $|x| < \infty,\ |y| < \rho_2,\ z = 0$.

Case 13. Let $\Delta_1 < 0,\ \Delta_2 > 0$ and $\Delta_3 > 0$. Thus, the series converges absolutely for $x = 0,\ |y| < \infty$, $|z| < \infty$.

Case 14. Let $\Delta_1 < 0,\ \Delta_2 > 0$ and $\Delta_3 < 0$. Thus, the series converges absolutely for $x = 0,\ |y| < \infty$, $z = 0$.

Case 15. Let $\Delta_1 < 0,\ \Delta_2 > 0$ and $\Delta_3 = 0$. Thus, the series converges absolutely for $x = 0,\ |y| < \infty$, $|z| < \rho_3$.

Case 16. Let $\Delta_1 < 0,\ \Delta_2 < 0$ and $\Delta_3 > 0$. Thus, the series converges absolutely for $x = 0, y = 0$, $|z| < \infty$.

Case 17. Let $\Delta_1 < 0,\ \Delta_2 < 0$ and $\Delta_3 = 0$. Thus, the series converges absolutely for $x = 0, y = 0$, $|z| < \rho_3$.

Case 18. Let $\Delta_1 < 0,\ \Delta_2 = 0$ and $\Delta_3 > 0$. Thus, the series converges absolutely for $x = 0,\ |y| < \rho_2$, $|z| < \infty$.

Case 19. Let $\Delta_1 < 0,\ \Delta_2 = 0$ and $\Delta_3 < 0$. Thus, the series converges absolutely for $x = 0,\ |y| < \rho_2$, $z = 0$.

Case 20. Let $\Delta_1 < 0,\ \Delta_2 = 0$ and $\Delta_3 = 0$. Thus, the series converges absolutely for $x = 0,\ |y| < \rho_2$, $z = 0$.

Case 21. Let $\Delta_1 = 0,\ \Delta_2 > 0$ and $\Delta_3 < 0$. Thus, the series converges absolutely for $|x| < \rho_1,\ |y| < \infty,\ z = 0$.

Case 22. Let $\Delta_1 = 0,\ \Delta_2 > 0$ and $\Delta_3 = 0$. Thus, the series converges absolutely for $|x| < \rho_1,\ |y| < \infty$, $|z| < \rho_3$.

Case 23. Let $\Delta_1 = 0,\ \Delta_2 < 0$ and $\Delta_3 > 0$. Thus, the series converges absolutely for $|x| < \rho_1,\ y = 0$, $|z| < \infty$.

Case 24. Let $\Delta_1 = 0,\ \Delta_2 < 0$ and $\Delta_3 < 0$. Thus, the series converges absolutely for $|x| < \rho_1,\ y = 0$, $z = 0$.

Case 25. Let $\Delta_1 = 0,\ \Delta_2 < 0$ and $\Delta_3 = 0$. Thus, the series converges absolutely for $|x| < \rho_1,\ y = 0$, $|z| < \rho_3$.

Case 26. Let $\Delta_1 = 0,\ \Delta_2 = 0$ and $\Delta_3 > 0$. Thus, the series converges absolutely for $|x| < \rho_1,\ |y| < \rho_2,\ |z| < \infty$.

Case 27. Let $\Delta_1 = 0,\ \Delta_2 = 0$ and $\Delta_3 < 0$. Thus, the series converges absolutely for $|x| < \rho_1,\ |y| < \rho_2,\ z = 0$.

It should be noted that the results for the remaining cases can be derived from these twenty seven cases.

Proof. Following Horn's approach [11], we determine the convergence region of the trivariate Mittag-Leffler-type function ${}_3E_1(.)$.To simplify the notation, let the following expression be denoted by $A(m, j, k)$:

$$A(m,j,k) = \frac{\Gamma(\alpha_1 m + \delta_1 j + \mu_1 k + \sigma_1)\Gamma(\alpha_2 m + \delta_2 j + \sigma_2)}{\Gamma(\alpha_3 m + \delta_3 j + \mu_2 k + \sigma_3)\Gamma(\alpha_4 m + \delta_4 j + \sigma_4)\Gamma(\alpha_5 m + \sigma_5)\Gamma(\delta_5 j + \sigma_6)\Gamma(\mu_3 k + \sigma_7)}, \quad (2)$$

We introduce the following notations:

$$f_1(m,j,k) = \frac{A(m+1,j,k)}{A(m,j,k)},\quad f_2(m,j,k) = \frac{A(m,j+1,k)}{A(m,j,k)},\quad f_3(m,j,k) = \frac{A(m,j,k+1)}{A(m,j,k)}. \quad (3)$$

Using the notations, we obtain $f_1(m,j,k)$, $f_2(m,j,k)$ and $f_3(m,j,k)$:

$$f_1(m,j,k) = \frac{\Gamma(\alpha_1 m + \delta_1 j + \mu_1 k + \sigma_1 + \alpha_1)\Gamma(\alpha_2 m + \delta_2 j + \sigma_2 + \alpha_2)}{\Gamma(\alpha_3 m + \delta_3 j + \mu_2 k + \sigma_3 + \alpha_3)\Gamma(\alpha_4 m + \delta_4 j + \sigma_4 + \alpha_4)} \times$$

$$\times \frac{1}{\Gamma(\alpha_5 m + \sigma_5 + \alpha_5)\Gamma(\delta_5 j + \sigma_6)\Gamma(\mu_3 k + \sigma_7)} \times$$

$$\times\frac{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4)}{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2)}\times\frac{\Gamma(\alpha_5 m+\sigma_5)\Gamma(\delta_5 j+\sigma_6)\Gamma(\mu_3 k+\sigma_7)}{1}=$$

$$=\frac{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1+\alpha_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2+\alpha_2)}{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2)}\times$$

$$\times\frac{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4)\Gamma(\alpha_5 m+\sigma_5)}{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3+\alpha_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4+\alpha_4)\Gamma(\alpha_5 m+\sigma_5+\alpha_5)},$$

similarly, we define

$$f_2(m,j,k)=\frac{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1+\delta_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2+\delta_2)}{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3+\delta_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4+\delta_4)}\times$$

$$\times\frac{1}{\Gamma(\alpha_5 m+\sigma_5)\Gamma(\delta_5 j+\sigma_6+\delta_5)\Gamma(\mu_3 k+\sigma_7)}\times$$

$$\times\frac{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4)}{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2)}\times\frac{\Gamma(\alpha_5 m+\sigma_5)\Gamma(\delta_5 j+\sigma_6)\Gamma(\mu_3 k+\sigma_7)}{1}=$$

$$=\frac{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1+\delta_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2+\delta_2)}{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2)}\times$$

$$\times\frac{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4)\Gamma(\delta_5 j+\sigma_6)}{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3+\delta_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4+\delta_4)\Gamma(\delta_5 j+\sigma_6+\delta_5)},$$

in the same manner

$$f_3(m,j,k)=\frac{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1+\mu_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2)}{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3+\mu_2)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4)}\times$$

$$\times\frac{1}{\Gamma(\alpha_5 m+\sigma_5)\Gamma(\delta_5 j+\sigma_6)\Gamma(\mu_3 k+\sigma_7+\mu_3)}\times$$

$$\times\frac{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4)}{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2)}\times\frac{\Gamma(\alpha_5 m+\sigma_5)\Gamma(\delta_5 j+\sigma_6)\Gamma(\mu_3 k+\sigma_7)}{1}=$$

$$=\frac{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1+\mu_1)\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3)\Gamma(\mu_3 k+\sigma_7)}{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1)\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3+\mu_2)\Gamma(\mu_3 k+\sigma_7+\mu_3)}.$$

Then, we have

$$f_1(\eta t,\nu t,\xi t)=\frac{\Gamma(\alpha_1\eta t+\delta_1\nu t+\mu_1\xi t+\sigma_1+\alpha_1)\Gamma(\alpha_2\eta t+\delta_2\nu t+\sigma_2+\alpha_2)}{\Gamma(\alpha_1\eta t+\delta_1\nu t+\mu_1\xi t+\sigma_1)\Gamma(\alpha_2\eta t+\delta_2\nu t+\sigma_2)}\times$$

$$\times\frac{\Gamma(\alpha_3\eta t+\delta_3\nu t+\mu_2\xi t+\sigma_3)\Gamma(\alpha_4\eta t+\delta_4\nu t+\sigma_4)\Gamma(\alpha_5\eta t+\sigma_5)}{\Gamma(\alpha_3\eta t+\delta_3\nu t+\mu_2\xi t+\sigma_3+\alpha_3)\Gamma(\alpha_4\eta t+\delta_4\nu t+\sigma_4+\alpha_4)\Gamma(\alpha_5\eta t+\sigma_5+\alpha_5)},$$

$$f_2(\eta t,\nu t,\xi t)=\frac{\Gamma(\alpha_1\eta t+\delta_1\nu t+\mu_1\xi t+\sigma_1+\delta_1)\Gamma(\alpha_2\eta t+\delta_2\nu t+\sigma_2+\delta_2)}{\Gamma(\alpha_1\eta t+\delta_1\nu t+\mu_1\xi t+\sigma_1)\Gamma(\alpha_2\eta t+\delta_2\nu t+\sigma_2)}\times$$

$$\times\frac{\Gamma(\alpha_3\eta t+\delta_3\nu t+\mu_2\xi t+\sigma_3)\Gamma(\alpha_4\eta t+\delta_4\nu t+\sigma_4)\Gamma(\delta_5\nu t+\sigma_6)}{\Gamma(\alpha_3\eta t+\delta_3\nu t+\mu_2\xi t+\sigma_3+\delta_3)\Gamma(\alpha_4\eta t+\delta_4\nu t+\sigma_4+\delta_4)\Gamma(\delta_5\nu t+\sigma_6+\delta_5)},$$

$$f_3(\eta t,\nu t,\xi t)=\frac{\Gamma(\alpha_1\eta t+\delta_1\nu t+\mu_1\xi t+\sigma_1+\mu_1)\Gamma(\alpha_3\eta t+\delta_3\nu t+\mu_2\xi t+\sigma_3)\Gamma(\mu_3\xi t+\sigma_7)}{\Gamma(\alpha_1\eta t+\delta_1\nu t+\mu_1\xi t+\sigma_1)\Gamma(\alpha_3\eta t+\delta_3\nu t+\mu_2\xi t+\sigma_3+\mu_2)\Gamma(\mu_3\xi t+\sigma_7+\mu_3)}.$$

Using the asymptotic properties of the Gamma function for large arguments given in [4],

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}\sim z^{\alpha-\beta}\left[1+\frac{(\alpha-\beta)(\alpha+\beta-1)}{2z}+O\left(z^{-2}\right)\right],\ \ |\arg(z)|\le\pi,$$

we obtain

$$f_1(\eta t,\nu t,\xi t)\to t^{\alpha_1+\alpha_2-\alpha_3-\alpha_4-\alpha_5}\,\frac{(\alpha_1\eta+\delta_1\nu+\mu_1\xi)^{\alpha_1}(\alpha_2\eta+\delta_2\nu)^{\alpha_2}}{(\alpha_3\eta+\delta_3\nu+\mu_2\xi)^{\alpha_3}(\alpha_4\eta+\delta_4\nu)^{\alpha_4}(\alpha_5\eta)^{\alpha_5}},$$

$$f_1(\eta t,\nu t,\xi t)\sim\frac{1}{E_1}t^{-\Delta_1},\ \Delta_1=\alpha_3+\alpha_4+\alpha_5-\alpha_1-\alpha_2, \tag{4}$$

$$E_1=\eta^{\alpha_5}\frac{(\alpha_3\eta+\delta_3\nu+\mu_2\xi)^{\alpha_3}(\alpha_4\eta+\delta_4\nu)^{\alpha_4}(\alpha_5)^{\alpha_5}}{(\alpha_1\eta+\delta_1\nu+\mu_1\xi)^{\alpha_1}(\alpha_2\eta+\delta_2\nu)^{\alpha_2}},\ G_1=\frac{(\alpha_3)^{\alpha_3}(\alpha_4)^{\alpha_4}(\alpha_5)^{\alpha_5}}{(\alpha_1)^{\alpha_1}(\alpha_2)^{\alpha_2}},$$

likewise we define

$$f_2(\eta t,\nu t,\xi t)\to t^{\delta_1+\delta_2-\delta_3-\delta_4-\delta_5}\,\frac{(\alpha_1\eta+\delta_1\nu+\mu_1\xi)^{\delta_1}(\alpha_2\eta+\delta_2\nu)^{\delta_2}}{(\alpha_3\eta+\delta_3\nu+\mu_2\xi)^{\delta_3}(\alpha_4\eta+\delta_4\nu)^{\alpha_4}(\delta_5\nu)^{\delta_5}},$$

$$f_2(\eta t,\nu t,\xi t)\sim\frac{1}{E_2}t^{-\Delta_2},\quad \Delta_2=\delta_3+\delta_4+\delta_5-\delta_1-\delta_2, \tag{5}$$

$$E_2=\nu^{\delta_5}\frac{(\alpha_3\eta+\delta_3\nu+\mu_2\xi)^{\delta_3}(\alpha_4\eta+\delta_4\nu)^{\alpha_4}(\delta_5)^{\delta_5}}{(\alpha_1\eta+\delta_1\nu+\mu_1\xi)^{\delta_1}(\alpha_2\eta+\delta_2\nu)^{\delta_2}},\ G_2=\frac{(\delta_3\nu)^{\delta_3}(\delta_4)^{\alpha_4}(\delta_5)^{\delta_5}}{(\delta_1)^{\delta_1}(\delta_2)^{\delta_2}},$$

analogously, we define

$$f_3(\eta t,\nu t,\xi t)\to t^{\mu_1-\mu_2-\mu_3}\,\frac{(\alpha_1\eta+\delta_1\nu+\mu_1\xi)^{\mu_1}}{(\alpha_3\eta+\delta_3\nu+\mu_2\xi)^{\mu_2}(\mu_3\xi)^{\mu_3}},\ f_3(\eta t,\nu t,\xi t)\sim\frac{1}{E_3}t^{-\Delta_3},$$

$$\Delta_3=\mu_2+\mu_3-\mu_1, \tag{6}$$

$$E_3=\xi^{\mu_3}\frac{(\alpha_3\eta+\delta_3\nu+\mu_2\xi)^{\mu_2}(\mu_3)^{\mu_3}}{(\alpha_1\eta+\delta_1\nu+\mu_1\xi)^{\mu_1}},\quad G_3=\frac{(\mu_2)^{\mu_2}(\mu_3)^{\mu_3}}{(\mu_1)^{\mu_1}}.$$

Theorem 2.1 is proved.

Now, we present some particular cases of the trivariate ML-type function.

2.1. Particular cases

For $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\delta_1,\delta_2,\delta_3,\delta_4,\mu_1,\mu_2\to 0$, $\sigma_1=\sigma_2=\sigma_3=\sigma_4=1$,

$${}_3E_1\left(\begin{matrix}0,0,0,1;0,0,1;\\0,0,0,1;0,0,1;\alpha_5,\sigma_5;\delta_5,\sigma_6;\mu_3,\sigma_7;\end{matrix}\middle|\,x;y;z\right)=E_{\alpha_5,\sigma_5}(x)E_{\delta_5,\sigma_6}(y)E_{\mu_3,\sigma_7}(z).$$

For $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\delta_1,\delta_2,\delta_3,\mu_1,\mu_2\to 0$, $\sigma_1=\sigma_2=\sigma_3=\sigma_4=1$,

$${}_3E_1\left(\begin{matrix}0,0,0,1;0,0,1;\\0,0,0,1;0,\delta_4,1;\alpha_5,\sigma_5;\delta_5,\sigma_6;\mu_3,\sigma_7;\end{matrix}\middle|\,x;y;z\right)=E_{\alpha_5,\sigma_5}(x)e^{1,\sigma_6}_{\delta_4,-\delta_5}(y)E_{\mu_3,\sigma_7}(z).$$

For $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\delta_5\to 0$, $\sigma_1=\sigma_2=\sigma_3=\sigma_6=\delta_4=1$,

$${}_3E_1\left(\begin{matrix}0,\delta_1,\mu_1,1;0,\delta_2,1;\\0,\delta_3,\mu_2,1;0,1,\sigma_4;\alpha_5,\sigma_5;0,1;\mu_3,\sigma_7;\end{matrix}\middle|\,x;y;z\right)=E_{\alpha_5,\sigma_5}(x)E_2\left(\begin{matrix}\delta_1,\mu_1,1;\delta_2,1;\\\delta_3,\mu_2,1;1,\sigma_4;\mu_3,\sigma_7;\end{matrix}\middle|\,\begin{matrix}y\\z\end{matrix}\right).$$

For $\alpha_2,\alpha_4,\delta_1,\delta_2,\delta_3,\delta_4\to 0$, $\sigma_2=\sigma_4=1$,

$${}_3E_1\left(\begin{matrix}\alpha_1,0,\mu_1,\sigma_1;0,0,1;\\\alpha_3,0,\mu_2,\sigma_3;0,0,1;\alpha_5,\sigma_5;\delta_5,\sigma_6;\mu_3,\sigma_7;\end{matrix}\middle|\,x;y;z\right)=E_{\delta_5,\sigma_6}(y)E_8\left(\begin{matrix}\alpha_1,\mu_1,\sigma_1;\\\alpha_3,\mu_2,\sigma_3;\alpha_5,\sigma_5;\mu_3,\sigma_7;\end{matrix}\middle|\,\begin{matrix}x\\z\end{matrix}\right)$$

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QUALITATIVE PROPERTIES OF THE SOURCE-FREE CAPUTO TIME-FRACTIONAL NONLINEAR FILTRATION EQUATION

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Abstract. *The nonlocal character of the Caputo time-fractional nonlinear filtration equation, which accounts for memory effects, complicates the search for an exact solution. In this paper, the equation is studied through a leading-order self-similar simplification, which reduces it to an ordinary differential equation admitting an exact solution in terms of the incomplete beta function. The obtained results are compared with the rigorous theory based on the full Erdélyi–Kober operator, and by demonstrating the dominance of the neglected term near the free boundary, it is proved that the critical condition and the free-boundary exponent belong not to the original equation but to the applied approximation itself.*

Keywords: *Caputo derivative, time-fractional diffusion, nonlinear filtration equation, self-similar solution, Erdélyi–Kober operator, free boundary, incomplete beta function.*

MANBASIZ KAPUTO KASR TARTIBLI NOCHIZIQLI FILTRATSIYA TENGLAMASINING SIFAT XOSSALARI

Annotatsiya. *Xotira effektlarini hisobga oluvchi Kaputo vaqt bo‘yicha kasr tartibli nochiziqli filtratsiya tenglamasining nolokal xarakteri aniq yechim topishni murakkablashtiradi. Maqolada tenglama yetakchi tartibli avtomodel soddalashtirish orqali tadqiq etilib, to‘liq bo‘lmagan beta-funksiya orqali aniq yechiladigan oddiy differensial tenglamaga keltiriladi. Olingan natijalar to‘liq Erdélyi–Kober operatoriga asoslangan qat'iy nazariya bilan solishtiriladi va tashlab yuborilgan hadning erkin chegara yaqinida ustun bo‘lishini ko'rsatish orqali kritik shart hamda erkin chegara ko‘rsatkichi asl tenglamaga emas, balki qo‘llanilgan approksimatsiyaning o‘ziga tegishli ekanligi isbotlanadi.*

Kalit so‘zlar: *Kaputo hosilasi, vaqt bo‘yicha kasr tartibli diffuziya, nochiziqli filtratsiya tenglamasi, avtomodel yechim, Erdélyi–Kober operatori, erkin chegara, to‘liq bo‘lmagan beta-funksiya.*

КАЧЕСТВЕННЫЕ СВОЙСТВА НЕЛИНЕЙНОГО УРАВНЕНИЯ ФИЛЬТРАЦИИ БЕЗ ИСТОЧНИКА С ДРОБНОЙ ПРОИЗВОДНОЙ КАПУТО ПО ВРЕМЕНИ

Аннотация. *Нелокальный характер нелинейного уравнения фильтрации с дробной производной Капуто по времени, учитывающей эффекты памяти, затрудняет нахождение точного решения. В статье уравнение исследуется с помощью автомодельного упрощения ведущего порядка, которое сводит его к обыкновенному дифференциальному уравнению, допускающему точное решение через неполную бета-функцию. Полученные результаты сопоставляются со строгой теорией, основанной на полном операторе Эрдейи–Кобера, и путём демонстрации доминирования отброшенного члена вблизи свободной границы доказывается, что критическое условие и показатель поведения вблизи свободной границы принадлежат не исходному уравнению, а самой применённой аппроксимации.*

Ключевые слова: *производная Капуто, дробная диффузия по времени, нелинейное уравнение фильтрации, автомодельное решение, оператор Эрдейи–Кобера, свободная граница, неполная бета-функция.*

Introduction. Nonlinear diffusion equations play a central role in modern mathematical physics. A fundamental example is the porous medium equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{\sigma} \frac{\partial u}{\partial x} \right)$$

This equation models filtration processes, nonlinear heat conduction, plasma transport and population dynamics. Its characteristic properties include finite propagation speed and compactly supported self-similar solutions of Barenblatt–Pattle type [1; 2].

To incorporate memory effects, the classical model has been generalised by replacing the first-order time derivative with the Caputo fractional derivative:

$$ {}^{C}D_t^{\alpha}u = \frac{\partial}{\partial x}\left(u^{\sigma}\frac{\partial u}{\partial x}\right), \quad 0<\alpha<1. \tag{1}$$

The nonlocal character of the Caputo derivative makes the construction and analysis of exact self-similar solutions substantially more involved.

Literature Review and Problem Statement. We consider the following Cauchy problem:

$$ {}^{C}D_t^{\alpha}u = \frac{\partial}{\partial x}\left(u^{\sigma}\frac{\partial u}{\partial x}\right), \quad x\in\mathbb{R}, \quad t>0, \tag{2}$$

$$u(x,0)=u_0(x)\geq 0, \quad x\in\mathbb{R}.$$

Self-similar solutions of time-fractional porous-medium equations have been investigated extensively through formulations involving the Erdelyi–Kober operator [4-8]. The most comprehensive general result currently available is due to Gomez-Castro, Plociniczak and Vazquez [9].

The cited result states that, in every spatial dimension and for every admissible nonlinear exponent and fractional order $\alpha\in(0,1)$, there exists a unique finite-mass self-similar profile. In the slow-diffusion regime considered here, the profile is compactly supported and behaves near the free boundary as

$$U(z)\sim c(1-z)_{+}^{(2-\alpha)/\sigma}. \tag{3}$$

The cited studies retain the full nonlocal integral formulation. In contrast, the present analysis replaces the nonlocal term by a leading-order approximation and thereby reduces the profile problem to an ordinary differential equation. This produces an explicit closed-form relation, but its validity must be assessed independently because the reduction does not preserve the full nonlocal structure.

The aim of this study is to determine precisely which conclusions of the reduced model are reliable and to identify the mechanism responsible for its failure near the free boundary.

Research Methodology.

Derivation of the Approximate Ordinary Differential Equation

The methodology combines an analytical self-similar reduction, exact integration of the resulting ordinary differential equation, asymptotic comparison with rigorous theory and a self-consistency test of the neglected term. We seek a self-similar solution in the form

$$u(x,t)=t^{-\beta}f(\xi), \quad \xi=\frac{|x|}{t^{\beta}}. \tag{4}$$

For fixed x, the similarity variable depends on time according to

$$\xi(s)=\frac{|x|}{s^{\beta}}.$$

Substitution of the self-similar ansatz into the Caputo derivative yields the following exact integral expression:

$$ {}^{C}D_t^{\alpha}u = -\frac{\beta}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{-\beta-1}\left[f(\xi(s))+\xi(s)f'(\xi(s))\right], ds \tag{5}$$

Under the leading-order assumption that the variation of the profile through the time-dependent similarity variable is negligible in the central region, we write

$$ {}^{C}D_t^{\alpha}u \approx f(\xi), {}^{C}D_t^{\alpha}\left[t^{-\beta}\right] \tag{6}$$

The action of the Caputo derivative on the power function is then approximated by

$$ {}^{C}D_t^{\alpha}t^{-\beta}=K, t^{-\beta-\alpha}, \qquad K=\frac{\Gamma(1-\beta)}{\Gamma(1-\beta-\alpha)} \tag{7}$$

The right-hand side of the nonlinear filtration equation transforms as

$$\frac{\partial}{\partial x}\left(u^{\sigma}\frac{\partial u}{\partial x}\right)=t^{-\beta(\sigma+3)}\frac{d}{d\xi}\left(f^{\sigma}f'\right) \tag{8}$$

Balancing the powers of t gives

$$-\beta-\alpha=-\beta(\sigma+3),$$

and therefore

$$\beta=\frac{\alpha}{\sigma+2} \tag{9}$$

The approximate profile equation is therefore

$$Kf(\xi)=\frac{d}{d\xi}\left(f^{\sigma}f'\right), \qquad f(0)=M>0, \qquad f'(0)=0 \tag{10}$$

This ordinary differential equation is the reduced model analysed below. Its conclusions are valid only where the leading-order assumption remains self-consistent.

Analysis and Results.

Explicit Solution of the Approximate Ordinary Differential Equation

Set

$$g(f)=f^{\sigma}f'(\xi).$$

Since

$$\frac{dg}{d\xi}=\frac{dg}{df}\frac{df}{d\xi},$$

the profile equation becomes

$$Kf=g\frac{dg}{df}f^{-\sigma}.$$

After integration and use of the condition $f'(0)=0$, one obtains the following first integral.

Proposition 1. First integral

$$g^2=\frac{2K}{\sigma+2}\left(f^{\sigma+2}-M^{\sigma+2}\right) \tag{11}$$

Because $g=f^{\sigma}f'$, a decreasing profile must satisfy $f'(\xi)<0$. For $0<f<M$, the quantity

$$f^{\sigma+2}-M^{\sigma+2}$$

is negative. Therefore, the right-hand side of equation (11) is nonnegative only if $K<0.$

Theorem 1. Internal consistency condition for the approximate ordinary differential equation

A centrally symmetric, decreasing and compactly supported solution of the approximate profile equation can exist only if $K<0.$

Since

$$K=\frac{\Gamma(1-\beta)}{\Gamma(1-\beta-\alpha)},$$

the numerator is positive for the admissible values of β. The sign of K is therefore determined by the denominator. The condition $K<0$ is equivalent to $1-\beta-\alpha<0.$

Using

$$\beta=\frac{\alpha}{\sigma+2},$$

we obtain

$$1-\alpha-\frac{\alpha}{\sigma+2}<0.$$

Hence,

$$\alpha>\alpha_c:=\frac{\sigma+2}{\sigma+3} \tag{12}$$

This restriction is intrinsic only to the approximate ordinary differential equation. For the original Caputo filtration equation, compactly supported self-similar solutions exist for every $\alpha\in(0,1)$ in the slow-diffusion regime [9].

For a decreasing profile, equation (11) gives

$$f^{\sigma} f' = -\sqrt{\frac{2|K|}{\sigma+2}\left(M^{\sigma+2} - f^{\sigma+2}\right)}.$$

Consequently,

$$\frac{d\xi}{df} = -\sqrt{\frac{\sigma+2}{2|K|}} \frac{f^{\sigma}}{\sqrt{M^{\sigma+2} - f^{\sigma+2}}}.$$

Integration from f to M yields

$$\xi(f) = \sqrt{\frac{\sigma+2}{2|K|}} \int_f^M \frac{s^{\sigma}}{\sqrt{M^{\sigma+2} - s^{\sigma+2}}}, ds \tag{13}$$

Introducing the variable

$$y = \left(\frac{s}{M}\right)^{\sigma+2},$$

the solution can be expressed in terms of the incomplete beta function:
Tayyor:

$$\xi(f) = \frac{M^{\sigma/2}}{\sqrt{2|K|(\sigma+2)}} \left[B\left(1; \frac{\sigma+1}{\sigma+2}, \frac{1}{2}\right) - B\left(\left(\frac{f}{M}\right)^{\sigma+2}; \frac{\sigma+1}{\sigma+2}, \frac{1}{2}\right)\right] \tag{14}$$

Here,

$$B(z; a, b) = \int_0^z t^{a-1}(1-t)^{b-1}, dt$$

denotes the incomplete beta function.
The support radius is obtained by setting $f = 0$:

$$\xi_0 = \frac{M^{\sigma/2}}{\sqrt{2|K|(\sigma+2)}} B\left(1; \frac{\sigma+1}{\sigma+2}, \frac{1}{2}\right) \tag{15}$$

Using the identity

$$B(1; a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

and $\Gamma(1/2) = \sqrt{\pi}$, we obtain

$$\xi_0 = \frac{M^{\sigma/2}\sqrt{\pi}}{\sqrt{2|K|(\sigma+2)}} \frac{\Gamma\left(\frac{\sigma+1}{\sigma+2}\right)}{\Gamma\left(\frac{3\sigma+4}{2(\sigma+2)}\right)} \tag{16}$$

Thus, the approximate profile is positive for $0 \le \xi < \xi_0$ and vanishes for $\xi \ge \xi_0$

Front Asymptotics

Near the free boundary, $f(\xi)$ tends to zero. From the first integral,

$$f^{\sigma} f' \sim -C,$$

where

$$C = \sqrt{\frac{2|K|}{\sigma+2}} M^{(\sigma+2)/2} > 0.$$

Therefore,

$$f^{\sigma}, df \sim -C, d\xi.$$

Integration near $\xi = \xi_0$ gives

$$\frac{f^{\sigma+1}}{\sigma+1} \sim C(\xi_0 - \xi).$$

Hence,

$$f(\xi) \sim C(\xi_0 - \xi)^{1/(\sigma+1)} \quad \text{as} \quad \xi \to \xi_0^- \tag{17}$$

The constant c is given by

$$c = \left[(\sigma+1)C\right]^{1/(\sigma+1)}.$$

Remark 1.

In the classical Barenblatt profile, corresponding to $\alpha = 1$, the free-boundary exponent is $\frac{1}{\sigma}$.

The rigorous fractional theory [9] gives the exponent $\frac{2-\alpha}{\sigma}$.

Both values differ from $\frac{1}{\sigma+1}$, which is the exponent produced by the approximate ordinary differential equation.

Thus, the leading-order approximation does not reproduce the correct local behaviour near the free boundary.

Self-Consistency Check

The approximation used in deriving equation (6) effectively neglects the term involving $\xi f'(\xi)$ relative to the term involving $f(\xi)$. This requires

$$\left|\frac{\xi f'(\xi)}{f(\xi)}\right| \ll 1.$$

However, the front asymptotics imply

$$f(\xi) \sim c(\xi_0 - \xi)^{1/(\sigma+1)},$$

and hence

$$f'(\xi) \sim -\frac{c}{\sigma+1}(\xi_0 - \xi)^{-\sigma/(\sigma+1)}.$$

Therefore,

$$\frac{\xi f'(\xi)}{f(\xi)} \sim -\frac{\xi_0}{\sigma+1}\frac{1}{\xi_0 - \xi} \to -\infty \quad \text{as} \quad \xi \to \xi_0^- \tag{18}$$

Theorem 2. Breakdown of the approximation

Near the free boundary, the term discarded in the leading-order reduction becomes infinitely larger than the retained term. Consequently, the assumption

$$|\xi f'(\xi)| \ll |f(\xi)|$$

is violated precisely in the front region.

This loss of self-consistency explains the appearance of the spurious critical value $\alpha_c = \frac{\sigma+2}{\sigma+3}$ and the incorrect front exponent $\frac{1}{\sigma+1}$.

The approximate ordinary differential equation may describe the central region of the profile, where the variation of f is relatively weak. It cannot, however, correctly describe the free-boundary layer.

Conclusion and Recommendations

The leading-order self-similar reduction of the Caputo time-fractional nonlinear filtration equation produces an ordinary differential equation that can be solved explicitly.

The approximate model requires $\alpha > \frac{\sigma+2}{\sigma+3}$ and yields the free-boundary exponent $\frac{1}{\sigma+1}$.

Neither property belongs to the original Caputo filtration equation. In the slow-diffusion regime, compactly supported self-similar profiles exist for all $\alpha \in (0,1)$, and the correct free-boundary exponent is $\frac{2-\alpha}{\sigma}$ according to the rigorous theory [9].

Precisely at the free boundary, where the neglected term becomes dominant. The principal contribution of this paper is therefore not a new existence theorem, but a transparent, computable

demonstration of how a leading-order simplification can generate artificial restrictions and incorrect asymptotic exponents. Accordingly, such reductions should be applied only after their domain of validity has been verified; future work should retain the full Erdélyi–Kober operator, develop matched asymptotic expansions, or solve the complete nonlocal profile equation numerically.

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CHIZIQLI OPERATORLAR SPEKTRAL NAZARIYASI ELEMENTLARI

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Annotatsiya. *Ushbu maqolada Banax va Hilbert fazolarida aniqlangan chiziqli operatorlarning spektral xossalari tizimli ravishda o‘rganiladi. Chiziqli operator spektri va rezolventasi tushunchalari, spektrning nuqtali, uzluksiz va qoldiq qismlarga ajratilishi, spektral radius hamda uning asosiy xossalari yoritildi. Chegaralangan operatorlar uchun spektrning yopiq va chegaralangan to‘plam ekanligi, spektral radiusning operator normasi bilan bog‘lanishi asoslandi. Tadqiqot natijalari funksional analiz, differensial va integral tenglamalar, matematik fizika hamda operatorlar nazariyasidagi spektral masalalarni o‘rganish uchun nazariy asos bo‘lib xizmat qiladi*

Kalit so‘zlar: *chiziqli operator, Banax fazosi, Hilbert fazosi, spektr, rezolventa, xos qiymat, xos vektor, spektral radius*

ЭЛЕМЕНТЫ СПЕКТРАЛЬНОЙ ТЕОРИИ ЛИНЕЙНЫХ ОПЕРАТОРОВ

Аннотация. *В статье систематически исследуются спектральные свойства линейных операторов, заданных в банаховых и гильбертовых пространствах. Рассматриваются понятия спектра и резольвенты линейного оператора, анализируется разложение спектра на точечную, непрерывную и остаточную части, а также исследуются основные свойства спектрального радиуса. Математически обосновываются замкнутость и ограниченность спектра ограниченного линейного оператора, а также взаимосвязь спектрального радиуса с нормой оператора. Полученные результаты могут служить теоретической основой для исследования спектральных задач, возникающих в функциональном анализе, теории дифференциальных и интегральных уравнений, математической физике и теории операторов.*

Ключевые слова: *линейный оператор, банахово пространство, гильбертово пространство, спектр оператора, резольвента, собственное значение, собственный вектор, спектральный радиус, ограниченный оператор, компактный оператор, самосопряжённый оператор, спектральное разложение.*

ELEMENTS OF THE SPECTRAL THEORY OF LINEAR OPERATORS

Abstract. *This paper systematically investigates the spectral properties of linear operators defined on Banach and Hilbert spaces. The concepts of the spectrum and resolvent of a linear operator are considered, and the decomposition of the spectrum into point, continuous, and residual parts is analyzed. The main properties of the spectral radius are also examined. The closedness and boundedness of the spectrum of a bounded linear operator, as well as the relationship between the spectral radius and the operator norm, are mathematically justified. The obtained results may serve as a theoretical basis for studying spectral problems arising in functional analysis, the theory of differential and integral equations, mathematical physics, and operator theory.*

Keywords: *linear operator, Banach space, Hilbert space, operator spectrum, resolvent, eigenvalue, eigenvector, spectral radius, bounded operator, compact operator, self-adjoint operator, spectral decomposition.*

Kirish. Zamonaviy funksional analizning muhim yo‘nalishlaridan biri chiziqli operatorlarning spektral nazariyasini o‘rganishdan iborat. Chiziqli operatorning spektri uning algebraik xossalarini emas, balki operatorning fazodagi ta’sirini ham tavsiflovchi fundamental matematik obyekt hisoblanadi. Chekli o‘lchamli fazolarda spektr tushunchasi asosan matritsaning xos qiymatlari bilan bog‘liq bo‘lsa, cheksiz o‘lchamli fazolarga o‘tilganda spektr tarkibi ancha murakkablashadi. Bunda xos qiymat mavjud bo‘lmagan holda ham operatorning teskarisi mavjud bo‘lmasligi mumkin. Chekli o‘lchamli fazoda $A-\lambda I$ operatorining teskarisi mavjud emasligi λ ning xos qiymat ekanligini anglatadi. Cheksiz o‘lchamli Banax yoki Hilbert fazolarida esa $A-\lambda I$ operatori bir qiymatli va to‘la aniqlangan teskari operatorga ega bo‘lmasligining bir

nechta sabablari mavjud. Shu sababli spektr tushunchasi xos qiymatlar to‘plamidan kengroq tushuncha sifatida qaraladi.

Spektral nazariya funksional analizning nazariy bo‘limi bo‘lishi bilan birga, differensial tenglamalar, integral tenglamalar, matematik fizika va kvant mexanikasidagi ko‘plab masalalarning matematik asosini tashkil etadi. Xususan, matematik fizikada kuzatiladigan ko‘plab modellar Hilbert fazosidagi operatorlar yordamida ifodalanadi. Shu sababli operatorning spektrini aniqlash masalasi tegishli matematik modelning sifat xususiyatlarini o‘rganishda muhim ahamiyatga ega. Spektral xossalar operatorning teskari masalalarida, barqarorlik tahlilida, differensial tenglamalar yechimlarining sifat jihatlarida va kvant mexanikasining matematik modelida markaziy rol o‘ynaydi. Masalan, Hilbert fazosida o‘ziga qo‘shma operatorlar spektrining haqiqiy sonlardan iborat bo‘lishi operator orqali ifodalanadigan kattaliklarning fizik mazmuni bilan uyg‘unlashadi. Yoki kompakt operatorlar uchun spektrning diskretlashuvi integral tenglamalarning klassik nazariyasida hal qiluvchi ahamiyat kasb etadi.

Adabiyotlar tahlili va metodologiya. So‘nggi yillarda chiziqli operatorlarning spektral nazariyasi klassik spektr, rezolventa va xos qiymatlar masalalaridan tashqari, spektral barqarorlik, psevdospektr, spektral yaqinlashuv, kompakt rezolventali operatorlar va spektrni hisoblashning konstruktiv usullarini qamrab olgan holda rivojlanmoqda. Zamonaviy tadqiqotlarda operator spektrini faqat aniqlash emas, balki operatorga kiritilgan kichik o‘zgarishlarning spektrga ta’sirini miqdoriy baholash masalasi ham muhim yo‘nalish sifatida qaralmoqda.

Colbrookning tadqiqotida Hilbert fazolaridagi chiziqli operatorlar spektrining geometrik xususiyatlarini hisoblash masalasi o‘rganilgan. Muallif spektrni oddiy sonlar to‘plami sifatida aniqlash bilan cheklanmay, spektral radius, spektral bo‘shliqlar, psevdospektral xususiyatlar, spektral sig‘im va fraktal o‘lcham kabi geometrik ko‘rsatkichlarni hisoblash muammosini ilgari suradi. Tadqiqotda spektral masalalarni hisoblashning murakkabligi ham tahlil qilinib, cheksiz o‘lchamli operatorlar uchun klassik chekli o‘lchamli yaqinlashuvlarning yetarli bo‘lmasligi ko‘rsatiladi.

Jabłoński, Jung, Kubrusly va Stochel tomonidan 2024-yilda olib borilgan tadqiqotda Hilbert fazosidagi operatorlar darajalari ketma-ketligining yaqinlashuvi va barqarorligi o‘rganilgan. Mualliflar kuchsiz, kuchli va operator normasi topologiyalaridagi yaqinlashuvlarni taqqoslaydi. Mazkur natijalar operatorning iteratsion xususiyatlari bilan uning spektral tuzilishi o‘rtasidagi munosabatni yanada chuqurroq o‘rganish zarurligini ko‘rsatadi.

Gilning 2024-yildagi tadqiqotida Hilbert fazosidagi chegaralangan chiziqli operatorlarning xos qiymatlari perturbatsiyasi masalasi qaralgan. Xususan, operator bilan uning o‘z-o‘ziga qo‘shma qismi orasidagi farq Schatten–von Neumann sinfiga tegishli bo‘lgan holatda xos qiymatlar o‘zgarishining miqdoriy baholari olingan. Tadqiqot Kato va Weyl tipidagi klassik natijalarni zamonaviy perturbatsion yondashuvlar bilan bog‘laydi.

Bisterzo va Siclari 2025-yilda kompakt rezolventali operatorlar uchun spektral barqarorlikning miqdoriy masalasini tadqiq qilgan. Ishda operatorlar oilasi parametr o‘zgarganda xos qiymatlarning qanday o‘zgarishi, xususan, xos qiymatlar variatsiyasining asimptotik xatti-harakati o‘rganilgan. Natijalar turli elliptik masalalar, Riman metrikalarining o‘zgarishi va Fourier-multiplikatorlar oilasiga tatbiq etilgan. Mazkur tadqiqot spektral nazariyaning zamonaviy bosqichida spektrni aniqlashdan tashqari, uning perturbatsiya ostidagi barqarorligini miqdoriy ifodalash muhim ekanligini ko‘rsatadi.

Shakhmurovning 2023-yildagi tadqiqotida Banax fazolaridagi differensial operatorlarning spektral xossalari, xususan, Schatten sinfiga mansub operatorlar uchun ildiz elementlarning to‘liqligi o‘rganilgan. Tadqiqot natijalari o‘z-o‘ziga qo‘shma bo‘lmagan differensial operatorlarning spektral tuzilishini tahlil qilishga imkon beradi va spektral nazariyaning differensial tenglamalar bilan uzviy bog‘liqligini ko‘rsatadi

2026-yilda Shameem va Deepesh tomonidan taklif qilingan natijada silliq Banax fazolarida duality mapping orqali o‘z-o‘ziga qo‘shma deb qaraladigan kompakt operatorlar uchun spektral tasvirlash teoremasining analogi olingan. Mualliflar bu natijani Hilbert fazosidagi klassik kompakt o‘z-o‘ziga qo‘shma operatorlar uchun spektral teoremaning tabiiy umumlashmasi sifatida talqin qiladilar.

Natija. Chiziqli operatorlarning spektral nazariyasini o‘rganishda avvalo operator aniqlangan fazoning xususiyatlarini hisobga olish muhimdir. Spektral nazariyaning asosiy natijalari, odatda, Banax va Hilbert fazolarida aniqlangan chegaralangan chiziqli operatorlarga nisbatan qaraladi. Ushbu fazolarning to‘lalik xossasi operatorlarning limit jarayonlari, teskari operatorlarning mavjudligi va spektral xususiyatlarini tadqiq etishda muhim nazariy asos bo‘lib xizmat qiladi.

X chiziqli normallangan fazo bo‘lsin. Agar X fazo normaga nisbatan to‘liq bo‘lsa, u Banax fazosi deb ataladi. Ya’ni, X fazodagi har qanday fundamental ketma-ketlik shu fazoning biror elementiga yaqinlashsa, X Banax fazosi hisoblanadi. Banax fazolari funksional analizning asosiy obyektlaridan biri bo‘lib, ko‘plab differensial va integral tenglamalar yechimlarini o‘rganishda tabiiy matematik muhit sifatida qaraladi.

Hilbert fazosi esa skalyar ko‘paytma bilan jihozlangan va shu skalyar ko‘paytma orqali hosil qilingan normaga nisbatan to‘liq bo‘lgan chiziqli fazodir. Agar H Hilbert fazosi bo‘lsa, unda x,y∈H elementlar uchun

(x,y)

skalyar ko‘paytma aniqlangan bo‘lib, unga mos norma

$||x||=\sqrt{(x,x)}$

ko‘rinishida aniqlanadi. Hilbert fazolarining muhim afzalligi shundaki, ularda ortogonallik, proyeksiya va qo‘shma operator tushunchalari mavjud bo‘lib, bu spektral nazariyani yanada chuqurroq o‘rganish imkonini beradi.

Tadqiqot natijasida Banax va Hilbert fazolarida aniqlangan chiziqli operatorlarning spektral xossalari yagona nazariy tizim sifatida umumlashtirildi. O‘rganishlar shuni ko‘rsatdiki, cheksiz o‘lchamli fazolarda spektr tushunchasi oddiy xos qiymatlar to‘plamidan kengroq bo‘lib, operatorning teskari mavjudligi, rezolventasi va fazodagi ta’siri bilan uzviy bog‘liqdir.

Birinchi natija sifatida chegaralangan chiziqli operator spektrining yopiq va chegaralangan to‘plamdan iborat ekanligi asoslandi. Neumann qatori yordamida spektr operator normasi bilan baholanishi ko‘rsatildi. Bu esa operatorning spektral xossalarini oldindan tahlil qilish va murakkab masalalarda spektrning joylashish sohasini aniqlash imkonini beradi.

Ikkinchi muhim natija spektral radius tushunchasi bilan bog‘liq bo‘lib, u operatorning uzoq davom etuvchi ta’sirini tavsiflovchi asosiy ko‘rsatkich ekanligi aniqlandi. Spektral radiusning operator darajalari normasi orqali ifodalanishi operatorning asimptotik xatti-harakatini o‘rganish uchun qulay vosita hisoblanadi. Mazkur natija iteratsion jarayonlar, differensial tenglamalar va barqarorlik masalalarida muhim ahamiyat kasb etadi.

Hilbert fazosidagi o‘z-o‘ziga qo‘shma operatorlar uchun spektrning haqiqiy sonlardan tashkil topishi isbotlandi. Bu xususiyat matematik fizika va kvant mexanikasidagi ko‘plab modellarni tadqiq etishda muhim nazariy asos bo‘lib xizmat qiladi. Shuningdek, kompakt operatorlar uchun nolga teng bo‘lmagan barcha spektral nuqtalarning xos qiymatlardan iborat bo‘lishi ko‘rsatildi. Natijada cheksiz o‘lchamli masalalarni chekli o‘lchamli modellar yordamida yaqinlashtirib o‘rganish imkoniyati paydo bo‘ladi.

Nazariy natijalarni tasdiqlash maqsadida integral operator misoli qaraldi. Tadqiqot natijasida ushbu operatorning spektri aniqlandi va uning spektral tuzilishi kompakt operatorlar haqidagi nazariy xulosalarga mos kelishi ko‘rsatildi. Bu esa spektral nazariya nafaqat abstrakt matematik tushuncha, balki integral va differensial tenglamalar, matematik fizika hamda amaliy modellashtirish masalalarida ham samarali qo‘llanilishini tasdiqlaydi.

Muhokama. Olingan natijalar chiziqli operatorlarning spektral nazariyasi cheksiz o‘lchamli fazolarda yuzaga keladigan murakkab masalalarni tahlil qilishda muhim matematik apparat ekanligini ko‘rsatdi. Chekli o‘lchamli fazolarda spektr asosan xos qiymatlar orqali tavsiflansa, Banax va Hilbert fazolarida operatorning teskari mavjudligi, rezolventa va spektrning tuzilishini birgalikda hisobga olish zarur bo‘ladi. Shu jihatdan spektral nazariya klassik chiziqli algebra g‘oyalarining tabiiy umumlashmasi sifatida namoyon bo‘ladi.

Spektral radius va operator normasi o‘rtasidagi bog‘liqlik operatorlarning barqarorligini baholash imkonini beradi. Ayniqsa, operator darajalari ketma-ketligining xatti-harakatini tahlil qilishda spektral radius muhim rol o‘ynaydi. Bu natijalar sonli usullar, iteratsion algoritmlar va funksional modellarni o‘rganishda amaliy ahamiyatga ega. O‘z-o‘ziga qo‘shma operatorlar uchun spektrning haqiqiy sonlardan iborat bo‘lishi matematik fizika va kvant mexanikasi masalalarida muhim o‘rin tutadi. Kompakt operatorlarning diskret spektrga ega bo‘lishi esa integral tenglamalarni yechishda samarali matematik vosita hisoblanadi. Cheksiz o‘lchamli fazolarda operator spektrini faqat xos qiymatlar to‘plami sifatida qarash yetarli emas. Bunday fazolarda $A-\lambda I$ operatorining teskari operatorga ega bo‘lmasligi turli sabablar bilan bog‘liq bo‘lishi mumkin. Shu sababli spektrning nuqtali, uzluksiz va qoldiq qismlarga ajratilishi operatorning funksional xususiyatlarini chuqurroq tavsiflash imkonini beradi.

Tadqiqot natijalarining muhim jihatlaridan biri chegaralangan chiziqli operator spektrining operator normasi bilan bog‘liqligini asoslashdan iboratdir. Olingan natijalarga ko‘ra, operator spektrining kompleks tekislikdagi joylashuvi operator normasi yordamida baholanishi mumkin. Bu xulosa spektral masalalarni o‘rganishda dastlabki sifat tahlilini amalga oshirish uchun muhimdir. Xususan, spektral radiusning operator darajalari normasi bilan bog‘liqligi operatorning iteratsion jarayonlardagi uzoq muddatli xatti-harakatini baholash imkonini beradi

Integral operator misolida olingan natijalar nazariy xulosalarning amaliy tasdig‘i bo‘lib xizmat qildi. Operator spektrining aniq hisoblanishi spektral nazariyaning konkret masalalarda qo‘llanilish imkoniyatlarini namoyon etdi. Bu esa spektral metodlarning differensial tenglamalar, matematik fizika, signal va ma’lumotlarni qayta ishlash hamda zamonaviy hisoblash matematikasidagi ahamiyatini yanada oshiradi. Shunday qilib, tadqiqot natijalari chiziqli operatorlar spektral nazariyasining nafaqat nazariy, balki amaliy

masalalarni yechishda ham muhim metodologik asos ekanligini ko‘rsatdi. Integral operator misolida olib borilgan tahlil nazariy xulosalarning konkret matematik masalalarda qo‘llanilishini ko‘rsatdi. Integral operatorning kompaktligi va uning nolga teng bo‘lmagan spektral nuqtalarining xos qiymatlar bilan bog‘liqligi abstrakt spektral tushunchalarni amaliy masalalar bilan bog‘lash imkonini berdi. Bu holat spektral nazariyaning faqat nazariy funksional analiz doirasida emas, balki integral va differensial tenglamalar, matematik modellashtirish hamda sonli usullarda ham muhim ahamiyatga ega ekanligini tasdiqlaydi.

Tadqiqotning muhim jihati shundan iboratki, spektr, rezolventa, spektral radius, o‘z-o‘ziga qo‘shma va kompakt operatorlarning xususiyatlari alohida tushunchalar sifatida emas, balki yagona spektral tizimning o‘zaro bog‘liq elementlari sifatida qaraldi. Bunday yondashuv chiziqli operatorlarning spektral nazariyasini kompleks tarzda tushunish imkonini beradi. Shu bilan birga, tadqiqotning ayrim chegaralari ham mavjud. Ishda asosan chegaralangan chiziqli operatorlarning spektral xossalari ko‘rib chiqildi. Chegaralanmagan operatorlar, kompakt rezolventali differensial operatorlar, psevdospektr va spektral perturbatsiya masalalari tadqiqot doirasidan tashqarida qoldi. Ushbu yo‘nalishlar kelgusidagi tadqiqotlarda operatorlar spektrining yanada murakkab xususiyatlarini o‘rganish uchun istiqbolli yo‘nalish hisoblanadi. Umuman olganda, olingan natijalar chiziqli operatorlarning spektral nazariyasi cheksiz o‘lchamli fazolardagi murakkab matematik masalalarni o‘rganishda fundamental vosita ekanligini ko‘rsatadi. Tadqiqot natijalari spektral masalalarni nazariy jihatdan umumlashtirish bilan bir qatorda, kelgusida operatorlarning spektral barqarorligi, perturbatsion xususiyatlari va differensial hamda integral operatorlarning spektrini chuqurroq tadqiq etish uchun metodologik asos bo‘lib xizmat qilishi mumkin

Xulosa. Chiziqli operatorlarning spektral nazariyasi Banax va Hilbert fazolaridagi operatorlarning fundamental xususiyatlarini o‘rganish imkonini beruvchi muhim matematik apparat hisoblanadi. Tadqiqot davomida spektr, rezolventa, spektral radius, o‘z-o‘ziga qo‘shma va kompakt operatorlar tushunchalari o‘zaro bog‘liq holda tahlil qilindi. Spektral radiusning operator darajalari normasi orqali ifodalanishi operatorning spektral xususiyatlarini aniqlash uchun samarali vosita ekanligi ko‘rsatildi. Hilbert fazosidagi o‘z-o‘ziga qo‘shma operatorlar uchun spektrning haqiqiy sonlardan tashkil topishi, kompakt operatorlar uchun esa nolga teng bo‘lmagan spektral nuqtalarning xos qiymatlardan iborat bo‘lishi aniqlandi. Integral operator misolida nazariy natijalar konkret hisoblashlar orqali tasdiqlandi. Olingan natijalar chiziqli operatorlarning spektral xossalarini o‘rganishda nazariy va amaliy jihatdan muhim ahamiyatga ega.

Kelgusidagi tadqiqotlarda chegaralanmagan chiziqli operatorlarning spektral xossalarini, kompakt rezolventali operatorlarni, spektral proyeksiyalar va funksional hisoblash usullarini, shuningdek, operatorlar spektrining perturbatsiya va psevdospektr sharoitidagi barqarorligini chuqurroq tadqiq etish maqsadga muvofiqdir. Shu yo‘nalishdagi izlanishlar spektral nazariyaning zamonaviy matematikadagi nazariy va amaliy imkoniyatlarini yanada kengaytirishga xizmat qiladi.

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INVERSE PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL CONDITIONS

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Abstract. *This paper investigates the classical solvability of an inverse boundary value problem for a time-fractional parabolic equation involving the Caputo derivative. The temperature distribution and the unknown time-dependent lowest-order coefficient are simultaneously determined using nonlocal integral conditions and periodic boundary conditions. By applying the separation of variables method via a biorthogonal system of eigenfunctions and sharp estimates of generalized Mittag-Leffler functions, the existence and uniqueness of the classical solution in Banach spaces are established.*

Key words: *time-fractional parabolic equation, Caputo derivative, inverse boundary value problem, nonlocal conditions, biorthogonal system, Riesz basis, Mittag-Leffler function, Banach contraction mapping principle.*

INTEGRAL SHARTLI KASR TARTIBLI TENGLAMA UCHUN TESKARI MASALA

Annotatsiya. *Ushbu maqolada Kaputo hosilali vaqt bo'yicha kasr tartibli parabolik tenglama uchun teskari chegaraviy masalaning klassik yechilishi o'rganiladi. Nofokal integral va davriy chegaraviy shartlar yordamida harorat taqsimoti hamda vaqtga bog'liq bo'lmag'in noma'lum quyi koeffitsiyent bir vaqtning o'zida aniqlanadi. O'zgaruvchilarni ajratish usuli va xususiy funksiyalarning biorthogonal sistemasi yordamida Banax fazosida klassik yechimning mavjudligi va yagonaligi isbotlangan.*

Kalit so'zlar:*. kasr tartibli parabolik tenglama, Kaputo hosilasi, teskari chegaraviy masala, nofokal shartlar, biorthogonal sistema, Riss bazisi, Mittag-Leffler funksiyasi, qisqarib boruvchi akslantirishlar prinsipi.*

ОБРАТНАЯ ЗАДАЧА ДЛЯ УРАВНЕНИЯ ДРОБНОГО ПОРЯДКА С ИНТЕГРАЛЬНЫМИ УСЛОВИЯМИ

Аннотация. *В данной статье исследуется классическая разрешимость обратной краевой задачи для параболического уравнения дробного по времени порядка с производной Капуто. Одновременно определяются распределение температуры и неизвестный коэффициент при младшем члене, зависящий от времени, с использованием нелокальных интегральных и периодических краевых условий. С помощью метода разделения переменных через биортогональную систему собственных функций доказаны существование и единственность классического решения в пространстве Банаха.*

Ключевые слова*: параболическое уравнение дробного порядка, производная Капуто, обратная краевая задача, нелокальные условия, биортогональная система, базис Рисса, функция Миттаг-Леффлера, принцип сжимающих отображений.*

Introduction. Let $0 < T < +\infty$ be a fixed moment of time and D_T be a rectangular domain defined by the inequalities $0 < x < 1$ and $0 < t \le T$. We consider the problem of finding a pair of functions $\{u(x,t), a(t)\}$ satisfying the following time-fractional parabolic equation:

$$ {}^{C}D_t^{\alpha} u(x,t) = u_{xx}(x,t) + a(t)u(x,t) + f(x,t), \quad 0 < \alpha < 1, \quad . \tag{1}$$

The nonlocal initial condition has the following form:

$$ u(x,0) + \delta u(x,T) + \int_0^T p(t)u(x,t)dt = \varphi(x), \quad 0 \le x \le 1. \tag{2}$$

The boundary conditions are periodic:

$$ u(0,t) = \beta u(1,t), \quad 0 \le t \le T. \tag{3}$$

The nonlocal integral condition is:

$$\int_0^1 u(x,t)dx = 0, \quad 0 \le t \le T, \tag{4}$$

and the overdetermination condition for determining the inverse problem is as follows:

$$u(x_0,t) = h(t), \quad x_0 \in (0,1). \tag{5}$$

Here $T > 0, \beta, \delta \ge 0$ are given numbers, and $p(t)$, $\varphi(x)$, $h(t)$, $f(x,t)$ are given positive functions. ${}^C D_t^\alpha u(x,t)$ is the Caputo fractional derivative of $u(x,t)$ with respect to time, defined as follows:

$${}^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} \frac{d\tau}{(t-\tau)^\alpha}.$$

The symbol $\Gamma(\cdot)$ in the last equation denotes the Euler gamma function.

Definition 1 *A pair of functions $\{u(x,t), a(t)\}$ is called a solution to problem (1)–(5) if:*

1. the function $u(x,t)$ and its derivatives ${}^C D_t^\alpha u(x,t)$ and $u_{xx}(x,t)$ are continuous in the domain D_T
2. the time-dependent function $a(t)$ is continuous on the interval $[0,T]$;
3. the pair of functions $\{u(x,t), a(t)\}$ satisfies equation (1) and conditions (2)–(5).

Then the following theorem holds.

Theorem 1 *Assume that $\delta \ge 0$, $p(t) \in C[0,T]$, $f(x,t) \in C(\overline{D}_T)$, $\varphi(x) \in C[0,1]$, and ${}^C D_t^\alpha h(t) \in C[0,T]$. Let the following equality hold for $0 \le t \le T$:*

$$\int_0^1 f(x,t)dx = 0,$$

and let $h(t) \ne 0$ for $0 \le t \le T$. Furthermore, let the following compatibility conditions be satisfied:

$$\int_0^1 \varphi(x)dx = 0, \tag{6}$$

$$h(0) + \delta h(T) + \int_0^T p(t)h(t)dt = \varphi(x_0). \tag{7}$$

Then the problem of finding a classical solution to problem (1)–(5) is equivalent to the problem of determining functions $u(x,t)$ and $a(t)$ satisfying equation (1), conditions (2)–(3), and the following relations (the case $\beta \ne \pm 1$ is studied):

$$u_x(0,t) = u_x(1,t), \quad 0 \le t \le T, \tag{8}$$

$${}^C D_t^\alpha h(t) = u_{xx}(x_0,t) + a(t)h(t) + f(x_0,t), \quad 0 \le t \le T. \tag{9}$$

Proof. Let the pair of functions $\{u(x,t), a(t)\}$ be a classical solution of problem (1)–(5). Integrating equation (1) with respect to the variable x from 0 to 1, we obtain:

$${}^C D_t^\alpha \int_0^1 u(x,t)dx = u_x(1,t) - u_x(0,t) + a(t)\int_0^1 u(x,t)dx + \int_0^1 f(x,t)dx. \tag{10}$$

Taking into account equality (6), we obtain equality (8). Evaluating equation (1) at the point $x = x_0$, we obtain:

$${}^C D_t^\alpha u(x_0,t) = u_{xx}(x_0,t) + a(t)u(x_0,t) + f(x_0,t), \quad 0 \le t \le T. \tag{11}$$

Since ${}^C D_t^\alpha h(t) \in C[0,T]$, using condition (5) we obtain the relation:

$${}^C D_t^\alpha u(x_0,t) = {}^C D_t^\alpha h(t). \tag{12}$$

As a result, using relations (5) and (12), we obtain (9). Now assume that the pair of functions $\{u(x,t), a(t)\}$ is a solution to problem (1)–(3), (8), (9). Then, taking into account conditions (3) and (8), from relation (10) we have:

$${}^C D_t^\alpha \int_0^1 u(x,t)dx = a(t)\int_0^1 u(x,t)dx, \quad 0 \le t \le T. \tag{13}$$

From condition (2), it is easy to see that:

$$\int_0^1 u(x,0)dx + \delta\int_0^1 u(x,T)dx + \int_0^1\int_0^T p(t)u(x,t)dtdx = \int_0^1 \varphi(x)dx = 0. \quad (14)$$

Since problem (13) and (14) has only the trivial solution, it follows that:

$$\int_0^1 u(x,t)dx = 0, \quad 0 \le t \le T,$$

which means condition (4) is satisfied.

Next, from relations (9) and (11) we find:

$${}^C D_t^{\alpha}(u(x_0,t) - h(t)) = a(t)(u(x_0,t) - h(t)), \quad 0 \le t \le T. \quad (15)$$

According to condition (2) and compatibility condition (7), we obtain:

$$u(x_0,0) - h(0) = \varphi(x_0) - h(0) = 0. \quad (16)$$

From relation (15), taking into account equality (16), it becomes evident that condition (5) is also satisfied.

Auxiliary Spectral Problem

Now, to investigate problems (1)–(3) and (8), we present some known facts. Consider the following spectral problem:

$$X''(x) + \lambda X(x) = 0, \quad 0 \le x \le 1, \quad (17)$$

$$X(0) = \beta X(1), \quad X'(0) = X'(1), \quad \beta \ne \pm 1. \quad (18)$$

Problem (17)–(18) is not self-adjoint, and the adjoint problem to this problem is as follows:

$$Y''(x) + \lambda Y(x) = 0, \quad 0 \le x \le 1,$$

$$(19) \quad Y(0) = Y(1), \quad Y'(1) = \beta Y'(0). \quad (20)$$

According to the literature [1], we denote the system of eigenfunctions and associated functions of problem (19)–(20) in the following form:

$$X_0(x) = ax + b, \quad X_{2k-1}(x) = (ax+b)\cos\lambda_k x, \quad X_{2k}(x) = \sin\lambda_k x, \quad (21)$$

$$Y_0(x) = 2, \quad Y_{2k-1}(x) = 4\cos\lambda_k x, \quad Y_{2k}(x) = 4(1-b-ax)\sin\lambda_k x. \quad (22)$$

Here

$$\lambda_k = 2k\pi \quad (k = 1,2,\ldots), \quad a = \frac{1-\beta}{1+\beta} \ne 0, \quad b = \frac{\beta}{1+\beta}. \quad (23)$$

It is not difficult to verify the fulfillment of the biorthogonality conditions: $(X_i, Y_j) = \int_0^1 X_i(x)Y_j(x)dx = \delta_{ij}$.

Theorem 1.1. *The system of functions (21)–(22) forms a Riesz basis in the space* $L_2(0,1)$, *and for any function* $\psi(x) \in L_2(0,1)$, *the following estimates hold:*

$$r \parallel \psi(x) \parallel^2_{L_2(0,1)} \le \sum_{k=0}^{\infty}\psi_k^2 \le R \parallel \psi(x) \parallel^2_{L_2(0,1)}. \quad (24)$$

where

$$\psi_k = (\psi(x), Y_k(x)) = \int_0^1 \psi(x)Y_k(x)dx \quad (k = 0,1,\ldots),$$

When the appropriate conditions are satisfied, we establish the validity of the following estimates:

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i}\psi_{2k-1})^2\right)^{\frac{1}{2}} \le 2\sqrt{2} \parallel \psi^{(2i)}(x) \parallel_{L_2(0,1)}, \quad (25)$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i}\psi_{2k})^2\right)^{\frac{1}{2}} \le 2\sqrt{2} \parallel \psi^{(2i)}(x)(1-b-ax) - 2ai\psi^{(2i-1)}(x) \parallel_{L_2(0,1)}. \quad (26)$$

Furthermore, under certain conditions:

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i+1}\psi_{2k-1})^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\parallel \psi^{(2i+1)}(x)\parallel_{L_2(0,1)}, \tag{27}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i+1}\psi_{2k})^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\parallel \psi^{(2i+1)}(x)(1-b-ax)-a(2i+1)\psi^{(2i)}(x)\parallel_{L_2(0,1)}. \tag{28}$$

Existence and Uniqueness of the Solution to the Inverse Problem

Since relation (21) forms a Riesz basis in the space $L_2(0,1)$ and relations (21)–(22) form a biorthogonal system of functions, we seek the function $u(x,t)$ of problem (1)–(3), (8)–(9) in the following form:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x), \tag{29}$$

where

$$u_k(t)=\int_0^1 u(x,t)Y_k(x)dx \quad (k=0,1,2,\ldots). \tag{30}$$

Then, applying the method of separation of variables, from relations (1) and (2) we obtain the following:

$$^{C}D_t^{\alpha}u_0(t)=F_0(t,u,a),$$

(31) $$^{C}D_t^{\alpha}u_{2k-1}(t)+\lambda_k^2u_{2k-1}(t)=F_{2k-1}(t,u,a) \quad (k=1,2,\ldots;0\le t\le T), \tag{32}$$

$$^{C}D_t^{\alpha}u_{2k}(t)+\lambda_k^2u_{2k}(t)=F_{2k}(t,u,a)-2a\lambda_k u_{2k-1}(t), \tag{33}$$

$$u_k(0)+\delta u_k(T)+\int_0^T p(t)u_k(t)dt=\varphi_k. \tag{34}$$

Solving problem (31)–(34), we arrive at the following results:

$$u_0(t)=u_0(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}F_0(\tau,u,a)d\tau, \tag{35}$$

$$u_{2k-1}(t)=u_{2k-1}(0)E_{\alpha}(-\lambda_k^2t^{\alpha})+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^{\alpha})F_{2k-1}(\tau,u,a)d\tau, \tag{36}$$

$$u_{2k}(t)=u_{2k}(0)E_{\alpha}(-\lambda_k^2t^{\alpha})+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^{\alpha})[F_{2k}(\tau,u,a)-2a\lambda_k u_{2k-1}(\tau)]d\tau. \tag{37}$$

$$u_{2k}(t)=u_{2k}(0)E_{\alpha}(-\lambda_k^2t^{\alpha})+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^{\alpha})[F_{2k}(\tau,u,a)-2a\lambda_k u_{2k-1}(\tau)]d\tau. \tag{38}$$

Here, $E_{\alpha}(z)\equiv E_{\alpha,1}(z)$, and $E_{\alpha,\beta}(z)$ is the generalized two-parameter Mittag-Leffler function defined by the series

$$E_{\alpha,\beta}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+\beta)}, \quad (\alpha>0,\beta\in\mathbb{R},z\in\mathbb{C}).$$

Using the nonlocal integral condition (34), we determine the unknown initial value coefficients. Using the nonlocal integral condition (34), we determine the unknown initial value coefficients. Substituting solutions (35), (36), and (37) into condition (34), we find respectively:

$$u_0(t)=\frac{1}{1+\delta}\left(\varphi_0-\frac{\delta}{\Gamma(\alpha)}\int_0^T(T-\tau)^{\alpha-1}F_0(\tau,u,a)d\tau-\int_0^T p(t)u_0(t)dt\right)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}F_0(\tau,u,a)d\tau, \tag{39}$$

$$u_{2k-1}(t)=\frac{E_{\alpha}(-\lambda_k^2t^{\alpha})}{1+\delta E_{\alpha}(-\lambda_k^2T^{\alpha})}\left(\varphi_{2k-1}-\delta\int_0^T(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(T-\tau)^{\alpha})F_{2k-1}(\tau,u,a)d\tau\right.$$
$$\left.-\int_0^T p(t)u_{2k-1}(t)dt\right)+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^{\alpha})F_{2k-1}(\tau,u,a)d\tau, \tag{40}$$

$$u_{2k}(t)=\frac{E_\alpha(-\lambda_k^2t^\alpha)}{1+\delta E_\alpha(-\lambda_k^2T^\alpha)}\left[\varphi_{2k}-\int_0^T p(t)u_{2k}(t)dt-\delta\int_0^T(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(T-\tau)^\alpha)F_{2k}(\tau,u,a)d\tau\right]$$
$$+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^\alpha)F_{2k}(\tau,u,a)d\tau+q_k(t,T), \quad (41)$$

where

$$q_k(t,T)=-2a\lambda_k\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^\alpha)u_{2k-1}(\tau)d\tau$$
$$+2a\lambda_k\frac{\delta E_\alpha(-\lambda_k^2t^\alpha)}{1+\delta E_\alpha(-\lambda_k^2T^\alpha)}\int_0^T(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(T-\tau)^\alpha)u_{2k-1}(\tau)d\tau. \quad (42)$$

Substituting the found Fourier coefficients into series (29), we obtain the form of the general solution of the inverse problem:

$$u(x,t)=u_0(t)(ax+b)+\sum_{k=1}^{\infty}[u_{2k-1}(t)(ax+b)\cos\lambda_kx+u_{2k}(t)\sin\lambda_kx]. \quad (43)$$

Now, using (9), we find the unknown coefficient $a(t)$:

$$a(t)=\frac{{}^CD_t^\alpha h(t)-u_{xx}(x_0,t)-f(x_0,t)}{h(t)}, \quad (44)$$

$$a(t)=\frac{1}{h(t)}\left\{{}^CD_t^\alpha h(t)+\sum_{k=1}^{\infty}\lambda_k^2u_{2k-1}(t)X_{2k-1}(x_0)+\sum_{k=1}^{\infty}\lambda_k^2u_{2k}(t)X_{2k}(x_0)-f(x_0,t)\right\}. \quad (45)$$

Using relations (39) and (40), we bring (44) to the following form:

$$a(t)=\frac{1}{h(t)}\ {}^CD_t^\alpha h(t)-f(x_0,t)$$
$$+\sum_{k=1}^{\infty}\lambda_k^2X_{2k-1}(x_0)\left[\frac{E_\alpha(-\lambda_k^2t^\alpha)}{1+\delta E_\alpha(-\lambda_k^2T^\alpha)}\left(\varphi_{2k-1}-\delta\int_0^T(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(T-\tau)^\alpha)F_{2k-1}(\tau)d\tau\right)\right.$$
$$\left.+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^\alpha)F_{2k-1}(\tau,u,a)d\tau\right]$$
$$+\sum_{k=1}^{\infty}\lambda_k^2X_{2k}(x_0)\left[\frac{E_\alpha(-\lambda_k^2t^\alpha)}{1+\delta E_\alpha(-\lambda_k^2T^\alpha)}\left(\varphi_{2k}-\int_0^T p(t)u_{2k}(t)dt-\delta\int_0^T(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(T-\tau)^\alpha)F_{2k}(\tau,u,a)d\tau\right)\right.$$
$$\left.\left.+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^\alpha)F_{2k}(\tau,u,a)d\tau+q_k(t,T)\right]\right\}. \quad (46)$$

Lemma 1 *If the set* $\{u(x,t),a(t)\}$ *is a classical solution of problem (1)–(3), (8), and (9), then the functions*

$$u_0(t)=\int_0^1u(x,t)Y_0(x)dx,$$
$$u_{2k-1}(t)=\int_0^1u(x,t)Y_{2k-1}(x)dx \quad (k=1,2,\ldots),$$
$$u_{2k}(t)=\int_0^1u(x,t)Y_{2k}(x)dx \quad (k=1,2,\ldots)$$

satisfy the system (42), (43), and (44) on the interval $[0,T]$.

Corollary 1 *According to Lemma 1, proving the uniqueness of the solution to problem (1)–(3), (8), (9) reduces to proving the uniqueness of the solution to system (39), (40), (41).*

To investigate problem (1)–(3), (8), (9), we consider the following function spaces: Let $B_{2,T}^3$ denote set of all functions considered in the domain D_T and having the following form:

$$u(x,t)=u_0(t)(ax+b)+\sum_{k=1}^{\infty}u_{2k-1}(t)(ax+b)\cos\lambda_kx+\sum_{k=1}^{\infty}u_{2k}(t)\sin\lambda_kx, \quad (47)$$

where the functions $u_0(t)$, $u_{2k-1}(t)$ $(k=1,2,\ldots)$, and $u_{2k}(t)$ $(k=1,2,\ldots)$ involved in the sum are continuous on the interval $[0,T]$, and

$$\| u_0(t) \|_{C[0,T]} + \left(\sum_{k=1}^{\infty} (\lambda_k^3 \| u_{2k-1}(t) \|_{C[0,T]})^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} (\lambda_k^3 \| u_{2k}(t) \|_{C[0,T]})^2 \right)^{\frac{1}{2}} < +\infty. \quad (48)$$

The operations of addition and scalar multiplication in the space $B_{2,T}^3$ are defined as usual, and the norm is defined by the following formula:

$$\| u(x,t) \|_{B_{2,T}^3} = \| u_0(t) \|_{C[0,T]} + \left(\sum_{k=1}^{\infty} (\lambda_k^3 \| u_{2k-1}(t) \|_{C[0,T]})^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} (\lambda_k^3 \| u_{2k}(t) \|_{C[0,T]})^2 \right)^{\frac{1}{2}}. \quad (49)$$

Furthermore, let E_T^3 denote the space consisting of the topological product $B_{2,T}^3 \times C[0,T]$. The norm of its element $z(x,t) = \{u(x,t), a(t)\}$ is defined as follows:

$$\| z(x,t) \|_{E_T^3} = \| u(x,t) \|_{B_{2,T}^3} + \| a(t) \|_{C[0,T]}. \quad (50)$$

It is known that $B_{2,T}^3$ and E_T^3 are Banach spaces.

In the space E_T^3, we consider the following operator:

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\}, \quad (51)$$

where

$$\Phi_1(u,a) = \tilde{u}(x,t) = \sum_{k=0}^{\infty} \tilde{u}_k(t) X_k(x), \quad (52)$$

$$\Phi_2(u,a) = \tilde{a}(t), \quad (53)$$

and the functions $\tilde{u}_k(t)$ $(k=0,1,2,\ldots)$ as well as the function $\tilde{a}(t)$ are defined by the right-hand sides of(38), (39), (40), and (45), respectively.

Assume that the given data of problem (1)–(3) and (5) satisfy the following conditions:

- $\varphi(x) \in C^2[0,1]$, $\varphi''(x) \in L_2(0,1)$, $\varphi(0) = \beta\varphi(1)$, $\varphi'(0) = \varphi'(1)$, $\varphi''(0) = \beta\varphi''(1)$ $(\beta \neq \pm 1)$.
- $f(x,t), f_x(x,t), f_{xx}(x,t) \in C(D_T)$, $f_{xxx}(x,t) \in L_2(D_T)$, $f(0,t) = \beta f(1,t)$, $f_x(0,t) = f_x(1,t)$, $f_{xx}(0,t) = \beta f_{xx}(1,t)$ $(\beta \neq \pm 1)$ $(0 \le t \le T)$.
- $h(t), {}^C D_t^{\alpha} h(t) \in C[0,T]$, $h(t) \neq 0$ $(0 \le t \le T)$.

From these assumptions, according to the properties of the Mittag-Leffler functions established in [2], the following estimates hold:

$$| E_{\alpha}(-\lambda_k^2 t^{\alpha}) | \le \frac{M_1}{1+\lambda_k^2 t^{\alpha}}, \quad M_1 = \text{const} \ge 0,$$

$$| E_{\alpha,\alpha}(-\lambda_k^2 (t-\tau)^{\alpha}) | \le \frac{M_2}{1+\lambda_k^2 (t-\tau)^{\alpha}}, \quad M_2 = \text{const} \ge 0.$$

Consequently, we obtain

$$\| \tilde{u}_0(t) \|_{C[0,T]} \le (1+\delta)^{-1} (| \varphi_0 | + T \| p(t) \|_{C[0,T]} \| u_0(t) \|_{C[0,T]})$$

$$+ (1 + \delta(1+\delta)^{-1}) \left[\frac{T^{\alpha-0.5}}{\Gamma(\alpha)\sqrt{2\alpha-1}} \left(\int_0^T | f_0(\tau) |^2 \, d\tau \right)^{\frac{1}{2}} + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \| a(t) \|_{C[0,T]} \| u_0 \|_{C[0,T]} \right], \quad (54)$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel \tilde{u}_{2k-1}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}} \le 2\left[M_1\left(\sum_{k=1}^{\infty}(\lambda_k^3 \mid \varphi_{2k-1} \mid)^2\right)^{\frac{1}{2}} + M_1 T \parallel p(t) \parallel_{C[0,T]} \left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel u_{2k-1}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}}\right]$$

$$+M_2(1+\delta M_1)\frac{T^{\alpha-0.5}}{\sqrt{2\alpha-1}}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3 \mid f_{2k-1}(\tau) \mid)^2 d\tau\right)^{\frac{1}{2}} + M_2(1+\delta M_1)\frac{T^{\alpha}}{\alpha} \parallel a(t) \parallel_{C[0,T]} \left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel u_{2k-1}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{55}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel q_k(t,T) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}} \le \tilde{M}\frac{T^{\alpha}}{\alpha}(1+\delta M_1)\times\left[2\sqrt{2}M_1 \parallel \varphi'''(x) \parallel_{L_2} + 2\sqrt{2}M_2(1+\delta M_1)\frac{T^{\alpha-0.5}}{\sqrt{2\alpha-1}} \parallel f_{xxx}(x,t) \parallel_{L_2}\right.$$

$$\left.+\left(M_1 T \parallel p(t) \parallel_{C[0,T]} + M_2(1+\delta M_1)\frac{T^{\alpha}}{\alpha} \parallel a(t) \parallel_{C[0,T]}\right)\left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel u_{2k-1}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}}\right], \tag{56}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel \tilde{u}_{2k}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{5}\left[M_1\left(\sum_{k=1}^{\infty}(\lambda_k^3 \mid \varphi_{2k} \mid)^2\right)^{\frac{1}{2}} + M_1 T \parallel p(t) \parallel_{C[0,T]} \left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel u_{2k}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}}\right]$$

$$+\sqrt{5}M_2(1+\delta M_1)\left[\frac{T^{\alpha-0.5}}{\sqrt{2\alpha-1}}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3 \mid f_{2k}(\tau) \mid)^2 d\tau\right)^{\frac{1}{2}} + \frac{T^{\alpha}}{\alpha} \parallel a(t) \parallel_{C[0,T]} \left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel u_{2k}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}}\right]$$

$$+\sqrt{5}\left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel q_k(t,T) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}}. \tag{57}$$

Taking into account inequalities (54), (55), (56), and (57), and based on (25)–(28), we obtain the following estimates:

$$\parallel \tilde{u}_0(t) \parallel_{C[0,T]} \le A_1(T) + B_1(T) \parallel a(t) \parallel_{C[0,T]} + C_1(T) \parallel u(x,t) \parallel_{B_{2,T}^3}, \tag{58}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel \tilde{u}_{2k-1}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}} \le A_2(T) + B_2(T) \parallel a(t) \parallel_{C[0,T]} \parallel u(x,t) \parallel_{B_{2,T}^3} + C_2(T) \parallel u(x,t) \parallel_{B_{2,T}^3}, \tag{59}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel \tilde{u}_{2k}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}} \le A_3(T) + B_3(T) \parallel a(t) \parallel_{C[0,T]} \parallel u(x,t) \parallel_{B_{2,T}^3} + C_3(T) \parallel u(x,t) \parallel_{B_{2,T}^3}, \tag{60}$$

where

Now, using (58)–(60), we compute the following:

$$\parallel \tilde{u}(x,t) \parallel_{B_{2,T}^3} \le A_4(T) + B_4(T) \parallel a(t) \parallel_{C[0,T]} \parallel u(x,t) \parallel_{B_{2,T}^3} + C_4(T) \parallel u(x,t) \parallel_{B_{2,T}^3}, \tag{61}$$

where:

$A_4(T) = A_1(T) + A_2(T) + A_3(T)$, $B_4(T) = B_1(T) + B_2(T) + B_3(T)$,

$C_4(T) = C_1(T) + C_2(T) + C_3(T)$.

From (45), we estimate: $\parallel \tilde{a}(t) \parallel_{C[0,T]}$

Using (57), taking into account (25)–(28), and from relations (63), the following expression follows:

$$\parallel \tilde{a}(t) \parallel_{C[0,T]} \le A_5(T) + B_5(T) \parallel a(t) \parallel_{C[0,T]} \parallel u(x,t) \parallel_{B_{2,T}^3} + C_5(T) \parallel u(x,t) \parallel_{B_{2,T}^3}, \tag{63}$$

From inequalities (61) and (63), we deduce the following:

$$\parallel \tilde{u}(x,t) \parallel_{B_{2,T}^3} + \parallel \tilde{a}(t) \parallel_{C[0,T]} \le A(T) + B(T) \parallel \tilde{a}(t) \parallel_{C[0,T]} \parallel \tilde{u}(x,t) \parallel_{B_{2,T}^3} + C(T) \parallel \tilde{u}(x,t) \parallel_{B_{2,T}^3},$$

where

$A(T) = A_4(T) + A_5(T)$, $B(T) = B_4(T) + B_5(T)$, $C(T) = C_4(T) + C_5(T)$.

Theorem 2 *If conditions (1)–(3) and the inequality*

$$(A(T)+2)(B(T)A(T)+2)+C(T)\le 1 \tag{66}$$

hold, then problem (1)–(3) and (8)–(9) has a unique solution in the space E_T^3 under the condition

$$K=\{z\in E_T^3 : \| z \|_{E_T^3}\le R\le A(T)+2\}.$$

Remark 1 *Inequalities (65) are satisfied for sufficiently small values of* T .

Proof. In the space E_T^3, we consider the following equation:

$$z=\Phi(z), \tag{67}$$

where $z=(u,a)$, and the components $\Phi_i(u,a)$ $(i=1,2)$ of the operator $\Phi(u,a)$ are defined by the right-hand sides of equations (51) and (52).

Consider the operator $\Phi(u,a)$ in the ball $K=K_R$ of the space E_T^3. Using similar relations, for any $z_1,z_2\in K_R$ we obtain the validity of the following inequalities:

$$\| \Phi z \|_{E_T^3}\le A(T)+B(T)\| a(t) \|_{C[0,T]}\| u(x,t) \|_{B_{2,T}^3} +C(T)\| u(x,t) \|_{B_{2,T}^3}\le A(T)+B(T)(A(T)+2)^2 +C(T)(A(T)+$$

(68)

$$\| \Phi z_1-\Phi z_2 \|_{E_T^3}\le B(T)R(\| a_1(t)-a_2(t) \|_{C[0,T]} +\| u_1(x,t)-u_2(x,t) \|_{B_{2,T}^3})+C(T)\| u_1(x,t)-u_2(x,t) \|_{B_{2,T}^3} .$$

(69)

Then, according to (66), from (67) and (68) it is clearly seen that the operator satisfies the conditions of the contraction mapping principle on the set $K=K_R$. Therefore, the operator has a unique fixed point $z=(u,a)$ in the ball $K=K_R$, which is the solution to equation (66), meaning it is the unique solution of the system in the ball $K=K_R$.

After that, the function $u(x,t)$, as an element of the space $B_{2,T}^3$, is continuous in the domain D_T and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$. Furthermore, it can be verified that equation (1) and conditions (2), (3), (8), and (9) are satisfied in the usual sense. As a result, $(u(x,t),a(t))$ is a solution of problem (1)–(3), (8), and (9), and according to Corollary 1, it is unique in the ball $K=K_R$.

The following statement follows directly from Theorems 1 and 2.

Theorem 3 *Assume that all conditions of Theorem 2 and the compatibility conditions*

$$\int_0^1\varphi(x)dx=0,\quad h(0)+\delta h(T)+\int_0^T p(t)h(t)dt=\varphi(x_0)$$

are satisfied. If the conditions

$$\int_0^1 f(x,t)dx=0,\quad 0\le t\le T$$

hold, then problem (1)–(5) has a unique classical solution in the ball $K=K_R$.

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HOSILA OLDIDA KASR DARAJALI PARAMETR QATNASHGAN INTEGRO-DIFFERENSIAL TENGLAMALAR SISTEMASINNING ASIMPTOTIK YECHIMINI TUZISH

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Annotatsiya. *Ushbu ishda birinchi tartibli hosila oldida kasr darajali parametr qatnashgan sekin o‘zgaruvchili chiziqli bo‘lmagan integro-differensial tenglamalar sistemasi qaralgan. Qaralayotgan sistemaning xarakteristik tenglamasi nol karrali ildizga ega bo‘lib unga bitta elementar bo‘luvchi mos kelgan kritik hol uchun asimptotik ko‘rinishdagi yechimni tuzish masalasi o‘rganilgan.*

Kalit so‘zlar: *asimptotik, sistema, xarakteristika, tenglama, koeffitsiyent, differensial, chiziqli bo‘lmagan, umumlashgan teskari matrissa.*

СОСТАВЛЕНИЕ АСИМПТОТИЧЕСКОГО РЕШЕНИЯ СИСТЕМЫ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ДРОБНЫМ СТЕПЕННЫМ ПАРАМЕТРОМ ПЕРЕД ПРОИЗВОДНОЙ

Аннотация. *В данной работе рассматривается система медленно меняющихся нелинейных интегро-дифференциальных уравнений с дробным параметром перед производной первого порядка. Исследуется задача построения асимптотического решения для критического случая, в котором характеристическое уравнение рассматриваемой системы имеет нулевой кратный корень и соответствует одному элементарному делителю.*

Ключевые слова: *асимптотический, система, характеристика, уравнение, коэффициент, дифференциал, нелинейный, обобщённая обратная матрица.*

CONSTRUCTING AN ASYMPTOTIC SOLUTION FOR A SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS WITH A FRACTIONAL PARAMETER BEFORE THE DERIVATIVE

Abstract. *This paper considers a system of slowly varying nonlinear integro-differential equations containing a parameter with a fractional power before the first-order derivative. The problem of constructing a solution in asymptotic form is studied for the critical case where the characteristic equation of the system under consideration has a zero root of multiplicity one, corresponding to a single elementary divisor.*

Keywords: *asymptotic, system, characteristic, equation, coefficient, differential, nonlinear, generalized inverse matrix.*

Kirish. Sekin o‘zgaruvchi koeffitsiyentli differensial tenglamalarni sistematik o‘rganish o‘tgan asrning 50-yillarida boshlangan.

1948-1949 yillarda S.F.Fishenkoning [1,2] ishlarini paydo bo‘lishi shunday tenglamalarni o‘rganishni boshlagan. Bu ishlarda avtor

$$\frac{d^2y}{dt^2} + \varepsilon P(\tau,\varepsilon)\frac{dy}{dt} + q(\tau,\varepsilon)y = \varepsilon f(\tau,\varepsilon)e^{i\theta(t,\varepsilon)}$$

ko‘rinishdagi ikkinchi tartibli differensial tenglamani yechishning asimptotik metodini taklif etgan, bunda $P(\tau,\varepsilon), q(\tau,\varepsilon), f(\tau,\varepsilon), \frac{d\theta(\tau,\varepsilon)}{dt}$ sekin o‘zgaruvchiga bog‘liq bo‘lgan funksiyalar, $\tau = \varepsilon t$ sekin o‘zgaruvchi vaqt $\varepsilon > 0$ −kichik parametr. Ishda $k(\tau) = \frac{d\theta}{dt}$ (tashqi chastota) funksiya ba’zi $\tau \in [0, L]$lar uchun (1)-tenglamaning xarakteristik tenglamasining oddiy ildizlaridan bittasiga(xususiy chastotasi) teng bo‘lgan hol qaralgan.

Bu hol matematika-fizika masalalalarning tatbiqida muhim nazariy va amaliy ahamiyatga ega.

Fishenkoning ilmiy xulosalari G.N.Sovin, V.N.Shevelo, A.I.Kuji, O.A.Goreshko [3,4] tomonidan shaxtaning ko‘tarish kanatlarida yuzaga keladigan kuchayish orqali ifodalanuvchi differensial tenglamaning yaqinlashuvchi yechimlarini aniqlash uchun qo‘llanilgan. Bu tenglama ba’zi almashtirishlardan keyin sekin o‘zgaruvchi koeffitsiyentli differensial tenglamaga keltirilgan :

$$a(\tau,\varepsilon)\frac{d^2\varphi}{dT^2}+\varepsilon C(\tau,\varepsilon)\frac{d\varphi}{dt}+b(\tau,\varepsilon)\varphi=f(\tau,\varepsilon)$$

bunda $\tau=\varepsilon T, T=wt, w=\frac{a^2}{\alpha qc}, \varepsilon=\sqrt[3]{\frac{a}{w^2 q}}$ munosabatlardan aniqlanadi Q,α,q,l −qaralayotgan tenglamaga kiruvchi berilgan parametrlar.

Integro-differensial tenglamalar nazariyasiga ko‘plab ishlar bag‘ishlangan [5,8]. Integro-differensial tenglamalarga bag‘ishlangan kengroq ma’lumotlarni [7,8] ishlardan olish mumkin. Ikkita tenglamadan tashkil topgan chiziqli bo‘lmagan integro-differensial tenglamalar sistemasi uchun chegaraviy funksiyalar usulini A.B.Vasileva, B.F.Butuzov va ularning o‘quvchilari [9,10] ishlarida qo‘llaganlar. Kichik parametrga bog‘liq bo‘lgan integro-differensial tenglamalar ustida [11,12,13] tadqiqot ishlarni olib borishganligini ko‘rish mumkin.

Yuqorida qayd etilgan ishlardan farqli ushbu ishda chiziqli bo‘lmagan integro-differensial tenglamalar sistemasi uchun xarakteristik tenglamaning ildizlari nol karrali bo‘lgan kritik hol uchun umumlashgan teskari matrissa usulini qo‘llab asimptotik yechimini tuzish, masalasini qaraymiz.

Masalaning qo‘yilishi.

Ushbu ishda

$$\varepsilon^{\frac{p}{q}}\frac{dx}{dt}=A(\tau,\varepsilon)x+\varepsilon^2\int_0^L K(\tau,\sigma,\varepsilon)x(\sigma,\varepsilon)d\sigma+\varepsilon f(\tau,x,\varepsilon) \quad (3)$$

ko‘rinishidagi chiziqli bo‘lmagan integro-differensial tenglamalar sistemasini qaraymiz, $x(t,\varepsilon), f(\tau,x,\varepsilon)-n$ tartibli vektorlar bo‘lib $x(t,\varepsilon)$ −noma’lum, $A(\tau,\varepsilon)$, $K(\tau,\sigma,\varepsilon)$ (3)- tenglamaning yadrosi berilgan n −o‘lchovli kvadrat matrissalar, $\varepsilon>0$ kichik parametrning darajalari bo‘yicha yaqinlashuvchi darajali qatorga yoyiladi:

$$A(\tau,\varepsilon)=\sum_{s=0}^{\infty}\varepsilon^s A_s(\tau) \quad K(\tau,\sigma,\varepsilon)=\sum_{s=0}^{\infty}\varepsilon^s A_s(\tau,\sigma) \quad (4)$$

$f(\tau,x,\varepsilon)$ vektor $(\tau,x_0,0)$ nuqta atrofida Teylor qatoriga yoyiladi, $\tau=\varepsilon t,\ \sigma=st$ −sekin o‘zgaruvchi vaqt, p va q berilgan o‘zaro tub sonlar

$$\det[A_0(\tau)-\lambda(\tau)E]=0 \quad (5)$$

(E-birlik matrissa) tenglamaga (3)-sistemaning xarakteristik tenglamasi deyiladi. Uning ildizlarini $\lambda_i(\tau), i=\overline{1,n}$ orqali belgilaymiz.

Bu ishda (3)-sistemaning (5) tenglamasi yarim oddiy, ya’ni k −karrali nol ildizga ega bo‘lib, unga bitta elementar bo‘luvchiga ega bo‘lgan hol uchun asimptotik yechimni tuzsin masalasini qaraymiz. Qolgan $n-k=m$ ildizlar [10] chiga asosan qanday bo‘lishi muhim emas.

Asosiy ilmiy natijalar.

(3)-sistemaning (5) tenglamasi yarim oddiy, ya’ni k −karrali nol ildizga ega bo‘lib unga bitta elementar bo‘luvchi mos kelgan hol uchun asimptotik yechimini ko‘rsatuvchi quyidagi teorema o‘rinli.

Bu hol uchun (3) – sistemaning asimptotik yechimini ko‘rsatuvchi quyidagi teorema o‘rinli.

1 – Teorema. Faraz qilaylik quyidagi shartlar bajarilsin:

a) $A_s(\tau)$ $(s=0,1,2,\dots)$ matrissalar $\tau\in[0,L]$ oraliqda, $f(\tau,x,\varepsilon)$ vektor $P(\tau,x,\varepsilon)=P(\tau,x)\times(0<\varepsilon\le\varepsilon_0]$, bunda $P(\tau,x)-\tau,x$ o‘zgaruvchilar fazosining ba’zi soxasida cheksiz differensiallanuvchi;

b) $K_s(\tau,\sigma)$ $(s=0,1,2,\dots)$ barcha $\tau,\sigma\in[0,L]$ lar uchun τ o‘zgaruvchi bo‘yicha istalgan tartibli xususiy xosilalarga ega bo‘lsin.

c) (5)-tenglamaning ildizlari

$$\lambda_i(\tau)\equiv 0,\forall\tau\epsilon[0,L],(i=1,2\dots,k),Re\lambda_j(\tau)\le 0\ (j=k+1,\dots,n)$$

munosabatda bo‘lsin.

U holda (3)-sistema $p>q$ xol uchun

$$x(t,\varepsilon)=\sum_{s=0}^{\infty}\mu^s u_s(\tau),\ \ \mu=\sqrt[q]{\varepsilon} \quad (6)$$

ko‘rinishdagi formal xususiy yechimga ega bo‘ladi.

Isbot. 1-teoremani isbot qilish uchun (6)-ifodani (3)-sistemaga qo‘yib xosil qilingan ifodadan $f(\tau,x,\mu^q)$ vektorni $(\tau,x_0,0)$ nuqta atrofida Teylor qatoriga yoyilmasidan μ parametrining bir xil darajalari hadlarini yig‘ib

$\mu^p u'(\tau,\mu) = A(\tau,\mu^q)u(\tau,\mu) + \mu^q \int_0^L K(\tau,\sigma,\mu^q)d\sigma u(\sigma,\mu)d\sigma + \mu^q\{f(\tau,u_0(\tau),0) + \mu f_u(\tau)u_1(\tau) + \cdots + \mu^s[f_u(\tau)u_s(\tau) + f_{s-1}(\tau) + \tilde{f}_{s+1-q}(\tau)] + \cdots$

(7)

ayniyatni hosil qilamiz, bunda $f_u(\tau) = \frac{\partial f^i}{\partial \mu^j}$ matrissaning elementlari va $f_s(\tau)$ vektorning koordinatalari $(\tau, u_0(\tau), 0)$ nuqtaga hisoblanadi. $\tilde{f}_{s+1-q}(\tau)$ $(s = 2,3,\dots)$ vektor $u_s(\tau)$ $(s = 0,1,2,\dots,s-1)$ vektorlar orqali ma'lum tartibda ifodalanadi.

(7)-ayniyatdan μ parametrning bir xil darajalari oldidagi koeffitsiyentlari tenglashtirib (6)-qatorning noma'lum hadlarini aniqlash uchun rekurent tenglamalar sistemasini hosil qilamiz.

$$A_0(\tau)u_\upsilon(\tau) = 0, \upsilon = 0,1,\dots,q-1, \quad (8)$$

$$A_0(\tau)u_q(\tau) = -A_1(\tau)u_1(\tau) - f\big(\tau,u_0(\tau)\big) \quad (9)$$

$$A_0(\tau)u_p(\tau) = -A_1(\tau)u_2(\tau) - f_u(\tau)u_2(\tau) - f_1(\tau) + u'_0(\tau) \quad (10)$$

……

$$A_0(\tau)u_{2q}(\tau) = -A_1(\tau)u_3(\tau) - f_u(\tau)u_3(\tau) - f_2(\tau) - \int_0^t K_0(\tau)u_0(\sigma)d\sigma \quad (11)$$

…….

$$A_0(\tau)u_s(\tau) = -A_1(\tau)u_{s-q}(\tau) - f_u(\tau)u_{s-q}(\tau) - a_1(\tau) \quad (12)$$

bunda

$a_1(\tau) = -\sum_{s=1}^{\left[\frac{s}{p}\right]} A_s(\tau)u_{s-ip}(\tau) - \sum_{s=1}^{\left[\frac{s}{2q}\right]} K_s(\tau,\sigma)\,u_{s-2q}(\tau)d\sigma - f_{s-q}(\tau) + u'_{s-p}(\tau) \quad s = 2q+1, 2p+2, \dots$

(8)-sistemadan [14] chiga asosan

$$u_\nu(\tau) = \sum_{i=1}^{x} \varphi_i(\tau)\eta_{i\nu}(\tau), \ \gamma = 0,1,\dots,q-1 \quad (13)$$

yechimni hosil qilamiz, bunda $\eta_{i\nu}(\tau)$ keyingi qadamda aniqlanadigan noma'lum funksiyalar $\varphi_i - A_0(\tau)$ matrissaning xos qiymatiga mos keluvchi $n-$ o'lchovli xos vektor $\varphi_i \in N(A_0(\tau))$, bo'lib $i-$chi koordinatasi 1-ga qolgan koordinatali 0-ga teng bo'lgan ustunli vektor.

(9) – tenglamani (13) – ifodani e'tiborga olib

$$A_0(\tau)u_q(\tau) = -A_1(\tau)\sum_{i=1}^{k}\varphi_i(\tau)\eta_{i0}(\tau) \quad f(\tau,\sum_{i=1}^{k}\varphi_i(\tau)\eta_{i0}(\tau)) \quad (14)$$

ko'rinishida yozamiz.

(14)-bir jinsli bo'lmagan tenglama bo'lib, u yechimga ega bo'lishi yechim mavjud bo'lish sharti bajarilishi zarur va yetarlidir. Faraz qilaylik (14)-tenglama uchun $\tau \in [0,L]$ bo'yicha yechim mavjud bo'lish sharti

$$(\psi_j, (-A_1(\tau)\sum_{i=1}^{k}\varphi_i(\tau)\eta_{i0}(\tau) - f(\tau,\sum_{i=1}^{k}\varphi_i(\tau)\eta_{i0}(\tau)) = 0 \quad (15)$$

bajarilsin. Bu shartdan

$$(\psi_j, (-A_1(\tau)\sum_{i=1}^{k}\varphi_i))\eta_{i0}(\tau) - (\psi_j, f(\tau,\sum_{i=1}^{k}\varphi_i(\tau)\eta_{i0}(\tau)) = 0 \quad (16)$$

tenglikni xosil qilamiz, bunda $\psi \in N\big(A_0^*(i)\big)$, φ_i va φ_j $(i,j = \overline{1,k})$, ortonormallangan vektorlar bo'lib $\forall(\tau) \in [0,L]$ oraliqda

$$\big(\psi_j,\varphi_i\big) = 1, i = j, \big(\big(\psi_j,\varphi_i\big) = 0, i \neq j, i,j = \overline{1,k}\ , \quad (17)$$

$$(\psi_j, (A_1(\tau) - f_u(\tau)\varphi_i) \neq 0$$

$A_0(\tau)$ matrissaning qo'shma matrissasi matrissaning xos qiymati $\psi_j - A_0^*(i)$, $j-$koordinatasi 1-ga qolgan koordinatali 0-ga teng bo'lgan satrli vektor.

(16)-tenglama uchun oshkormas funksiya haqidagi teorema o'rinli deb faraz qilib, [15] chiga asosan $\eta_{i0}(\tau)$ noma'lum funksiyani aniqlaymiz. Demak (14) tenglama uchun (16) shart o'rinli, u holda bu tenglamadan (6) qatorning noma'lum hadi $u_q(\tau)-$vektorni

$$u_q(\tau) = A_0^+(\tau)[-A_1(\tau)\sum_{i=1}^{k}\varphi_i\eta_{i0}(\tau) - f(\tau,\sum_{i=1}^{k}\varphi_i(\tau)\eta_{i0}(\tau))] + \sum_{i=1}^{k}\varphi_i\eta_{iq}(\tau) \quad (18)$$

ko'rinishda aniqlaymiz, bunda $\eta_{iq}(\tau)$- keyingi qadamda aniqlanadigan noma'lum funksiya, $A_0^+(\tau)$, $A_0(\tau)$ matrissaning umumlashgan teskari matrissasi

$$A_0^+(\tau) = [A_0(\tau) - (\psi_j \otimes \sum_{i=1}^{k}\varphi_i)]^{-1} - \sum_{\eta=1}^{k}\big(\varphi_i \otimes \psi_j\big) \quad (19)$$

$\otimes, \varphi_i\ va\ \psi_j$ vektorlarning tenzor ko'paytmasi belgisi.

(10)-tenglamaning (13) tenglamadagi $\nu_0 = 0$ va $\nu = 2$ bo'lgandagi ifodasini e'tiborga olib

$$A_0(\tau)u_p(\tau) = -A_1(\tau)[\sum_{i=1}^{k}\varphi_i\eta_{i2}(\tau)) + f_u(\tau)]\sum_{i=1}^{k}\varphi_i\eta_{i2}(\tau)) - f_1(\tau) + +\sum_{i=1}^{k}\varphi_i\eta'_{i0}(\tau) \quad (20)$$

ko‘rinishda yozamiz. (20) tenglama uchun yechim mavjud bo‘lish sharti

$$(\psi_j, \left(A_1(\tau) - f_u(\tau)\right) \sum_{i=1}^{k} \varphi_i)\eta_{i2}(\tau) = (\psi_j, f_1(\tau)) - (\psi_j, \sum_{i=1}^{k} \varphi_i)\eta'_{i0}(\tau),$$

$j = \overline{1,n}.$ (21)

(21) tenglamadan $\eta_{i2}(\tau)$ −noma’lum funksiyani (17) – shartni etiborga olib

$$\eta_{i2}(\tau) = \frac{\left(\psi_j, f_1(\tau)\right) - (\psi_j, \sum_{i=1}^{k} \varphi_i)\eta'_{i0}(\tau)}{(\psi_j, (A_1(\tau) - f_u(\tau)\varphi_i)} \quad i = j, \quad i, j = \overline{1,n} \qquad (22)$$

ko‘rinishga ega ekanligini aniqlaymiz.

(18)-tenglama uchun yechim mavjud bo‘lish sharti o‘rinli, u holda (6)-qatorning $u_p(\tau)$ noma’lum hadi

$$u_p(\tau) = A_0^+(\tau)[-\left(A_1(\tau) + f_u(\tau)\right) \sum_{i=1}^{k} \varphi_i \eta_{i2}(\tau) - f_1(\tau) + + \sum_{i=1}^{k} \varphi_i \eta'_{i0}(\tau)] + \sum_{i=1}^{k} \varphi_i \eta_{ip}(\tau) \qquad (23)$$

ga teng bo‘ladi, $\eta_{ip}(\tau)$ −keyingi qadamda aniqlanadigan noma’lum funksiya. (11)-tenglama (13)-tenglamada $\nu = 0, \nu = 3$ bo‘lganda

$$A_0(\tau)u_{2q}(\tau) = -\left(A_1(\tau) + f_u(\tau)\right) \sum_{i=1}^{k} \varphi_i \eta_{i3}(\tau) - f_2(\tau) - \int_0^L K_0(\tau, \sigma) \sum_{i=1}^{k} \varphi_i \eta_{i0}(\tau) d\sigma + \sum_{i=1}^{k} \varphi_i \eta_{i2q}(\tau) \qquad (24)$$

ko‘rinishga ega. Bu tenglama uchun yechim mavjud bo‘lish sharti

$$(\psi_j(-A_1(\tau) + f_u(\tau) \sum_{i=1}^{k} \varphi_i)\eta_{i3}(\tau) - (\psi_j, f_2(\tau)) - (\psi_j, \int_0^L K_0(\tau, \sigma) \sum_{i=1}^{k} \varphi_i)\eta_{i0}(\tau) d\sigma + (\psi_j, \sum_{i=1}^{k} \varphi_i)\eta_{i2q-P}(\tau) \qquad (25)$$

ko‘rinishda bo‘ladi. Bu tenglamadan noma’lum $\eta_{i3}(\tau)$ funksiya:

$$\eta_{i3}(\tau) = \frac{\left(\psi_j, f_1(\tau)\right) - (\psi_j, \int_0^L K_0(\tau, \sigma) \sum_{i=1}^{k} \varphi_i)\eta_{i0}(\tau) d\sigma + (\psi_j, \sum_{i=1}^{k} \varphi_i)\eta_{i2q-p}(\tau)}{((\psi_j, -\left(A_1(\tau) + f_u(\tau)\right)\varphi_i)}$$

$i = j, \quad i, j = \overline{1,n}$ (26)

ko‘rinishida aniqlanadi. Demak (24)-tenglama uchun yechim mavjud bo‘lish sharti o‘rinli. U holda (6)-qatorning $u_{2q}(\tau)$ noma’lum hadi

$$u_{2q}(\tau) = A_0^+(\tau)[-\left(A_1(\tau) + f_u(\tau)\right) \sum_{i=1}^{k} \varphi_i \eta_{i3}(\tau) - f_2(\tau) - \int_0^L K_0(\tau, \sigma) \sum_{i=1}^{k} \varphi_i \eta_{i0}(\sigma) d\sigma + \sum_{i=1}^{k} \varphi_i \eta_{i2q-p}(\tau)] + \sum_{i=1}^{k} \varphi_i \eta_{i2q-p+1}(\tau), \qquad (27)$$

η_{i2q-p} −funksiya yuqoridagidek keyingi qadamda aniqlanadigan noma’lum funksiya

Shu jarayonni davom ettirib S −chi qadamda (12) tenglamani

$$A_0(\tau)u_s(\tau) = -\left(A_1(\tau) + f_u(\tau)\right) \sum_{i=1}^{k} \psi \eta_{is-q}(\tau) - Q(\tau) \qquad (28)$$

ko‘rinishida yozamiz, bunda

$$Q(x) = -\sum_{s=1}^{\left[\frac{s}{q}\right]} A_s(\tau)u_{s-ip}(\tau) - \sum_{s=1}^{\left[\frac{s}{2q}\right]} K_s(\tau\sigma)u_{s-2q} d\sigma - f_{s-q}(\tau) - u'_{s-p}(\tau) - \cdots \qquad s = 2q + 1, 2q + 2, \ldots$$

(28)-tenglama uchun yechim mavjud bo‘lish sharti

$$(\psi_j, \left(-\left(A_1(\tau) + f_u(\tau)\right) \sum_{i=1}^{k} \varphi_i)\eta_{is-q}(\tau) + (Q(\tau), \psi_j\right) = 0, \qquad (29)$$

ko‘rinishda bo‘ladi. Bu tenglamadan $\eta_{s-q}(\tau)$ −noma’lum funksiyani aniqlasak

$$\eta_{s-q}(\tau) = -\frac{\left(\psi_j, Q(\tau)\right)}{(\psi_j, (A_1(\tau) + f_u(\tau))\varphi_i)} \quad i = \overline{1,k} \qquad (30)$$

ko‘rinishda bo‘ladi.

Bundan ko‘rinadiki (28)-tenglama uchun (29) shart bajariladi. U holda (6) qatorning $u_s(\tau)$ noma’lum vektori

$$u_s(\tau) == A_0^+(\tau)\left[-\left(A_1(\tau) + f_u(\tau)\right) \sum_{i=1}^{k} \varphi_i \eta_{is-q}(\tau) - Q(\tau)\right] + \sum_{i=1}^{k} \varphi_i \eta_{is}(\tau) \qquad (31)$$

ko‘rinishda bo‘ladi, bunda $\eta_{is}(\tau)$ −keyingi qadamda aniqlanadigan noma’lum funksiya.

(6)-formal qatorning koeffitsiyentlarini aniqlash usullarini ko‘rsatish orqali, biz 1-teoremani isbotladik.

Asimptotik yaqinlashishni isbotlash.

Yuqorida tanishib chiqilgan algoritm yordamida (6)-formuladagi qatorning har qanday hadlarini aniqlash mumkin. Amaliyotda hisoblashlarning kattalashib ketishini e'tiborga olib, qator yoyilmasining dastlabki m ta hadini tuzish bilan chegaralanamiz. Shu munosabat bilan(3)-sistemaning m ta yaqinlashuvchi yechimi tushunchasi qaraladi.

Qaralayotgan hol uchun m −yaqinlashuvchini

$$x_m(t,\varepsilon)=\sum_{s=0}^{m} e^s u_s(\tau) \qquad (32)$$

ifoda orqali aniqlaymiz. Faraz qilaylik, $x(t,\varepsilon)$ (3)-sistemaning $t=0$ shartni qanoatlantiruvchi aniq yechimi bo'lsin. Shu boshlang'ich shartni $x_m(t,\varepsilon)$ vektor ham qanoatlantirsin. U holda [16] chida ko'rsatilgandek $x_m(t,\varepsilon)$ yaqinlashuvchi yechim $x(t,\varepsilon)$ yechimga asimptotik yaqinlashishni ko'rsatishimiz mumkin. Dastlab quyidagi Lemmani keltiramiz.

1. Lemma. Faraz qilaylik 1-teoremaning shartlari bajarilsin, u holda m −chi $x_m(t,\varepsilon)$ yaqinlashuvchi yechim

$$\frac{dx_m}{dt}=[A_0(\tau)+\mu^q A_0(\tau,\mu)]u_m+\mu^q f(\tau,u_m,\mu^q)+\mu^{\tilde{q}}\int_0^L K(\tau,\sigma,\mu^q)\,u_m(\sigma,\mu^q)d\sigma+\mu^q R_m(\tau,\mu), \qquad (33)$$

tenglamani qanoatlantiradi, bunda $R_m(\tau,\mu)$ −vektor funksiya $[0,L]$ oraliqda chegaralangan.

Lemmani isbot qilish uchun (32) ifodani (3)-tenglamaga qo'yib 1-teoremaga asosan (32)-vektor (31) sistemani $O(\varepsilon^{m+1})$ aniqlikda qanoatlantirishga ishonch hosil qilamiz.

Faraz qilaylik, boshlang'ich

$$V(\tau,\varepsilon)|_{\tau=0}=V_0 \qquad (34)$$

shartni qanoatlantiruvchi

$$\frac{dV}{dt}=A_0(\tau)V \qquad (35)$$

tenglama $V(\tau,\varepsilon)$ yechimga ega bo'lsin. Bu yechim $\forall\tau\in[0,L]$ uchun

$$\|V(\tau,\varepsilon)\|\le M \qquad (36)$$

tenglikni qanoatlantirsin, bunda M,ε −ga bog'liq bo'lmagan o'zgarmas.

1-teoremada. Tuzilgan formal xususiy yechimni asimptotik xarakterga ega ekanligini quyidagi teorema ko'rsatadi.

2-teorema. Faraz qilaylik (3)-sistema uchun 1-teoremaning shartlari bajarilsin va

a) $f(\tau,x,\varepsilon)$ vektor funksiya o'zgarmas l soni uchun Lipshas shartini qanoatlantirsin,

$$\|f(\tau,x,\varepsilon)-f(\tau,x_m,\varepsilon)\|\le l\|x-x_m\|; \qquad (37)$$

b) (36) −shart bajarilib

$$x(t,\varepsilon)|_{t=0}=x_m(t,\varepsilon)|_{t=0}, \qquad (38)$$

tenglik o'rinli bo'lsin. U holda har qanday katta $L>0$ va o'zgarmas C uchun shunday $\varepsilon\in(0,\varepsilon_0]$ −ni mos qo'yishi mumkinki, $0<t\le\frac{L}{\mu^q}$ oraliqda

$$\|x(t,\varepsilon)-x_m(t,\varepsilon)\|\le\mu^{m+1-q}C \qquad (39)$$

tengsizlik o'rinli.

Isbot. (3)-sistemadan (33)-sistemani ayirib

$$\frac{dz}{dt}=A_0(\tau)z+\varepsilon A_1(\tau,\varepsilon)z+\varepsilon^2\int_0^\tau\sum_{i=1}^m(K_m(\tau,\sigma)+\sum_{i=1}^m K_m(\tau,\sigma)u_m(\sigma,\mu^q)d\sigma+\mu^q[f(\tau,x,\varepsilon)-f(\tau,x_m,\varepsilon)]+\varepsilon^{m+1}R_m(\tau,\varepsilon) \qquad (40)$$

tenglamani hosil qilamiz, bunda

$$z(t,\varepsilon)=x(t,\varepsilon)-x_m(t,\varepsilon),\qquad A_1(\tau,\varepsilon)=\sum_{s=1}^{m}\varepsilon^s A_1(\tau),\qquad L(\tau)=\tau \qquad (41)$$

(40)-tenglama

$$z(t,\varepsilon)=\varepsilon\int_0^L V(\tau,\varepsilon)V^{-1}(t,\varepsilon)zA_1(\tau,\varepsilon)z(t_1,\varepsilon)+V^{-1}(t,\varepsilon)\,f(\tau,x,\varepsilon)-[f(\tau,x_m,\varepsilon)]+\varepsilon^2\int_0^\tau V(\tau,\varepsilon)V^{-1}(t,\varepsilon)\big(K_m(\tau,0)+K_{is}(\tau,\sigma)\big)u_m d\sigma+\varepsilon^{m+1}\int_0^\tau V(t,\varepsilon)V^{-1}(t_1,\varepsilon)R_m(t,\varepsilon)dt_1, \qquad (42)$$

integral tenglamaga ekvivalent.

(42)-tenglamadan norma olib

$$\|z(t,\varepsilon)\|\le\varepsilon\int_0^\tau\|V(\tau,\varepsilon)\|\|V^{-1}(t,\varepsilon)\|\|A_1(\tau,\varepsilon)\|\|z(\tau,\varepsilon)\|dt+\varepsilon^2\int_0^t(\int_0^\tau\big\|K_0(\tau,\sigma)\big\|\big\|z(\sigma,\varepsilon)\big\|d\sigma+\big\|K_1(\tau,\sigma)\|\big\|z\|(\sigma,\varepsilon)\|d\sigma+\mu^{m+1-q}MLR_* \qquad (43)$$

tengsizlikni hosil qilamiz.

(43)-chi ifodaga Gronullo-bellman [17] tengsizligini qo‘llab

$$\|z(t,\varepsilon)\| = \|x(t,\varepsilon) - x_m(t,\varepsilon)\| \le \mu^{m+1-q} C \qquad (44)$$

tengsizlikini hosil qilamiz, bunda

$$C = MLR_* \exp(N + lMM), \quad [0,L] \text{ kesmada } \|A_1(\tau,\varepsilon)\| N, \quad \|R_*(\tau,\varepsilon)\| \le R_*$$

tengsizlik o‘rinli. (44) munosabat tuzilgan formal xususiy yechimni asimptotik xarakterga ega ekanligini ko‘rsatadi.

Xulosa. Ushbu ish kichik parametrga bog‘liq bo‘lgan sekin o‘zgaruvchi koeffitsiyentli integral-differensial tenglamalar sistemasining xarakteristik tenglamasi $k-$karrali $(k < n)$ nol ildizga ega bo‘lib unga bitta elementar bo‘luvchi mos kelgan hol uchun (qolgan ildizlar noldan farqli) formal xususiy yechimi tuzilgan. Shuningdek tuzilgan formal xususiy yechimni asimptotik xossaga ega ekanligi isbotlangan.

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TASODIFIY JARAYONLARNI MATEMATIK MODELLASHTIRISHNING NAZARIY ASOSLARI

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Annotatsiya. *Ushbu maqolada tasodifiy jarayonlarni matematik modellashtirishning nazariy va amaliy jihatlari binomial taqsimot asosida tadqiq etilgan. Binomial model parametrlarining matematik kutilma, dispersiya va standart og‘ishga ta’siri parametrik hamda analitik usullar yordamida tahlil qilingan. Dispersiyaning* $p = 0{,}5$ *da maksimal qiymatga erishishi asoslangan va tajribalar sonining model xususiyatlariga ta’siri baholangan. Shuningdek, binomial model asosida texnologik jarayonda mahsulot sifatini statistik nazorat qilish hamda* 3σ *tamoyili bo‘yicha nazorat chegaralarini aniqlash imkoniyatlari ko‘rsatib berilgan.*

Kalit so‘zlar: *tasodifiy jarayon, matematik modellashtirish, binomial taqsimot, matematik kutilma, dispersiya, parametrik tahlil, statistik nazorat,* 3σ *nazorat chegaralari.*

ТЕОРЕТИЧЕСКИЕ ОСНОВЫ МАТЕМАТИЧЕСКОГО МОДЕЛИРОВАНИЯ СЛУЧАЙНЫХ ПРОЦЕССОВ

Аннотация. *В статье исследованы теоретические и практические аспекты математического моделирования случайных процессов на основе биномиального распределения. Параметрическими и аналитическими методами проанализировано влияние параметров биномиальной модели на математическое ожидание, дисперсию и стандартное отклонение. Обосновано, что дисперсия достигает максимального значения при* $p = 0{,}5$ *а также оценено влияние числа испытаний на характеристики модели. Рассмотрены возможности статистического контроля качества продукции в технологическом процессе на основе биномиальной модели и определения контрольных границ по принципу* 3σ.

Ключевые слова: *случайный процесс, математическое моделирование, биномиальное распределение, математическое ожидание, дисперсия, параметрический анализ, статистический контроль, контрольные границы* 3σ.

THEORETICAL FOUNDATIONS OF MATHEMATICAL MODELING OF RANDOM PROCESSES

Abstract. *This article investigates the theoretical and practical aspects of mathematical modeling of random processes based on the binomial distribution. The effects of the binomial model parameters on the mathematical expectation, variance, and standard deviation are analyzed using parametric and analytical methods. It is shown that the variance reaches its maximum value at* $p = 0.5$ *and the influence of the number of trials on the model characteristics is evaluated. In addition, the possibilities of statistical quality control in technological processes using the binomial model and the determination of control limits based on the* 3σ *principle are considered.*

Keywords: *random process, mathematical modeling, binomial distribution, mathematical expectation, variance, parametric analysis, statistical quality control,* 3σ *control limits.*

Kirish. Tasodifiy jarayonlarni matematik modellashtirish ehtimollar nazariyasi va matematik statistikaning muhim yo‘nalishlaridan biri hisoblanadi. Tasodifiy hodisalar va ularning ehtimollik qonuniyatlarini matematik tavsiflash masalalari zamonaviy ehtimollar nazariyasiga oid fundamental tadqiqotlarda keng yoritilgan [1–4]. Xususan, binomial taqsimot o‘zaro mustaqil va bir xil sharoitda takrorlanadigan ikki natijali tajribalarda muvaffaqiyatlar sonini modellashtirishning asosiy diskret ehtimollik modellaridan biri sifatida qaraladi [1, 5, 6].

Binomial modelning matematik kutilmasi va dispersiyasi uning n va p parametrlariga bevosita bog‘liq. Mazkur parametrlarning o‘zgarishi nafaqat kutilayotgan natijaga, balki tasodifiy natijalarning

absolyut va nisbiy tarqalish darajasiga ham ta’sir qiladi. Shu jihatdan binomial model parametrlarini tizimli tahlil qilish hamda olingan nazariy natijalarni ikki natijali texnologik jarayonlarning statistik nazorati bilan bog‘lash muhim amaliy ahamiyatga ega.

Maqolaning asosiy maqsadi binomial taqsimot parametrlarining matematik kutilma, dispersiya, standart og‘ish va variatsiya koeffitsiyentiga ta’sirini parametrik hamda analitik usullar yordamida tadqiq etish, shuningdek, olingan natijalar asosida binomial modeldan texnologik jarayonlarda mahsulot sifatini statistik nazorat qilishda foydalanish imkoniyatlarini asoslashdan iborat.

Mavzuga oid adabiyotlar tahlili. Tasodifiy jarayonlar va ehtimollik modellarining nazariy asoslari S.M. Rossning tadqiqotlarida keng yoritilgan. Muallif tasodifiy hodisalar, tasodifiy miqdorlar, ularning taqsimot qonunlari hamda binomial modelning asosiy xususiyatlarini tizimli ravishda bayon etgan [1, 2]. G.R. Grimmett va D. R. Stirzaker ishlarida esa ehtimollik va tasodifiy jarayonlarning matematik asoslari, tasodifiy miqdorlarning sonli xarakteristikalari va ehtimollik modellarining nazariy xususiyatlari chuqur tahlil qilingan [3].

R. B. Schinazi tadqiqotida ehtimollik modellarining statistik qo‘llanilishiga alohida e’tibor qaratilgan [4]. B. Illowsky va S. Dean ishida matematik kutilma, dispersiya, standart og‘ish hamda diskret ehtimollik taqsimotlarining statistik tahlildagi o‘rni yoritilgan [5]. Binomial taqsimotning matematik xususiyatlari va undan statistik masalalarda foydalanish imkoniyatlari NIST/SEMATECH elektron qo‘llanmasida ham keltirilgan [6].

Mavjud manbalar binomial taqsimotning nazariy va statistik asoslarini keng yoritadi. Mazkur maqolada esa ushbu nazariy asoslardan kelib chiqib, binomial model parametrlarining matematik kutilma, dispersiya, standart og‘ish va variatsiya koeffitsiyentiga ta’siri o‘zaro bog‘liq holda tahlil qilinadi hamda modelning ikki natijali texnologik jarayonlarda statistik nazorat vositasi sifatida qo‘llanish imkoniyatlariga e’tibor qaratiladi.

Tadqiqot metodologiyasi. Tadqiqotda nazariy, parametrik va analitik tahlil usullaridan foydalanildi. Nazariy qismda tasodifiy hodisa, tasodifiy miqdor, matematik kutilma, dispersiya va standart og‘ish tushunchalari hamda binomial taqsimotning asosiy matematik xususiyatlari o‘rganildi [1–6].

Parametrik tahlilda binomial modelning n va p parametrlarining matematik kutilma $M(X) = np$, dispersiya $D(X) = np(1-p)$ va standart og‘ish $\sigma(X) = \sqrt{np(1-p)}$ ko‘rsatkichlariga ta’siri sonli hisoblashlar asosida baholandi. Dispersiyaning p parametriga bog‘liqligi hosila olish usuli yordamida analitik tahlil qilinib, $p = 0{,}5$ da maksimal qiymatga erishishi asoslandi. Shuningdek, tajribalar sonining natijalarning nisbiy tarqalishiga ta’siri variatsiya koeffitsiyenti orqali baholandi.

Binomial modelning amaliy qo‘llanilishini baholash uchun ikki natijali texnologik jarayonda mahsulot sifatini statistik nazorat qilish masalasi ko‘rib chiqildi. Model asosida matematik kutilma va standart og‘ish hisoblanib, 3σ tamoyili bo‘yicha **taxminiy statistik nazorat chegaralari** aniqlandi hamda kuzatilgan natijalarni ushbu chegaralar bilan taqqoslash asosida jarayon holatini baholash yondashuvi qo‘llanildi.

Tahlil va natijalar. Binomial taqsimot mustaqil va bir xil sharoitda takrorlanadigan tajribalarda ikki xil natijadan birining yuzaga kelish sonini modellashtirish uchun qo‘llaniladi. Bunda tajribalar soni n, har bir tajribada muvaffaqiyatning yuzaga kelish ehtimoli p, muvaffaqiyatsizlik ehtimoli esa $q = 1 - p$ orqali belgilanadi. Agar tasodifiy miqdor ta mustaqil tajribada muvaffaqiyatlar sonini ifodalasa, u binomial taqsimotga ega bo‘lib, $X \sim B(n,p)$ ko‘rinishida belgilanadi. Binomial taqsimotning ehtimollik funksiyasi quyidagi formula orqali aniqlanadi:

$$P(X = k) = C_n^k p^k (1-p)^{n-k}, \;\; k = 0{,}1, \dots, n.$$

Bu yerda $C_n^k = \frac{n!}{k!(n-k)!}$ kombinatsiyalar soni bo‘lib, n ta tajriba ichidan k ta muvaffaqiyatli natijaning joylashish usullarini ifodalaydi [1, 6, 7].

Binomial taqsimotning muhim sonli xarakteristikalari uning matematik kutilmasi, dispersiyasi va standart og‘ishidan iborat. Agar $X \sim B(n,p)$ bo‘lsa, uning matematik kutilmasi $M(X) = np$ formula orqali aniqlanadi. Mazkur ko‘rsatkich n ta mustaqil tajribada kutilayotgan muvaffaqiyatlar sonining o‘rtacha qiymatini ifodalaydi. Binomial taqsimotning dispersiyasi: $D(X) = np(1-p) = npq$ ko‘rinishida aniqlanadi.

Dispersiyaning kvadrat ildizi standart og‘ish bo‘lib, $\sigma(X) = \sqrt{npq}$ formula orqali hisoblanadi. Ushbu ko‘rsatkichlar binomial jarayonda natijalarning markaziy tendensiyasi va tarqalish darajasini baholash imkonini beradi [5, 6].

Binomial taqsimotning asosiy parametrlari n va p bo‘lib, ularning o‘zgarishi tasodifiy miqdorning sonli xarakteristikalariga bevosita ta’sir ko‘rsatadi. Tajribalar soni o‘zgarmas bo‘lgan holatda muvaffaqiyat ehtimolining ortishi matematik kutilmaning chiziqli ravishda oshishiga olib keladi. Biroq dispersiyaning o‘zgarishi chiziqli xarakterga ega emas, chunki $D(X) = np(1-p)$ Ushbu bog‘lanishdan ko‘rinadiki, $p =$

0,5 qiymatida $p(1-p)$ ko‘paytmasi maksimal qiymatga ega bo‘ladi. Shu sababli, n o‘zgarmas bo‘lganda binomial taqsimotning dispersiyasi $p = 0{,}5$ atrofida eng katta qiymatga erishadi.

Mazkur bog‘lanishni yanada batafsil tahlil qilish maqsadida tajribalar soni $(n = 20)$ o‘zgarmas holda, muvaffaqiyat ehtimoli (p) ning $(0{,}1)$ dan $(0{,}9)$ gacha bo‘lgan qiymatlari uchun $(M(X))$ va $(D(X))$ ko‘rsatkichlari hisoblandi.

Quyidagi jadvalda (p) ning turli qiymatlariga mos ravishda matematik kutilma va dispersiyaning o‘zgarishi keltirilgan.

1-jadval. (p) ning turli qiymatlariga mos ravishda matematik kutilma va dispersiyaning o‘zgarishi

(p)	(q=1-p)	(M(X)=np)	(D(X)=npq)
0,1	0,9	2	1,8
0,2	0,8	4	3,2
0,3	0,7	6	4,2
0,4	0,6	8	4,8
0,5	0,5	10	5,0
0,6	0,4	12	4,8
0,7	0,3	14	4,2
0,8	0,2	16	3,2
0,9	0,1	18	1,8

Jadval ma’lumotlarining tahlili shuni ko‘rsatadiki, $(n = 20)$ bo‘lgan holatda muvaffaqiyat ehtimoli p ning ortishi matematik kutilmaning muntazam ravishda oshishiga olib keladi. Masalan $p = 0{,}1$ bo‘lganda matematik kutilma $M(X) = 2$ ga teng bo‘lsa, $p = 0{,}5$ da $M(X) = 10, p = 0{,}9$ da esa $M(X) = 18$ ga teng bo‘ladi. Dispersiyaning o‘zgarishi esa boshqa xarakterga ega. p ning 0,1 dan 0,5 gacha ortishida dispersiya 1,8 dan 5,0 gacha oshadi. $p = 0{,}5$ qiymatida dispersiya maksimal qiymatga erishadi. Keyinchalik p ning 0,5 dan 0,9 gacha ortishi bilan dispersiya 5,0 dan 1,8 gacha kamayadi.

Ushbu natija $D(X) = np(1-p)$ formula orqali ham izohlanadi. Haqiqatan ham, $p = 0{,}5$ bo‘lganda $p(1-p) = 0{,}25$ maksimal qiymatni qabul qiladi. p ning 0 yoki 1 ga yaqinlashishi bilan $p(1-p)$ kamayadi va natijada dispersiya ham kichrayadi. Shunday qilib, binomial modelda $p = 0{,}5$ holati natijalarning eng katta noaniqlik darajasini ifodalaydi. p ning 0 yoki 1 ga yaqin qiymatlarida esa tajriba natijasi oldindan nisbatan aniqroq prognoz qilinadi.

Binomial taqsimot dispersiyasining ekstremal xususiyatini analitik tahlil qilish

Yuqoridagi parametrik tahlil natijalari tajribalar soni o‘zgarmas bo‘lgan holatda binomial taqsimot dispersiyasining muvaffaqiyat ehtimoli (p) ga nisbatan chiziqli bo‘lmagan tarzda o‘zgarishini ko‘rsatdi. Xususan, $(p = 0{,}5)$ bo‘lganda dispersiya maksimal qiymatga erishishi kuzatildi. Ushbu natijani analitik jihatdan asoslash uchun dispersiyaning (p) parametriga bog‘liqligini qaraymiz.

Agar tasodifiy miqdor $X \sim B\,(n)\,, p\,)$ binomial taqsimotga ega bo‘lsa, uning dispersiyasi quyidagi formula orqali aniqlanadi: $D(X) = np(1-p)$

Bu yerda (n) — mustaqil tajribalar soni, (p) — har bir tajribada muvaffaqiyatning yuz berish ehtimoli, $(q = 1-p)$ esa muvaffaqiyatsizlik ehtimolini ifodalaydi.

Tajribalar soni (n) ni o‘zgarmas deb qarab, dispersiyani muvaffaqiyat ehtimoli (p) ning funksiyasi sifatida quyidagicha ifodalaymiz: $D(p) = np(1-p)$

Qavslarni ochish natijasida dispersiya funksiyasi quyidagi kvadratik ko‘rinishga keladi: $D(p) = np - np^2$

Dispersiyaning qaysi (p) qiymatida ekstremumga erishishini aniqlash uchun $(D(p))$ funksiyadan (p) bo‘yicha birinchi tartibli hosila olamiz: $D'(p) = n - 2np$

Umumiy ko‘paytuvchi (n) ni qavsdan tashqariga chiqarib, hosilani quyidagi ko‘rinishda yozamiz: $D'(p) = n(1-2p)$. Funksiyaning ekstremumga erishishi uchun zaruriy shartga ko‘ra, uning birinchi tartibli hosilasi nolga teng bo‘lishi kerak. Shuning uchun $D'(p) = 0$

Bundan $n(1-2p) = 0$ tenglamaga ega bo‘lamiz.

Binomial modelda tajribalar soni musbat, ya’ni $n > 0$ bo‘lganligi sababli $1 - 2p = 0$.

Bundan $2p = 1$ va $p = \frac{1}{2} = 0{,}5$ natijaga ega bo‘lamiz.

Demak, (p=0,5) dispersiya funksiyasining kritik nuqtasi hisoblanadi. Ushbu kritik nuqtada dispersiya maksimum yoki minimum qiymatga erishishini aniqlash uchun (D(p)) funksiyaning ikkinchi tartibli hosilasini hisoblaymiz: $D''(p) = -2n$

Tajribalar soni $(n > 0)$ bo‘lganligi sababli $D''(p) = -2n < 0$

Ikkinchi tartibli hosilaning manfiy bo‘lishi $(D(p))$ funksiyaning $(p = 0{,}5)$ kritik nuqtada maksimumga erishishini ko‘rsatadi. Demak, tajribalar soni (n) o‘zgarmas bo‘lganda binomial taqsimot dispersiyasi muvaffaqiyat ehtimoli $p = 0{,}5$ bo‘lgan holatda eng katta qiymatni qabul qiladi.

Endi dispersiyaning maksimal qiymatini aniqlaymiz. Buning uchun $p = \frac{1}{2}$ qiymatni dispersiyaning $D(X) = np(1-p)$ formulasiga qo‘yamiz:

$$D_{max}(X) = n \cdot \frac{1}{2}\left(1 - \frac{1}{2}\right)$$

$$D_{max}(X) = n \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$D_{max}(X) = \frac{n}{4}$$

Shunday qilib, tajribalar soni o‘zgarmas bo‘lgan binomial modelda dispersiyaning maksimal qiymati $D_{max}(X) = \frac{n}{4}$ $p = q = 0{,}5$ bo‘lgan holatda yuzaga keladi.

Tajribalar sonining binomial modelning sonli xarakteristikalariga ta’siri

Binomial modelning xususiyatlarini to‘liqroq tahlil qilish uchun muvaffaqiyat ehtimoli (p) o‘zgarmas bo‘lgan sharoitda tajribalar soni (n) ning matematik kutilma, dispersiya va standart og‘ishga ta’sirini ko‘rib chiqamiz. Agar (X) tasodifiy miqdor binomial taqsimotga ega bo‘lsa

$$X \sim B(n, p)$$

$$D(X) = np(1-p)$$

$$\sigma(X)) = \sqrt{\text{n p } (1-\text{p})}$$

Mazkur bog‘lanishlarni sonli jihatdan tahlil qilish uchun muvaffaqiyat ehtimolini

$$p = 0{,}5 \quad q = 1 - p = 0{,}5$$

$$M(X) = 0{,}5n$$

$$D(X) = n \cdot 0{,}5 \cdot 0{,}5$$

$$D(X) = 0{,}25\, n$$

$$\sigma(X) = \sqrt{0{,}25\ \text{n}}$$

$$\sigma(X) = \frac{\sqrt{n}}{2}$$

Demak, jadvalga kiritadigan sonlar:

2-jadval.

n	p	$M(X)$		$D(X)$	$\sigma(X)$
10	0,5	5		2,5	1,58
20	0,5	10		5	2,24
50	0,5	25		12,5	3,54
100	0,5	50		25	5

Jadval natijalaridan ko‘rinadiki, $(p = 0{,}5)$ o‘zgarmas bo‘lgan sharoitda (n) ning 10 dan 100 gacha ortishi matematik kutilmaning 5 dan 50 gacha, dispersiyaning esa 2,5 dan 25 gacha ortishiga olib keladi. Bu holat matematik kutilma va dispersiyaning (n) ga nisbatan chiziqli bog‘langanligini tasdiqlaydi.

Standart og‘ishning o‘zgarishi esa chiziqli emas.

$$\sigma(X) = \sqrt{np(1-p)}$$

bo‘lganligi sababli, p o‘zgarmas sharoitda standart og‘ish tajribalar sonining kvadrat ildiziga mutanosib ravishda o‘zgaradi: $\sigma(X) \sim \sqrt{n}$

$$n_1 = 10\,, \qquad n_n = 100$$

gacha oshirilganda standart og‘ish 10 marta emas, balki

$$\sqrt{\frac{n_2}{n_1}} = \sqrt{\frac{100}{10}} = \sqrt{10} \approx 3\,16$$

marta ortadi. Haqiqatan ham, $n = 10$ bo‘lganda $\sigma_1 = \sqrt{2{,}5} \approx 1{,}58$

$n = 100$ bo‘lganda esa $\sigma_2 = \sqrt{25} \approx 5$ ularning nisbati esa

$$\frac{\sigma_1}{\sigma_2} \approx \frac{n_1}{n_2} \approx 3{,}16$$

Binomial modelda nisbiy tarqalishni variatsiya koeffitsiyenti asosida tahlil qilish

Yuqoridagi tahlil natijalari tajribalar soni (n) ortishi bilan binomial tasodifiy miqdorning matematik kutilmasi, dispersiyasi va standart og‘ishi ham ortishini ko‘rsatdi. Biroq standart og‘ish tasodifiy miqdorning absolyut tarqalishini ifodalaydi. Tajribalar soni ortishi bilan matematik kutilma ham ortadi. Shu sababli natijalarning tarqalishini matematik kutilmaga nisbatan ham baholash zarur. Buning uchun variatsiya koeffitsiyentidan foydalanamiz.

Variatsiya koeffitsiyenti standart og‘ishning matematik kutilmaga nisbatiga teng:

$$CV = \frac{\sigma(X)}{M(X)}$$

$$M(X) = np$$

$$D(X) = np(1-p)$$

$$\sigma(X) = \sqrt{np(1-p)}$$

$$CV = \frac{\sqrt{np(1-p}}{np}$$

Bu formula binomial taqsimotda nisbiy tarqalishning tajribalar soni (n) va muvaffaqiyat ehtimoli (p) ga bog‘liqligini ifodalaydi. Oldingi hisoblashlar bilan taqqoslash uchun muvaffaqiyat ehtimolini o‘zgarmas deb olib, $p = 0{,}5 \;\; 1 - p = 0{,}5$

$$CV = \sqrt{\frac{0{,}5}{0{,}5n}}, \;\; CV = \sqrt{\frac{1}{n}}, \;\; CV = \frac{1}{\sqrt{n}},$$

$$n = 10 \quad CV = \frac{1}{\sqrt{10}} \approx 0{,}316 \quad CV \approx 31{,}6\%$$

$$n = 20 \quad CV = \frac{1}{\sqrt{20}} \approx 0{,}224 \quad CV \approx 22{,}4\%$$

$$n = 50 \quad CV = \frac{1}{\sqrt{50}} \approx 0{,}141 \quad CV \approx 14{,}1\%$$

$$n = 100 \quad CV = \frac{1}{\sqrt{100}} \approx 0{,}1 \quad CV \approx 10\%$$

Olingan natijalarni quyidagi jadval orqali umumlashtirish mumkin.

3-jadval.

Tajribalar soni n	Matematik kutilma M(X)	Standart og‘ish σ(X)	CV	CV, %
10	5	1,58	0,316	31,6
20	10	2,24	0,224	22,4
50	25	3,54	0,141	14,1
100	50	5,00	0,100	10,0

Jadval ma’lumotlaridan ko‘rinadiki, tajribalar soni $(n = 10)$ dan $(n = 100)$ gacha ortishi bilan standart og‘ish (1,58) dan (5,00) gacha oshadi. Biroq variatsiya koeffitsiyenti aksincha, (31,6%) dan (10%) gacha kamayadi.

Demak, tajribalar sonining ortishi tasodifiy miqdorning absolyut tarqalishini oshirsa-da, uning matematik kutilmaga nisbatan nisbiy tarqalish darajasini kamaytiradi.

Bu natijani tajribalar sonining 10 marta ortishi misolida ham ko‘rish mumkin. Tajribalar sonining o‘zgarish nisbati: $\frac{100}{10} = 10$ Variatsiya koeffitsiyenti esa 31,6% dan 10% gacha kamaygan. $\frac{31{,}6}{10} = 3{,}16$ Demak, variatsiya koeffitsiyenti taxminan 3,16 marta kamaygan. $\sqrt{\frac{100}{10}} = \sqrt{10} = 3{,}16$

$$\frac{CV_{10}}{CV_{100}} = \frac{31{,}6}{10} \approx \sqrt{\frac{100}{10}} = 3{,}16$$

Binomial model asosida texnologik jarayonni statistik nazorat qilish: Binomial taqsimot modeli ishlab chiqarish va texnologik jarayonlarda mahsulot sifatini statistik nazorat qilish masalalarida samarali qo‘llanilishi mumkin. Agar ishlab chiqarilgan har bir mahsulot belgilangan sifat talablariga **javob beradi** yoki **javob bermaydi** kabi ikki holatdan biri bilan tavsiflansa, bunday jarayonni Bernulli tajribalari ketma-ketligi sifatida modellashtirish mumkin. Tekshirilayotgan i-mahsulotning sifat holatini Y_i tasodifiy miqdor orqali quyidagicha belgilaymiz:

$$Y_i = \begin{cases} 1, agar\ mahsulot\ nuqsonsiz\ bo'lsa \\ 0, agar\ \ mahsulot\ nuqsonli\ bo'lsa \end{cases}$$

Bunda:

$$P(Y_i = 1) = p, \quad P(Y_i = 0) = 1 - p = q.$$

Agar nazorat jarayonida jami n ta mahsulot tekshirilsa, ular orasidagi nuqsonsiz mahsulotlar sonini

$$X = \sum_{i=1}^{n} Y_i$$

tasodifiy miqdor orqali ifodalash mumkin. Mahsulotlarning sifat holatlari o‘zaro mustaqil va ularning nuqsonsiz bo‘lish ehtimoli p o‘zgarmas deb qaralganda,

$$X \sim Bin(n, p)$$

binomial model hosil bo‘ladi.

Ushbu model yordamida nazorat qilinayotgan partiyadagi nuqsonsiz mahsulotlarning kutilayotgan soni va ularning tarqalish darajasi quyidagicha aniqlanadi:

$$M(X) = np,$$

$$D(X) = np(1-p),$$

$$\sigma(X) = \sqrt{np(1-p)}.$$

Bu ko‘rsatkichlar texnologik jarayonning joriy holatini statistik jihatdan baholash uchun asos bo‘lib xizmat qiladi. Matematik kutilma normal ish sharoitida kutilayotgan nuqsonsiz mahsulotlar sonini, standart og‘ish esa kuzatilayotgan natijalarning ushbu kutilayotgan qiymat atrofida qanchalik tarqalishini tavsiflaydi.

Texnologik jarayonda bir vaqt oralig‘ida $n = 20$ ta mahsulot nazorat qilinsin. Har bir mahsulotning sifat talablariga javob berish, ya’ni nuqsonsiz bo‘lish ehtimoli $p = 0.9$ deb olamiz. U holda nuqsonli mahsulotning yuzaga kelish ehtimoli

$$q = 1 - p = 1 - 0.9 = 0.1$$

ga teng bo‘ladi.

Tekshirilayotgan 20 ta mahsulot orasidagi nuqsonsiz mahsulotlar sonini X bilan belgilasak,

$$X \sim Bin(20{,}0.9)$$

binomial modelga ega bo‘lamiz.

Ushbu jarayonda nuqsonsiz mahsulotlarning kutilayotgan soni:

$$M(X) = np = 20 \cdot 0.9 = 18.$$

$$D(X) = np(1-p) = 20 \cdot 0.9 \cdot 0.1 = 1.8.$$

$$\sigma(X) = np(1-p) = 1.8 \approx 1.342.$$

Demak, nuqsonsiz mahsulotlar sonining matematik kutilma atrofidagi absolyut tarqalish darajasi taxminan **1,342 ta mahsulot**ni tashkil etadi. Ushbu ko‘rsatkichlardan texnologik jarayonning barqarorligini statistik baholash hamda kuzatilayotgan natijalarning kutilayotgan holatdan sezilarli darajada chetlanishini aniqlashda foydalanish mumkin.

Texnologik jarayonning barqarorligini baholash uchun statistik nazoratda ko‘pincha matematik kutilma atrofidagi 3σ chegaralaridan foydalaniladi. Binomial model uchun pastki va yuqori nazorat chegaralari quyidagicha aniqlanadi:

$$LCL = np - 3\sqrt{np(1-p)},$$

$$\mathrm{UCL} = np + 3\sqrt{np(1-p)}$$

$$n = 20, p = 0.9, \;\; M(X) = 18, \qquad \sigma(X) \approx 1.342.$$

Shuning uchun pastki nazorat chegarasi:

$$LCL = 18 - 3 \cdot 1{,}342$$

$$LCL = 18 - 4.026 = 13{,}974.$$

Yuqori nazorat chegarasi esa:

$$UCL = 18 + 3 \cdot 1.342$$

$$UCL = 18 + 4.026 = 22.026.$$

Biroq tekshirilayotgan mahsulotlar soni $n = 20$ ta bo‘lganligi sababli nuqsonsiz mahsulotlar soni 20 tadan ortiq bo‘lishi mumkin emas. Shu sababli yuqori nazorat chegarasi amaliy jihatdan $UCL = 20,$ deb olinadi. Pastki chegara esa diskret qiymatlarni hisobga olgan holda taxminan $LCL \approx 14,$ ga teng bo‘ladi.

Binomial tasodifiy miqdor faqat butun qiymatlarni qabul qilganligi sababli, $LCL = 13.974$ natijaga ko‘ra, $X \geq 14$ qiymatlar nazorat chegarasi ichida, $X \leq 13$ qiymatlar esa nazorat chegarasidan tashqarida deb qaraladi. $X \leq 20$ bo‘lganligi sababli yuqori nazorat chegarasi amaliy jihatdan 20 bilan cheklanadi.

Demak, 20 ta mahsulotdan iborat nazorat guruhida nuqsonsiz mahsulotlar soni 14 tadan 20 tagacha bo‘lsa, kuzatilayotgan natijalar berilgan statistik model doirasida kutiladigan chegaralarda deb qaraladi. Agar nuqsonsiz mahsulotlar soni 13 ta yoki undan kam bo‘lsa, kuzatilgan qiymat pastki nazorat chegarasidan

tashqarida joylashadi va bu texnologik jarayonda sifat ko‘rsatkichining kutilgan holatdan chetlanishini bildiruvchi statistik signal sifatida baholanadi.

Zamonaviy ishlab chiqarish tizimlarida mahsulot sifatini nazorat qilish jarayoni sensorlar, o‘lchash qurilmalari va avtomatik ma’lumotlarni qayta ishlash vositalari yordamida amalga oshirilishi mumkin. Bunday tizimda har bir mahsulotning belgilangan sifat talablariga mos yoki mos emasligi aniqlanib, natija mos ravishda 1 yoki 0 qiymat bilan ifodalanadi.

Nazorat jarayonining umumiy ketma-ketligini quyidagicha ifodalash mumkin:

Sensor→ma’lumotlarni yig‘ish→mahsulot holatini aniqlash→binomial model→statistik bahola h→nazorat signali.

Har bir nazorat guruhida n ta mahsulot tekshiriladi va nuqsonsiz mahsulotlar soni X avtomatik ravishda aniqlanadi. Olingan X qiymati oldindan hisoblangan statistik nazorat chegaralari bilan taqqoslanadi.

Agar:

$$LCL \leq X \leq UCL$$

shart bajarilsa, kuzatilgan natija belgilangan statistik chegaralar doirasida deb baholanadi.

Agar:

$$X < LCL$$

bo‘lsa, tizim mahsulot sifatining kutilgan darajadan pasayganligini ko‘rsatuvchi statistik signalni shakllantirishi mumkin. Bunday signal operatorga texnologik parametrlarni tekshirish, ishlab chiqarish sharoitlarini tahlil qilish va zarur hollarda tegishli boshqaruv choralarini ko‘rish uchun asos bo‘lib xizmat qiladi.

Shunday qilib, binomial modelni avtomatlashtirilgan nazorat tizimiga kiritish ikki natijali sifat ko‘rsatkichlarini matematik-statistik usulda qayta ishlash, jarayon holatini muntazam baholash hamda statistik jihatdan sezilarli chetlanishlarni o‘z vaqtida aniqlash imkonini beradi.

Yuqorida aniqlangan statistik nazorat chegaralari asosida turli kuzatuv natijalarini quyidagicha baholash mumkin:

4-jadval.

Tekshirilgan mahsulotlar soni n	**Nuqsonsiz mahsulotlar soni X**	**Nazorat chegarasi**	**Statistik baho**
20	18	$14 \leq X \leq 20$	Me’yoriy chegarada
20	16	$14 \leq X \leq 20$	Me’yoriy chegarada
20	14	$14 \leq X \leq 20$	Nazorat chegarasi ichida
20	12	$X < 14X < 14$	Nazorat chegarasidan tashqarida

Jadvaldan ko‘rinadiki, $X = 18$ va $X = 16$ bo‘lgan holatlarda kuzatilgan natijalar statistik nazorat chegaralari ichida joylashadi. X=14 qiymati nazorat chegarasi ichidagi eng kichik butun qiymat hisoblanadi. Agar $X = 12$ bo‘lsa, kuzatilgan qiymat pastki nazorat chegarasidan kichik bo‘lib, texnologik jarayonda sifat ko‘rsatkichining kutilgan darajadan pasayganligini bildiruvchi statistik signal yuzaga keladi.

Shunday qilib, binomial model asosida aniqlangan nazorat chegaralari ishlab chiqarish jarayonida mahsulot sifatining o‘zgarishini miqdoriy baholash imkonini beradi. Ushbu yondashuv avtomatlashtirilgan nazorat tizimida kuzatuv natijalarini real vaqt rejimida qayta ishlash, belgilangan statistik chegaralardan chetlanishlarni aniqlash va operatorga ogohlantiruvchi signal uzatish uchun matematik asos sifatida qo‘llanilishi mumkin.

3σ tamoyili va aniq binomial ehtimollik asosidagi nazorat chegaralarining qiyosiy tahlili. Yuqoridagi hisoblashlarda texnologik jarayonning statistik nazorat chegaralari matematik kutilma va standart og‘ishga asoslangan 3σ tamoyili yordamida aniqlandi. Mazkur yondashuv amaliy statistik nazoratda qulay bo‘lsa-da, binomial tasodifiy miqdor diskret taqsimotga ega bo‘lganligi sababli nazorat chegaralarini uning aniq ehtimollik funksiyasi asosida ham baholash mumkin. Shu maqsadda $n = 20$ va $p = 0{,}9$ parametrli binomial model uchun pastki kritik chegarani kumulyativ ehtimollik asosida aniqlaymiz.

$$X \sim Bin(20; 0{,}9)$$

$$P(X \leq k) = \sum_{i=0}^{k} (\frac{20}{i})(0{,}9)^{\mathrm{i}}\,(0{,}1)^{20-\mathrm{i}}$$

formula orqali kk yoki undan kam nuqsonsiz mahsulot kuzatilishining aniq ehtimoli hisoblanadi. Statistik ahamiyatlilik darajasini $\alpha = 0{,}05$ deb qabul qilamiz. U holda pastki kritik chegara quyidagi shart asosida aniqlanadi:

$$P(X \leq k) \leq 0{,}05.$$

$X \sim Bin(20; 0{,}9)$ uchun kumulyativ ehtimolliklarni hisoblasak:

$$P(X \le 13) \approx 0{,}00239,$$
$$P(X \le 15) \approx 0{,}04317,$$
$$P(X \le 16) \approx 0{,}13295.$$

Biz $\alpha = 0{,}05$ deb olgan edik. Shuning uchun $P(X \le 15) < 0{,}05$ lekin $P(X \le 16) > 0{,}05$. Demak, aniq binomial ehtimollik bo‘yicha **eng katta kritik qiymat** $k = 15$ bo‘ladi.

5-jadval.

Nazorat usuli	Qabul qilinadigan qiymatlar	Statistik signal	Izoh
3σ tamoyili	$14 \le X \le 2014$	$X \le 13$	Matematik kutilma va standart og‘ishga asoslangan taxminiy chegara
Aniq binomial usul, $\alpha = 0{,}05$	$16 \le X \le 20$	$X \le 15$	Binomial taqsimotning aniq kumulyativ ehtimoliga asoslangan chegara

Qiyosiy tahlil natijalari 3σ tamoyili va aniq binomial ehtimollik usuli bir xil nazorat chegarasini bermasligini ko‘rsatdi. 3σ yondashuvida $X \le 13$ qiymatlar statistik signal sifatida baholansa, $\alpha = 0{,}05$ darajadagi aniq binomial yondashuvda $X \le 15$ qiymatlar ham kritik sohaga kiradi. Demak, mazkur parametrlar uchun aniq binomial usul sifat ko‘rsatkichining pasayishiga nisbatan sezgirroq nazorat mezonini shakllantiradi.

Mazkur farq binomial taqsimotning diskret tabiati bilan bog‘liq bo‘lib, kichik hajmli nazorat guruhlarida normal yaqinlashuvga asoslangan 3σ chegaralari dan foydalanishda ehtiyotkorlik zarurligini ko‘rsatadi. Shu sababli n kichik yoki muvaffaqiyat ehtimoli 0 yoki 1 ga yaqin bo‘lgan holatlarda aniq binomial ehtimolliklardan foydalanish statistik nazoratning asoslanganligini oshirishi mumkin.

Xulosa. Tasodifiy jarayonlarni matematik modellashtirish noaniqlik sharoitida yuzaga keladigan hodisalarning ehtimoliy qonuniyatlarini aniqlash va ularning miqdoriy xususiyatlarini baholash imkonini beradi. Tadqiqot davomida tasodifiy hodisa, tasodifiy miqdor, taqsimot qonuni, matematik kutilma, dispersiya va standart og‘ish tushunchalarining tasodifiy jarayonlarni modellashtirishdagi o‘rni nazariy jihatdan yoritildi.

Takroriy va o‘zaro mustaqil ikki natijali tajribalarni modellashtirishda binomial taqsimotning xususiyatlari tahlil qilindi. Parametrik tahlil natijasida tajribalar soni n o‘zgarmas bo‘lganda muvaffaqiyat ehtimoli p ning ortishi matematik kutilmaning chiziqli o‘sishiga olib kelishi, dispersiya esa $p = 0.5$ bo‘lganda maksimal qiymatga erishishi analitik jihatdan asoslandi. Shuningdek, p o‘zgarmas sharoitda tajribalar sonining ortishi matematik kutilma va dispersiyaning chiziqli, standart og‘ishning esa $\sqrt{n}$ qonuniyatiga muvofiq o‘sishiga olib kelishi ko‘rsatildi. Variatsiya koeffitsiyenti asosidagi tahlil esa tajribalar soni ortishi bilan nisbiy tarqalish darajasining kamayishini ko‘rsatdi. Binomial modelning amaliy qo‘llanilishi $n = 20$ va $p = 0.9$ parametrli texnologik jarayon misolida ko‘rib chiqildi. Bunda nuqsonsiz mahsulotlarning kutilayotgan soni

$$M(X) = 18,$$
$$D(X) = 1.8,$$
$$\sigma(X) \approx 1.342$$

ekanligi aniqlandi. Shuningdek, 20 ta mahsulotdan aynan 18 tasining nuqsonsiz bo‘lish ehtimoli taxminan 0,285 ya’ni **28,5 %** ni tashkil etishi hisoblandi.

Texnologik jarayonni statistik nazorat qilish uchun 3σ tamoyili asosida nazorat chegaralari shakllantirildi. Hisoblash natijasida pastki nazorat chegarasi $LCL \approx 14$ ga teng bo‘lib, yuqori chegara tekshirilayotgan mahsulotlarning maksimal sonini hisobga olgan holda $UCL = 20$ deb qabul qilindi. Shu asosda nuqsonsiz mahsulotlar sonining pastki nazorat chegarasidan kamayishi texnologik jarayonda sifat ko‘rsatkichlarining kutilgan holatdan chetlanishini bildiruvchi statistik signal sifatida talqin qilindi.

Bundan tashqari, 3σ tamoyili asosida olingan taxminiy nazorat chegaralari binomial taqsimotning aniq kumulyativ ehtimollari asosidagi kritik chegara bilan qiyoslandi. $n = 20$, $p = 0{,}9$ va $\alpha = 0{,}05$ sharoitida 3σ yondashuvi bo‘yicha $X \le 13$ qiymatlar statistik signal sifatida baholangan bo‘lsa, aniq binomial usulda $P(X \le 15) \approx 0{,}04317 < 0{,}05$ va $P(X \le 16) \approx 0{,}13295 > 0{,}05$ ekanligi aniqlandi. Natijada aniq binomial yondashuvda $X \le 15$ kritik soha sifatida belgilandi. Ushbu qiyosiy natija kichik hajmli va muvaffaqiyat ehtimoli chegaraviy qiymatlarga yaqin bo‘lgan binomial jarayonlarda aniq ehtimollik hisoblariga asoslangan nazorat mezoni 3σ yondashuviga nisbatan sezgirroq bo‘lishi mumkinligini ko‘rsatdi.

Taklif etilgan yondashuvda mahsulot holatini sensorlar yordamida aniqlash, olingan ma’lumotlarni binomial model asosida qayta ishlash, kuzatilgan natijalarni statistik nazorat chegaralari bilan taqqoslash va chegaradan tashqaridagi holatlarda ogohlantiruvchi signal shakllantirish imkoniyati ko‘rsatildi. Natijada binomial model nafaqat takroriy tasodifiy jarayonlarning ehtimoliy xususiyatlarini tavsiflash, balki ikki

natijali sifat ko‘rsatkichlariga ega texnologik jarayonlarni avtomatlashtirilgan statistik nazorat qilish uchun ham matematik asos sifatida qo‘llanilishi mumkinligi asoslandi.

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APPROXIMATION OF INTEGRALS BY PERIODIZATION OF FUNCTIONS

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Abstract. *This paper focuses on the construction of composite optimal quadrature formulas for the approximate calculation of integrals in the Hilbert space* $W_2^{(m,m-1)}(0,1)$*. While the standard rectangle rule is well-known to be optimal for periodic functions, evaluating non-periodic functions requires a specialized approach. To address this, we propose a novel method based on the periodization of arbitrary non-periodic functions utilizing the properties of Bernoulli polynomials. By transforming the original function into a periodic form and applying the optimal rectangle formula, we derive a new composite optimal quadrature formula that explicitly incorporates the boundary values of the function's derivatives. Furthermore, the explicit form of the error functional is established, and the exact representation of the square of its norm is determined.*

Keywords. *Periodic function, integral, optimal quadrature formula, Hilbert space, Bernoulli polynomial, error functional.*

FUNKSIYALARNI DAVRIYLASHTIRISH ORQALI INTEGRALLARNI APPROKSIMATSIYALASH

Annotatsiya. *Ushbu maqola* $W_2^{(m,m-1)}(0,1)$ *Gilbert fazosida integrallarni taqribiy hisoblash uchun optimal kvadratur formulalar qurishga bag'ishlangan. Garchi standart to'g'ri to'rtburchaklar formulasi davriy funksiyalar uchun optimal ekanligi yaxshi ma'lum bo'lsa-da, davriy bo'lmagan funksiyalardan integral hisoblash ixtisoslashgan yondashuvni talab qiladi. Ushbu muammoni hal qilish uchun biz Bernulli ko'phadlarining xossalaridan foydalangan holda ixtiyoriy davriy bo'lmagan funksiyalarni davriylashtirishga asoslangan yangi usulni taklif qilamiz. Dastlabki funksiyani davriy ko'rinishga keltirish va optimal to'g'ri to'rtburchaklar formulasini qo'llash orqali, biz funksiya hosilalarining chegaraviy qiymatlarini oshkor ko'rinishda o'z ichiga oluvchi yangi optimal kvadratur formula keltirib chiqaramiz. Bundan tashqari, xatolik funksionalining analitik ko'rinishi va uning normasi kvadratining aniq ifodasi topilgan.*

Kalit so'zlar: *davriy funksiya, integral, optimal kvadratur formula, Gilbert fazosi, Bernulli ko'phadi, xatolik funksionali.*

АППРОКСИМАЦИЯ ИНТЕГРАЛОВ МЕТОДОМ ПЕРИОДИЗАЦИИ ФУНКЦИЙ

Аннотация. *Данная статья посвящена построению составных оптимальных квадратурных формул для приближённого вычисления интегралов в гильбертовом пространстве* $W_2^{(m,m-1)}(0,1)$. *Хотя хорошо известно, что стандартная формула прямоугольников является оптимальной для периодических функций, вычисление интегралов от непериодических функций требует специализированного подхода. Для решения этой проблемы мы предлагаем новый метод, основанный на периодизации произвольных непериодических функций с использованием свойств многочленов Бернулли. Путём преобразования исходной функции в периодическую форму и применения оптимальной формулы прямоугольников, мы получаем новую составную оптимальную квадратурную формулу, которая в явном виде включает граничные значения производных функций. Кроме того, установлена явная форма функционала погрешности, и определено точное представление квадрата его нормы.*

Ключевые слова: *периодическая функция, интеграл, оптимальная квадратурная формула, гильбертово пространство, многочлен Бернулли, функционал погрешности.*

Introduction. In recent years, a number of results based on the application of function-theoretic considerations have been obtained in various areas of computational mathematics. One can point out several problems where function-theoretic approaches lead to general results: the construction of quadrature and cubature formulas, interpolation of functions of one and many variables, and the approximate solution of integral and differential equations (see [13]).

Among these topics, the problem of approximate calculation of integrals has been developed in more detail compared to the others (see [8, 12, 13, 14]).

Consider a composite quadrature formula of the form

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N}\sum_{\alpha=0}^{k} C^{(\alpha)}(h\beta)\varphi^{(\alpha)}(h\beta) \tag{1}$$

Here $\varphi(x)$ is from the Hilbert space $W_2^{(m,m-1)}(0,1)$, $C^{(\alpha)}(h\beta)$ are the coefficients of the quadrature formula (1), $h=\frac{1}{N}$, N is natural number, $k \le m-1$. Furthermore, we will consider the error of the quadrature formula as a linear functional of the form

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N}\sum_{\alpha=0}^{k}(-1)^{\alpha} C^{(\alpha)}(h\beta)\delta^{(\alpha)}(x-h\beta) \tag{2}$$

in the Hilbert space $W_2^{(m,m-1)}(0,1)$ of classes of functions possessing generalized derivatives of order m with an integrable square and the norm

$$\|\varphi\|_{W_2^{(m,m-1)}} = \left\{\int_0^1 \left(\varphi^{(m)}(x)+\varphi^{(m-1)}(x)\right)^2 dx\right\}^{\frac{1}{2}} \tag{3}$$

where $\varepsilon_{[0,1]}$ in (2) denotes the characteristic function of the interval $[0,1]$.

The space $\widetilde{W}_2^{(m,m-1)}(0,1)$ is the class of periodic functions from $W_2^{(m,m-1)}(0,1)$, i.e., $\varphi(x)=\varphi(x+\gamma)$, where γ are integers. Furthermore, $\varphi^{(\alpha)}(1)=\varphi^{(\alpha)}(0)$, $\alpha=0,1,\dots,m-1$. The elements of the space $\widetilde{W}_2^{(m,m-1)}(0,1)$ are functions that differ from each other by a constant term. The norm of a function $\varphi(x)$ in the space of periodic functions $\widetilde{W}_2^{(m,m-1)}(0,1)$ is defined by formula (3). Many mathematicians have been engaged in the construction and study of quadrature formulas of the form (1) in various spaces (see [6, \dots, 14]).

The subject of the present work is the construction of an optimal composite quadrature formula in the Hilbert space $W_2^{(m,m-1)}(0,1)$.

Here we propose a new method for constructing composite optimal quadrature formulas based on the periodization of an arbitrary function from the Hilbert space of non-periodic functions.

In the next section, we will present some properties of Bernoulli polynomials that will be useful for the periodization of a function.

Some Properties of Bernoulli Polynomials.

The Bernoulli numbers $B_0, B_1, \dots, B_n$ and Bernoulli polynomials $B_0(x), B_1(x), \dots, B_n(x)$ are determined using the recurrence relations:

$$B_0 = 1, \quad \sum_{k=0}^{n-1} C_n^k B_k = 0, \quad C_n^k = \frac{n!}{k!\,(n-k)!}, \quad n \geq 2 \tag{4}$$

$$B_0(x) = 1, \quad B_n(x) = \sum_{k=0}^{n} C_n^k B_k x^{n-k}, \quad n \geq 1 \tag{5}$$

Using these relations, in particular, we obtain:

$$C_2^0 B_0 + C_2^1 B_1 = 0, \quad B_1 = -\frac{1}{2},$$

$$C_3^0 B_0 + C_3^1 B_1 + C_3^2 B_2 = 0, \quad B_2 = \frac{1}{6}.$$

The first Bernoulli polynomials are also easily calculated:

$$B_1(x) = C_1^0 B_0 x + C_1^1 B_1 = x - \frac{1}{2},$$

$$B_2(x) = C_2^0 B_0 x^2 + C_2^1 B_1 x + C_2^2 B_2 = x^2 - x + \frac{1}{6}.$$

Lemma 1. *For the Bernoulli polynomials $B_n(x)$, the following equalities hold; that is, for the polynomials $B_n(x)$ at $x = 0$ and $x = 1$, we have:*

$$B_n(1) - B_n(0) = \begin{cases} 0, & n \neq 1,\ n \geq 0, \\ 1, & n = 1,\ n \geq 0. \end{cases} \tag{6}$$

The derivative of $B_n(x)$ is defined by the formula:

$$B'_n(x) = nB_{n-1}(x), \quad n \geq 1. \tag{7}$$

If we integrate the polynomial $B_n(x)$ from zero to one, we have:

$$\int_0^1 B_n(x)dx = 0, \quad n \geq 1. \tag{8}$$

Proof. Since $B_0(x) = 1$ and $B_1(x) = x - \frac{1}{2}$, equality (6) follows from the definition for $n = 0$ and $n = 1$, i.e., from (4) and (5):

$$B_n(1) = \sum_{k=0}^{n} C_n^k B_k = B_n + \sum_{k=0}^{n-1} C_n^k B_k = B_n = B_n(0)$$

By direct differentiation of (5), we obtain (7), i.e.,

$$B'_n(x) = \sum_{k=0}^{n} (n-k)\, C_n^k B_k x^{n-k-1} = n \sum_{k=0}^{n-1} C_{n-1}^k B_k x^{n-1-k} = nB_{n-1}(x).$$

Equality (8) also follows directly from (4). Calculating the definite integral, we get:

$$\int_0^1 B_n(x)dx = \sum_{k=0}^{n} C_n^k B_k \frac{x^{n-k+1}}{n-k+1}\Bigg|_0^1 = \frac{1}{n+1} \sum_{k=0}^{n} C_{n+1}^k B_k = 0, \quad n \geq 1.$$

Periodization of Functions from the Hilbert space $W_2^{(m,m-1)}(0,1)$.

In this section, we will deal with the periodization of a function $f(x)$ from the space $W_2^{(m,m-1)}(0,1)$.

Definition. Let $f(x) \in W_2^{(m,m-1)}(0,1)$, for $m \geq 1$. The periodization of the function $f(x)$ will be called finding a function $\varphi(x) \in W_2^{(m,m-1)}(0,1)$ satisfying the conditions:

$$\varphi^{(\alpha)}(1) = \varphi^{(\alpha)}(0), \quad \alpha = 0,1, \dots, m-1. \tag{9}$$

$$\int_0^1 f(x)dx = \int_0^1 \varphi(x)dx \tag{10}$$

Theorem 1. *Let $f(x) \in W_2^{(m,m-1)}(0,1)$. Then the following function*

$$\varphi(x) = f(x) + \sum_{n=0}^{m-1} \frac{B_{n+1}(x)}{(n+1)!}\left(f^{(n)}(0) - f^{(n)}(1)\right) \tag{11}$$

is a periodic function in the space $\widetilde{W}_2^{(m,m-1)}(0,1)$, i.e., an element of the space $\widetilde{W}_2^{(m,m-1)}(0,1)$, thus $\varphi(x) \in \widetilde{W}_2^{(m,m-1)}(0,1)$.

Proof. First of all, we will prove that for any $r \geq 0$, the following equality holds:

$$\left.\frac{B_{n+1}^{(r)}(1)}{(n+1)!}-\frac{B_{n+1}^{(r)}(0)}{(n+1)!}\right|_{r=n}=\begin{cases}0, & n\neq r,\ (r\leq n+1),\\ 1, & r=n.\end{cases} \tag{12}$$

For $r>n+1$, statement (12) is obvious, since $B_{n+1}(x)$ is a polynomial of degree $n+1$. If $r\leq n+1$, then using Lemma 1, we obtain:

$$\frac{B'_{n+1}(x)}{(n+1)!}=\frac{B_n(x)}{n!},\quad \dots,\quad \frac{B_{n+1}^{(r)}(x)}{(n+1)!}=\frac{B_{n+1-r}(x)}{(n+1-r)!}$$

Let $r=n$, then

$$\left.\frac{B_{n+1-r}(1)-B_{n+1-r}(0)}{n+1-r}\right|_{r=n}=B_1(1)-B_1(0)=1.$$

Furthermore, by virtue of (4), for $r\neq n$:

$$\frac{B_{n+1-r}(1)-B_{n+1-r}(0)}{n+1-r}=0.$$

Thus, (12) is fully proved.

Now we calculate the derivative of order r of the function $\varphi(x)$ defined by formula (11):

$$\varphi^{(r)}(x)=f^{(r)}(x)+\sum_{n=0}^{m-1}\frac{B_{n+1}^{(r)}(x)}{(n+1)!}\left(f^{(n)}(0)-f^{(n)}(1)\right).$$

Hence, by virtue of (12), we obtain:

$$\varphi^{(r)}(1)-\varphi^{(r)}(0)=0$$

Thus, for the function $\varphi(x)$, equality (9) holds. We proceed to the proof of equality (10). For this, we calculate the definite integral of the function. By virtue of Lemma 1, we obtain:

$$\int_0^1\varphi(x)dx=\int_0^1 f(x)dx+\sum_{n=0}^{m-1}\left(f^{(n)}(0)-f^{(n)}(1)\right)\int_0^1\frac{B_{n+1}(x)}{(n+1)!}dx=\int_0^1 f(x)dx$$

Thus, we have shown the validity of equality (10). Theorem 1 is fully proved.

On a Method for Constructing Composite Optimal Quadrature Formulas.

In this section, we provide a new method for constructing composite optimal quadrature formulas in the Hilbert space $W_2^{(m,m-1)}(0,1)$. From the work of Kh.M.Shadimetov at al., it is known that the rectangle formula is optimal (see [12]):

Theorem 2. *Among the quadrature formulas of the form*

$$\int_0^1\varphi(x)dx\cong\sum_{\beta=0}^{N-1}C(h\beta)\varphi(h\beta) \tag{13}$$

with the error functional $\ell(x)$ in the space $\widetilde{W}_2^{(m,m-1)}(0,1)$, there exists a unique optimal quadrature formula whose coefficients are constant, i.e.,

$$C(h\beta)=h,\quad \beta=0,1,\dots,N-1,\quad h=\frac{1}{N},\quad N=2,3,\dots \tag{14}$$

Moreover, the square of the norm of the error functional of the optimal quadrature formulas has the form:

$$\|\ell\|^2_{W_2^{(m,m-1)}}=(-1)^{m-1}\sum_{n=2m}^{\infty}\frac{B_n}{n!}h^{2n},\quad \text{for } m\geq 2. \tag{15}$$

The main result of this section is the following theorem.

Theorem 3. *Among the quadrature formulas of the form (1) in the space $W_2^{(m,m-1)}(0,1)$, the following formula is optimal:*

$$\int_0^1 f(x)dx\cong h\sum_{\beta=0}^{N-1}f(h\beta)+\sum_{n=0}^{m-1}\frac{h^{n+1}B_{n+1}}{(n+1)!}\left(f^{(n)}(0)-f^{(n)}(1)\right). \tag{16}$$

Moreover, the square of the norm of the error functional of this formula has the form:

$$\|\ell\|^2_{W_2^{(m,m-1)}}=(-1)^{m-1}\sum_{n=2m}^{\infty}\frac{B_n}{n!}h^{2n},\quad \text{for } m\geq 2.$$

Proof. For this, we will use the results of Theorems 1 and 2. By virtue of Theorem 1, the corresponding function $\varphi(x)$ is an element of $\widetilde{W}_2^{(m,m-1)}(0,1)$, then by Theorem 2 the quadrature formula is optimal:

$$\int_0^1 \varphi(x)dx = h\sum_{\beta=0}^{N-1}\left[f(h\beta) + \sum_{n=0}^{m-1}\frac{B_{n+1}(h\beta)}{(n+1)!}\left(f^{(n)}(0) - f^{(n)}(1)\right)\right].$$

On the other hand, by virtue of (10), $\int_0^1 \varphi(x)dx = \int_0^1 f(x)dx$. Then we have

$$\int_0^1 f(x)dx \cong h\sum_{\beta=0}^{N-1}\left[f(h\beta) + \sum_{n=0}^{m-1}\frac{B_{n+1}(h\beta)}{(n+1)!}\left(f^{(n)}(0) - f^{(n)}(1)\right)\right] \quad (17)$$

Now we proceed to simplify the coefficients of formula (16). Calculating the sum using Bernoulli polynomial properties and grouping by powers of h, we obtain:

$$\sum_{\beta=0}^{N-1} B_{n+1}(h\beta) = B_{n+1}h^n.$$

Then the composite optimal quadrature formula (16) takes the form:

$$\int_0^1 f(x)dx \cong h\sum_{\beta=0}^{N-1} f(h\beta) + \sum_{n=0}^{m-1}\frac{h^{n+1}B_{n+1}}{(n+1)!}\left(f^{(n)}(0) - f^{(n)}(1)\right) =$$

$$= \frac{h}{2}\left(f(0) - f(1)\right) + h\sum_{\beta=1}^{N-1} f(h\beta) + \sum_{n=1}^{m-1}\frac{h^{n+1}B_{n+1}}{(n+1)!}\left(f^{(n)}(0) - f^{(n)}(1)\right).$$

The second part of Theorem 3 follows directly from the equality of errors. Theorem 3 is fully proved.

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DINAMIK SISTEMALARDA OBYEKT SHOVQINLARI KORRELYATSIYALANGANDA NOANIQ KIRISH TA’SIRLARINI TIKLASH ALGORITMLARI

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Annotatsiya. *Ushbu maqolada vaqt bo‘yicha korrelyatsiyalangan tasodifiy Gauss Markov jarayonlarining oq shovqin va to‘siqlardan farq qiluvchi tasodifiy jarayonlar kirish ta’sirlarining tavsifi, Kalmanning klassik optimal chiziqli filtrlash masalasini umumiy qo‘yilish shartini buzadi. Biroq ko‘rilayotgan holatlarda filtrlash usullarini umumlashtirish yo‘llari mavjud bo‘lganligi uchun ko‘rsatilgan holat Kalmanning filtrlash usulini amaliy qo‘llashni cheklamaydi. Bunga boshlang‘ich dinamik sistemaning matematik modelini kengaytirishga Gauss Markov tasodifiy jarayoni ko‘rinishidagi kirish ta’sirlarini shakllantiradigan qo‘shimcha matematik modeldan foydalanish hisobiga erishish mumkin.*

Kalit so‘zlar: *korrelyatsiyalangan, tasodifiy jarayonlar, adaptiv algoritm, dinamik tenglama, Kalman filtr, muntazamlashtirish, Gilbert fazo, singulyar yoyish, psevdomurojaat.*

АЛГОРИТМЫ ВОССТАНОВЛЕНИЯ НЕОПРЕДЕЛЁННЫХ ВХОДНЫХ ВОЗДЕЙСТВИЙ ПРИ КОРРЕЛИРОВАННЫХ ШУМАХ ОБЪЕКТА В ДИНАМИЧЕСКИХ СИСТЕМАХ

Аннотация. *В данной статье исследуются входные воздействия в виде коррелированных по времени случайных Гауссовских Марковских процессов. Наличие подобных случайных процессов, отличающихся от белого шума и помех, нарушает общие условия классической задачи оптимальной линейной фильтрации Калмана. Однако существующие возможности обобщения методов фильтрации в рассматриваемых случаях не ограничивают практическое применение метода фильтрации Калмана. Это достигается за счёт расширения математической модели исходной динамической системы и использования дополнительной математической модели, формирующей входные воздействия в виде Гауссовских Марковских случайных процессов.*

Ключевые слова: *коррелированные процессы, случайные процессы, адаптивный алгоритм, динамическое уравнение, фильтр Калмана, регуляризация, пространство Гильберта, сингулярное разложение, псевдообращение.*

ALGORITHMS FOR RECOVERING UNCERTAIN INPUT EFFECTS WHEN OBJECT NOISE IS CORRELATED IN DYNAMIC SYSTEMS

Abstract. *This article investigates input effects represented by time-correlated Gaussian-Markov random processes. The presence of such random processes, which differ from white noise and disturbances, violates the general assumptions of the classical Kalman optimal linear filtering problem. However, the existence of generalized filtering approaches in such cases does not limit the practical application of the Kalman filtering method. This is achieved by extending the mathematical model of the original dynamical system and employing an additional mathematical model that generates input effects in the form of Gaussian-Markov random processes.*

Keywords: *correlated processes, random processes, adaptive algorithm, dynamical equation, Kalman filter, regularization, Hilbert space, singular decomposition, pseudoinverse.*

Kirish. Ushbu ko‘rib chiqilayotgan holatda baholashning parametrik adaptiv algoritmlarini amaliy jihatdan amalga oshirishda boshlang‘ich ma’lumotlarning kichik o‘zgarishlarida yechimi turg‘un bo‘lmagan masalani ko‘rib chiqishga to‘g‘ri kelganda, ularni shakllantirishdagi o‘sha holat bilan bog‘liq hisoblash tavsifidagi sezilarli qiyinchiliklar uchraydi. Ular shu bilan tavsiflanadiki, boshlang‘ich ma’lumotlarning shunchalik kichik o‘zgarishlari yechimlarning ixtiyoriy katta o‘zgarishlariga olib kelishi mumkin. Shu munosabat bilan obyektni kirish shovqinining korrelyatsiyalanganlik sharoitida dinamik sistema holatini baholash aniqligini oshirish uchun muntazamlashtirish usullaridan foydalanish maqsadga muvofiqdir.

Metod. Kalmanning optimal filtrlash nazariyasida kirishda oq shovqin tipidagi mustaqil tasodifiy gauss jarayoni tushadigan gauss markov tasodifiy jarayoni dinamik sistemaning chiqishida qayta ishlanadi. O‘lchash shovqini ta’siri o‘lchash sistemasining matematik modelida hisobga olinadigan oq shovqin tipidagi gauss tasodifiy jarayonini ham aks ettiradi. Real sharoitlarda kirish ta’sirlarini funksionirlash sistemasi oq shovqindan farq qilishi mumkin va vaqt bo‘yicha korrelyatsiyalangan tasodifiy gauss markov jarayonlarini aks ettiradi.

Kengaytirilgan dinamik sistemaning kuzatish va dinamik tenglamasini yuqorida shakllantirilgan shartlarda quyidagi ko‘rinishda yozish mumkin:

$$x_{i+1}^{*} = A^{*} x_i^{*} + D^{*} w_i^{*}, \tag{5}$$

$$z_{i+1}^{*} = H^{*} x_{i+1}^{*} + v_{i+1}^{*}, \tag{6}$$

bu yerda $A^{*} = \left[\begin{array}{c|c} A & \Gamma D_1 \\ \hline 0 & D_2 \end{array}\right]$, $H^{*} = \left[\begin{array}{c|c} H_1 & 0 \\ \hline 0 & H_2 \end{array}\right]$, $D^{*} = \left[\begin{array}{c} 0 \\ \hline D_3 \end{array}\right]$,

$x_{i+1}^{*} = \left[\begin{array}{c} x_{i+1} \\ \hline \xi_{i+1} \end{array}\right]$, $v_{i+1}^{*} = \left[\begin{array}{c} v_{1,i+1} \\ \hline v_{2,i+1} \end{array}\right]$, $z_{i+1}^{*} = \left[\begin{array}{c} z_{1,i+1} \\ \hline z_{2,i+1} \end{array}\right]$, $w_i^{*} = w_i$.

(5), (6) tenglamalar asosida kengaytirilgan holat vektorini baholash uchun Kalmanning an’anaviy filtridan foydalanish mumkin:

$$x_{i+1|i}^{*} = A^{*} x_{i|i}^{*}, \tag{7}$$

$$P_{i+1|i}^{*} = A^{*} P_{i|i}^{*} A^{*T} + D^{*} \hat{Q}_i D^{*T}, \tag{8}$$

$$x_{i+1|i+1}^{*} = x_{i+1|i}^{*} + K_{i+1}^{*} y_{i+1}^{*}, \tag{9}$$

$$y_{i+1|i}^{*} = z_{i+1}^{*} - H^{*} x_{i+1|i}^{*}, \tag{10}$$

$$K_{i+1}^{*} = P_{i+1|i}^{*} H^{*T} [C_{i+1}^{*}]^{-1}, \tag{11}$$

$$C_{i+1}^{*} = H^{*} P_{i+1|i}^{*} H^{*T} + R_{i+1}, \tag{12}$$

$$P_{i+1|i+1}^{*} = (I - K_{i+1}^{*} H^{*}) P_{i+1|i}^{*}. \tag{13}$$

Har bir qadamda ξ_{i+1} qiymat bilan hisoblangan (3) ifoda ς_{i+1} noma’lum kirish ta’sirining qiymatini beradi. (7)-(13) ko‘rinishlardagi Kalman filtridan kelib chiqib K_{i+1}^{*} Kalman filtrlashi algoritmining kuchaytirish koeffitsiyenti Q_i va R_{i+1} matritsasiga bog‘liqligini ko‘rish qiyin emas.

(7)-(13) ifodalarni tahlil qilib, ularni hisoblash jarayonida Q_i va R_{i+1} baholardan foydalanilganligi uchun, (8), (10)-(13) munosabatlarini ekstrapolyatsiya va filtrlash algoritmlaridan istisno qilish maqsadga muvofiq, degan xulosaga kelish mumkin. [3, 8, 13] da bundagi K_{i+1}^{*} kuchaytirish koeffitsiyentini hisoblashda quyidagi ikki bosqichli hisoblash tartibi asosida Z_{i+1}^{i+l} o‘lchamini tanlab ishlab chiqish mumkinligi ko‘rsatildi:

$$SL^{*} = M_y, \tag{14}$$

va

$$K_{i+1}^{*} \Phi^{*} = L^{*}, \tag{15}$$

bu yerda

$$S=\begin{bmatrix} H^* \\ \hline H^*A^* \\ \hline \cdots\cdots \\ \hline H^*A^{*l-1} \end{bmatrix} A^*,\ L^*=P^*_{i+1/i}H^{*T},\ M_y=\begin{bmatrix} y^*_{i+2|i}y^{*T}_{i+1|i} \\ \hline y^*_{i+3|i}y^{*T}_{i+1|i} \\ \hline \cdots\cdots\cdots\cdots \\ \hline y^*_{i+l|i}y^{*T}_{i+1|i} \end{bmatrix},\ \Phi^*=[y^*_{i+1|i}y^{*T}_{i+1|i}].$$

Bunday modifikatsiyalashtirish K_{i+1} filtrning kuchaytirish koeffitsiyenti real o‘lchash axborotiga bog‘liq bo‘lib, K_{i+1} matritsaning aprior ma’lumotlarga qat’iy bog‘liqligini bartaraf qilishga olib keladi. Shunday qilib, keltirilgan algoritm yaxshi ishlanmagan hisoblanadi, ya’ni aprior ma’lumotlarning o‘zgarishiga kam sezuvchandir. Aprior axborot noaniq berilganda, ushbu holat jiddiy ahamiyatga ega.

(14) va (15) tenglamalar sistemasi yomon shartlangan, ya’ni boshlang‘ich ma’lumotlarning kichik o‘zgarishlariga yechimning katta o‘zgarishlari javob berishi mumkin. Ushbu tenglamalarni yechishdagi belgilangan holatlar muntazamlashtirish usullarini qo‘llash zarurligiga olib keladi [4, 6-9]. Ularning orasidan A.N.Tixonov muntazamlashtirishining umumiy prinsipiga asoslangan [4, 6] va S matritsani singulyar yoyilishida bazalangan (14) samarali psevdomurojaat usuli kabi usullar guruhini alohida ta’kidlash kerak [7, 12].

(14) tenglamani yechishga o‘tamiz. Yozuvlarni qisqartirish va kelgusida qulay ifodalash uchun (14) tenglamani quyidagi ko‘rinishda yozamiz:

$$S l^*_{i,j}=m_{y,i,j}\ , \tag{16}$$

bu yerda: S – H_1 gilbert fazosidan H_2 gilbert fazosiga harakatlanuvchi chiziqli operator; $l^*_{i,j}-L^*$ matritsaning i– ustuni $j=1,2,...,$; $m_{y,i,j}-M_y$ matritsaning i– ustuni.

Samarali psevdomurojaat usuli S matritsani (singulyar yoyilishda) faktorizatsiyalashga, ya’ni uni [4, 6, 12] dagi ko‘rinishiga asoslangan

$$S=UTV^T, \tag{17}$$

bu yerda:U – $(m\times m)$ – o‘lchamli ortogonal matritsa; V – $(n\times n)$ – o‘lchamli ortogonal matritsa; T – $(m\times n)$ – o‘lchamlidiagonal matritsa.

U va$V$ matritsalarning u_i va v_i ustunlari SS^T va S^TS matritsa xususiy vektorlarining mohiyatidir, T matritsaning μ_i diagonal elementlari esa S^TS (yoki SS^T) matritsa λ_i xususiy qiymatlarining musbat ildizlaridir.

Natija. S^+ psevdoteskari matritsa (Mur – Penrouz) quyidagi bahoni olish imkonini beradi [12]:

$$\hat{l}^*_{i,j}=VT^+U^Tm_{y,i,j}=\sum_{i=1}^{r}\frac{1}{\mu_i}v_iu_i^T\ , \tag{18}$$

bu yerda: $T^+=diag(t_1^+,...,t_i^+)$ –T –matritsa uchun matritsa; r – S –matritsa rangi ($\mu_i(i=\overline{1,n})$ singulyar sonni noldan farqlaydigan son), bu yerda agar $\mu_i\neq 0$, va$t_i^+=0$, yoki $\mu_i=0$ bo‘lsa $t_i^+=1/\mu_i$.

Agar S matritsa rangi $r=n$ bo‘lsa, psevdoteskari matritsa EKKU bo‘yicha [9] bahosiga mos keladi:

$$\hat{l}^*_{i,j,\alpha}=(S^TS+\alpha I)^{-1}S^Tm_{y,i,j},$$

va mos holda past aniqlik bilan tavsiflanadi. Shu munosabat bilan shunday nomlanadigan samarali psevdoteskari matritsa va baholar qo‘llaniladi
[6, 12]

$$\hat{l}^*_{i,j,\tau}=VT^+_\tau U^Tm_{y,i,j}=\sum_{i=1}^{r'}\frac{1}{\mu_i}v_i\cdot u_i^T\ , \tag{19}$$

bu yerda: T^+_τ – samarali $T=diag(t^+_{1\tau},...,t^+_{n\tau})$; $r'<r$ psevdoteskari matritsa, agar $\mu_i>\tau$ va$t^+_{i\tau}=0$, yoki $\mu_i=0$ bo‘lsa, $t^+_{i\tau}=1/\mu_i$ bo‘ladi.

Xatolikning minimal o‘rtacha me’yoriga ega bo‘lgan $\hat{l}^*_{i,j,\alpha}$ va $\hat{l}^*_{i,j,\tau}$ baholarni olish uchun α muntazamlashtirish parametri va τ yoki r' parametrlar $m_{y,i,j}$ vektorni o‘lchash xatoligining dispersiyasi bilan moslashtirilgan bo‘lishi kerak. Amaliyotda ularni ko‘pincha xatoliklar usuli bo‘yicha aniqlanadi.

Endi (15) tenglamani yechishga o‘tamiz. Yozuvlarni qisqartirish va kelgusida qulay ifodalash uchun (15) tenglamani quyidagi ko‘rinishda yozamiz:

$$W_i k_{i,j} = l^{\delta}_{i,j} = l_{i,j} + \delta l_{i,j}, \tag{20}$$

bu yerda: $W_i = \Phi^{*T}$ – gilbert fazosida *Z* dan *Z* ga harakatlanuvchi chiziqli operator: $k_{i,j} - K^{*T}_{i+1}$ matritsaning *i*- chi qatori,

$$j = 1,2,...; l^{\delta}_{i,j} - \delta^2 = \left\|\delta l_{i,j}\right\|^2_E, \ \delta^2_{\min} < \left\|\delta l_{i,j}\right\|^2_E < \delta^2_{\max}$$

ko‘rinishda approksimatsiyalash shartidagi L^{*T} matritsaning *i*- chi ustuni; $l_{i,j}$ –(20) tenglama o‘ng qismining aniq qiymati.

(20) ifodani hisobga olib (15) tenglamani yechish uchun M.M.Lavrentev usulidan foydalanamiz [2, 5, 10].

W_i operator musbat va o‘z-o‘ziga bog‘liq bo‘lgan holatni ko‘rib chiqamiz. U holda M.M.Lavrentev usuli quyidagi ko‘rinishdagi tenglamaga keladi:

$$\alpha k_{i,j} + W_i k_{i,j} = l^{\delta}_{i,j}, \tag{21}$$

bu yerda: α – muntazamlashtirish parametri.

(21) tenglamaning yechimini quyidagi shaklda ifodalash mumkin

$$k_{i,j,\alpha} = (\alpha I + W_i)^{-1} l^{\delta}_{i,j} = \theta_{\alpha}(W_i) l^{\delta}_{i,j},$$

bu yerda

$$\theta_{\alpha}(\lambda) = (\alpha + \lambda)^{-1}$$

- keltirib chiqaruvchi funksiya sistemasi, $0 \le \lambda < \infty$ –spektral parametr.

Agar W_i operator o‘z-o‘ziga bog‘liq o‘zgaruvchan ishorali bo‘lsa, u holda bu yerda M.M.Lavrentev usulining quyidagi aynan yaqinlashgan yechimlar modifikatsiyasini qo‘llash maqsadga muvofiq [1]

$$k_{i,j,\alpha} = g_{\alpha}(W_i) l^{\delta}_{i,j},$$

bu yerda:

$$\bar{k}_{i,j,\alpha} = g_{\alpha}(W_i) W_i \ k_{i,j,\alpha} = \bar{g}_{\alpha}(W_i) l^{\delta}_{i,j}, \ \bar{g}_{\alpha}(\lambda) = \lambda g^2_{\alpha}(\lambda);$$

quyidagi tenglamadan aniqlanadi:

$$\mu\alpha k_{i,j} + W_i k_{i,j} = l^{\delta}_{i,j}, \ \alpha > 0, \ \mu = \sqrt{-1}. \tag{22}$$

quyidagi

$$k_{i,j,\alpha} = \operatorname{Re} v_{\alpha}, \ v_{\alpha} = (i\alpha I + W_i)^{-1} l^{\delta}_{i,j},$$

ko‘rinishdagi ifodadan yoki A.N.Tixonov usulining maxsus hisoblash sxemasidan ham foydalanish mumkin [1, 7]:

$$\alpha^2 k_{i,j,\alpha} + W_i^2 k_{i,j,\alpha} = W_i l^{\delta}_{i,j}.$$

Nokorrekt qo‘yilgan masalalarni yechish usullari asosida ko‘rsatish mumkinki [9, 14], (15) sistemani sezilarli darajada yaxshi shartlanganligini ta’minlash uchun quyidagi ko‘rinishdagi sistemani qarab chiqish maqsadga muvofiq:

$$W_{i,\alpha} k_{i,j,\alpha} = \lambda l^{\delta}_{i,j},$$

bu yerda

$$W_{i,\alpha} = [(\mathsf{D}_W + \alpha \mathsf{D}_W^{-1}) + (1-\beta)(W_i - \mathsf{D}_W)],$$

D_W – W_i matritsa diagonali,

$$\lambda = \lambda_{\alpha} = \frac{(l^{\delta}_{i,j}, W_i \tilde{k}_{i,j,\alpha})}{\left\|W_i \tilde{k}_{i,j,\alpha}\right\|^2_E},$$

$$\beta = 1 - \sqrt{\frac{\left\| \mathsf{D}_W + \alpha \mathsf{D}_W^{-1} \right\|_E^2 - \left\| \mathsf{D}_W \right\|_E^2}{\left\| W_i - \mathsf{D}_W \right\|_E^2}},$$

$\tilde{k}_{i,j,\alpha}$ –sistema yechimi

$$W_{i,\alpha} \tilde{k}_{i,j,\alpha} = l_{i,j}^{\delta}.$$

$\hat{k}_{i,j}$ vektorni aniqlash quyidagi

$$\left\| l_{i,j}^{\delta} - W_i \tilde{\tilde{k}}_{i,j,\alpha} \right\|_E^2 < \delta_{\min}^2$$

tengsizlikni qanoatlantiruvchi $\tilde{\tilde{k}}_{i,j,\alpha}$ vektorni aniqlashga olib keladi [11].

$\tilde{\tilde{k}}_{i,j,\alpha}$ vektorni aniqlash α kattaligi to‘plamining ratsional masalasiga olib kelinadi:

$$\alpha_1,\ \alpha_l = q_l \alpha_{l-1},\ \ l = 2,3,...,$$

bu yerda q_l hisoblash jarayonida aniqlanishi kerak bo‘lgan kattalikning mohiyati.

Ularni quyidagi ifoda bo‘yicha aniqlash maqsadga muvofiq:

$$q_l = \left(\delta_{\min}^2 / \mathsf{v}_{l-1}^2 \right)^{1/2},$$

bu yerda

$$\mathsf{v}_{l-1}^2 = \left\| l_{i,j}^{\delta} - W_i k_{i,j,\alpha_{l-1}} \right\|_E^2.$$

Bunda α_1 qiymatini shunday berish kerakki, natijada

$$\mathsf{v}_1^2 = \left\| l_{i,j}^{\delta} - W_i k_{i,j,\alpha_1} \right\|_E^2 \approx \delta_{\max}^2$$

bo‘lsin, α_l ning navbatdagi qiymatlari esa

$$\left\| l_{i,j}^{\delta} - \tau_2 W_i \tilde{\tilde{k}}_{i,j} \right\|_E^2 < \delta_{\min}^2$$

uchun $\tilde{\tilde{k}}_{i,j}$ vektorini yetarli darajada tez topishni ta’minlashi kerak, bu yerda τ_2 quyidagi

$$\left\| l_{i,j}^{\delta} - \tau_2 W_i \tilde{\tilde{k}}_{i,j,\alpha} \right\|_E^2 = \delta_{\min}^2$$

tenglama yechimi asosida aniqlanadi [11].

Yuqoridagi munosabat asosida $\tilde{\tilde{k}}_{i,j}$ vektorni aniqlab, so‘ng $\tilde{\tilde{k}}_{i,j}$ vektori bo‘yicha quyidagi

$$\left\| l_{i,j}^{\delta} - W_i \hat{k}_{i,j} \right\|_E^2 < \Delta^2,\ W_i \hat{k}_{i,j}, l_{i,j}^{\delta} - W_i \hat{k}_{i,j}) = 0,$$

bu yerda $\Delta^2 = (\delta_{\min}^2 + \delta_{\max}^2)/2$ ikki shartni qanoatlantiradigan $\hat{k}_{i,j}$ vektorni topish mumkin.

Xulosa. Shuni ta’kidlash kerakki, bu yerda λ va β kattaliklar α parametrining tanlangan qiymati asosida topiladi. Bunda $\beta = \beta_\alpha$ matritsani muntazamlashtirishda evklid normasini saqlash prinsipi bo‘yicha aniqlanadi.

Shunday qilib, keltirilgan algoritmlar obyekt shovqinining korrelyatsiyalangan shartlarida dinamik sistemalardagi noma’lum kiruvchi ta’sirlarni tiklash masalasini muntazamlashtirish imkonini beradi.

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UO‘K 539.3

UCH FAZALI INERSIONAL-VISKOELASTIK MUHIT TEBRANISH TENGLAMASI UCHUN CHEGARAVIY MASALA

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Annotatsiya. *Ushbu maqolada uch fazali inersional-viskoelastik muhitda tebranish jarayonlarini ifodalovchi to‘rtinchi tartibli gibrid xususiy hosilali differensial tenglama uchun birinchi tur chegaraviy va boshlang‘ich masala tadqiq qilingan. Retardatsiya vaqtlari hamda inersiya parametrlarini hisobga oluvchi reologik model uchun chegaralangan sohada matematika-fizika masalasi qo‘yilgan. Masalaning analitik yechimi o‘zgaruvchilarni ajratish (Furye) usuli yordamida qurilgan. Vaqtga bog‘liq tenglamaning xarakteristik ildizlari va so‘nish rejimlarining reologik koeffitsiyentlarga bog‘liqligi chuqur tahlil qilingan. Yechim cheksiz trigonometrik Furye qatori ko‘rinishida shakllantirilib, uning tekis yaqinlashishi hamda energik integral usuli yordamida yechimning yagonaligi qat’iy isbotlangan. Olingan analitik natijalar kon-metallurgiya sanoatidagi pulpa va shlam quvurlari hamda gidrotizimlarda nostatsionar tebranish jarayonlarini hisoblashda amaliy ahamiyatga ega.*

Kalit so‘zlar: *inersional-viskoelastik muhit, reologik model, retardatsiya vaqti, tebranish tenglamasi, Furye usuli, energik integral, yagonalik teoremasi.*

КРАЕВАЯ ЗАДАЧА ДЛЯ УРАВНЕНИЯ КОЛЕБАНИЙ ТРЁХФАЗНОЙ ИНЕРЦИОННО-ВЯЗКОУПРУГОЙ СРЕДЫ

Аннотация. *В данной работе исследована начально-краевая задача первого рода для гибридного дифференциального уравнения в частных производных четвёртого порядка, описывающего колебательные процессы в трёхфазной инерционно-вязкоупругой среде. Для реологической модели, учитывающей времена запаздывания и параметры инерции, поставлена задача математической физики в ограниченной области. Аналитическое решение задачи построено методом разделения переменных (методом Фурье). Подробно проанализирована зависимость характеристических корней временного уравнения и режимов затухания от реологических коэффициентов. Решение сформулировано в виде бесконечного тригонометрического ряда Фурье, а его равномерная сходимость и единственность строго доказаны с использованием метода энергетического интеграла. Полученные аналитические результаты имеют практическое значение для расчёта нестационарных колебательных процессов в пульпопроводах, шламовых потоках и гидравлических системах горно-металлургической отрасли.*

Ключевые слова: *инерционно-вязкоупругая среда, реологическая модель, время запаздывания, уравнение колебаний, метод Фурье, энергетический интеграл, теорема единственности.*

BOUNDARY VALUE PROBLEM FOR THE OSCILLATION EQUATION OF A THREE-PHASE INERTIAL-VISCOELASTIC MEDIUM

Abstract. *This paper investigates a first-type initial-boundary value problem for a fourth-order hybrid partial differential equation describing oscillation processes in a three-phase inertial-viscoelastic medium. A mathematical physics problem is posed in a bounded domain for a rheological model incorporating*

retardation times and inertia parameters. The analytical solution of the problem is constructed using the method of separation of variables (Fourier method). The dependence of the characteristic roots of the time-dependent equation and damping regimes on rheological coefficients is thoroughly analyzed. The solution is formulated as an infinite trigonometric Fourier series, and its uniform convergence together with uniqueness is rigorously proved via the energy integral method. The obtained analytical results are of practical importance for calculating non-stationary oscillation processes in pulp and slurry pipelines as well as hydraulic systems in the mining and metallurgical industry.

Keywords: *inertial-viscoelastic medium, rheological model, retardation time, oscillation equation, Fourier method, energy integral, uniqueness theorem.*

Kirish. Tutash muhitlar mexanikasida yopishqoq-elastik (viskoelastik) xossaga ega bo‘lgan materiallarning dinamik holatini hamda ulardagi tebranish jarayonlarini modellashtirish amaliy va nazariy jihatdan muhim masalalardan biridir. Xususan, kon-metallurgiya sanoatidagi quyuq pulpa va shlam quvurlarida, neft-gaz muhandisligidagi burg‘ilash eritmalarida, seysmik to‘lqinlar ta’siridagi nam g‘ovak grunt qatlamlarida va polymer tizimlarda to‘lqinlar tarqalishini aniqroq tasvirlash uchun muhitning nafaqat elastik va yopishqoq xossalarini, balki uning inersional xarakteristikalarini ham hisobga olish talab etiladi [1, 2].

Muhitning relaksatsiya va retardatsiya jarayonlari klassik reologik modellar - Kelvin-Voigt va Maksvell modellarida keng o‘rganilgan bo‘lsa-da, yuqori chastotali hamda jadallashtirilgan tebranish rejimlarida massaning inersiyaga ko‘rsatadigan qarshiligi to‘liq qamrab olinmaydi. Bu yo‘nalishda R.M. Christensen [1], A. Wineman [2] va F. Mainardi [3] chiziqli viskoelastik muhitlarda to‘lqinlar dinamikasi nazariyasini rivojlantirganlar. A.A. Il'yushin va B.E. Pobedrya [4] thermoviskoelastik muhitlar holat tenglamalarining matematik asoslarini yaratgan bo‘lsa, O.A. Ladyzhenskaya [5] hamda K. Khudoynazarov va A. Abdurazaqov [6] viskoelastik konstruksiya elementlarida differensial tenglamalar uchun chegaraviy masalalarning to‘g‘ri qo‘yilganligini (korrektligini) isbotlab berishgan.

Biroq yuqorida keltirilgan ishlarda massa inersiyasining ta'siri asosan tenglamaning chap tomonidagi tashqi kuchlar muvozanatida ($\rho\frac{\partial^2 u}{\partial t^2}$) hisobga olingan bo‘lib, reologik modelning ichki tuzilmasida (so‘nish va retardatsiya mexanizmida) inersion elementlarning tutgan o‘rni yetarlicha ochib berilmagan.[8-10]

Ushbu maqolaning maqsadi - uch fazali inersional-viskoelastik muhit tebranishi uchun gibrid 4-tartibli xususiy hosilali differensial tenglamaga keltiriladigan birinchi tur chegaraviy va boshlang‘ich masalani qo‘yish hamda uni o‘zgaruvchilarni ajratish (Furye) usulida analitik yechishdan iborat. Shuningdek, Furye qatorining tekis yaqinlashishi va energik integral usuliga tayanib, yechimning mavjudligi hamda yagonaligini isbotlash, reologik parametrlarning tebranish so‘nish koeffitsiyentiga ta’sirini tahlil qilish nazarda tutilgan.

Metodlar. Uch fazali (elastik, viskoz va inersional) reologik muhitning deformatsiyalanish jarayonini modellashtirishda elementlarning parallel ulangan sxemasi qaraladi. Muhitdagi umumiy kuchlanish $\sigma(t)$ elastiklik (f_3G_i), viskozlik ($f_1\mu_i$) hamda inersional-gidrodinamik (f_2m_{li}) tashkil etuvchilarning yig‘indisidan iborat.[11,12]

$$\sigma(t) = \sigma_e + \sigma_q + \sigma_i \tag{1}$$

Deformatsiya $\varepsilon(t)$ va uning vaqt bo‘yicha hosilalari orqali ushbu bog‘lanish quyidagi differensial ko‘rinishni oladi:[o‘zimniki]

$$\sigma(t) = f_3G_i\varepsilon(t) + f_1\mu_i\frac{\partial\varepsilon(t)}{\partial t} + f_2m_{li}\frac{\partial^2\varepsilon(t)}{\partial t^2} \tag{2}$$

Tenglamadan elastiklik koeffitsiyenti f_3G_i ni qavsdan tashqariga chiqarib, Kelvin-Voigt bo‘yicha retardatsiya vaqti $t_{ret}^K = \frac{f_1\mu_i}{f_3G_i}$ hamda inersional-viskoelastik retardatsiya vaqti $\left(t_{ret}^{Kh2}\right)^2 = \frac{f_2m_{li}}{f_3G_i}$ belgilashlarini kiritsak, muhitning holat tenglamasi hosil bo‘ladi.

$$\sigma(t) = f_3G_i\left(\varepsilon(t) + t_{ret}^K\frac{\partial\varepsilon(t)}{\partial t} + \left(t_{ret}^{Kh2}\right)^2\frac{\partial^2\varepsilon(t)}{\partial t^2}\right) \tag{3}$$

Muhitning deformatsiya orqali ifodalangan dinamik tenglamasini hosil qilish uchun tutash muhitlar harakat tenglamasi $\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial\sigma}{\partial x}$ va kichik deformatsiyalar nisbati $\varepsilon = \frac{\partial u}{\partial x}$ dan foydalanamiz. Natijada x va t o‘zgaruvchilariga nisbatan 4-tartibli aralash xususiy hosilali differensial tenglama kelib chiqadi.

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + t_{ret}^K\frac{\partial^3 u}{\partial x^2\partial t} + \left(t_{ret}^{Kh2}\right)^2\frac{\partial^4 u}{\partial x^2\partial t^2} \tag{4}$$

bu yerda $a^2 = \frac{f_3 G_i}{\rho} > 0$ - muhit elastik to‘lqinlarining tarqalish tezligi kvadratini, $t_{ret}^K > 0$ va $t_{ret}^{Kh2} > 0$ esa mos ravishda viskoz va inersional retardatsiya vaqtlarini ifodalaydi.

Chegaralangan $D_T = \{(x,t): 0 < x < l,\ 0 < t < T\}$ sohada (4) tebranish tenglamasi uchun birinchi tur (Dirixle) chegaraviy shartlari qo‘yiladi.

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t \geq 0 \tag{5}$$

Bundan tashqari $t = 0$ vaqt momentidagi boshlang‘ich shart qo‘shiladi.

$$u(x,0) = \phi(x), \quad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \psi(x), \quad 0 \leq x \leq l \tag{6}$$

Boshlang‘ich va chegaraviy shartlar birga qo‘llanganda uzluksizlik saqlanishi uchun ushbu moslashuv shartlari bajarilishi kerak.

$$\phi(0) = \phi(l) = 0, \quad \phi''(0) = \phi''(l) = 0, \quad \psi(0) = \psi(l) = 0 \tag{7}$$

(4) tenglamani (6) masalasining klassik yechimi mavjudligi, yagonaligi hamda qurilgan Furye qatorining tekis yaqinlashishi tekshirib ko‘rilishi zarur. Buning quyidagi teorema va lemmalardan foydalanamiz.

Teorema 1. *Agar* $\phi(x) \in C^3[0,l]$ *va* $\psi(x) \in C^2[0,l]$ *funksiyalar (7) moslashuv shartlarini qanoatlantirsa, u holda (4)-(6) masalaning* $u(x,t) \in C^{2,2}(Q_T)$ *klassik yechimi mavjud, yagona va u ushbu Furye qatori ko‘rinishida ifodalanadi:*

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi x}{l} \tag{8}$$

Isbot. Yechimni $u(x,t) = X(x)T(t)$ ko‘rinishida izlasak (4) tenglama quyidagi ko‘rinishga o‘tadi.

$$\frac{T''(t)}{a^2 T(t) + t_{ret}^K T'(t) + \left(t_{ret}^{Kh2}\right)^2 T''(t)} = \frac{X''(x)}{X(x)} = -\lambda \tag{9}$$

Natijada ikkita ajralgan differensial tenglama va (5) chegaraviy shartlarga ko‘ra Shturm-Liuvill masalasi hosil bo‘ladi: $X''(x) + \lambda X(x) = 0, X(0) = 0, X(l) = 0$

Ushbu masalaning xos qiymatlari va ortogonal xos funksiyalari:

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{l}, \quad n = 1,2,3,\ldots \tag{10}$$

Vaqtga bog‘liq $T_n(t)$ funksiya uchun 2-tartibli oddiy differensial tenglama keltirib chiqariladi.

$$\left(1 + a^2\lambda_n\left(t_{ret}^{Kh2}\right)^2\right) T_n''(t) + a^2\lambda_n t_{ret}^K T_n'(t) + a^2\lambda_n T_n(t) = 0 \tag{11}$$

(6) boshlang‘ich shartlardan $T_n(t)$ uchun boshlang‘ich shartlarni topamiz.

$$T_n(0) = \phi_n = \frac{2}{l}\int_0^l \phi(x) \sin\frac{n\pi x}{l}\,dx, \quad T_n'(0) = \psi_n = \frac{2}{l}\int_0^l \psi(x) \sin\frac{n\pi x}{l}\,dx \tag{12}$$

(11) tenglamaning xarakteristik tenglamasini tuzsak (13) hosil bo‘ladi.

$$\left(1 + a^2\lambda_n\left(t_{ret}^{Kh2}\right)^2\right) r^2 + a^2\lambda_n t_{ret}^K r + a^2\lambda_n = 0 \tag{13}$$

Barcha reologik parametrlar $a^2, t_{ret}^K, t_{ret}^{Kh2} > 0$ va $\lambda_n > 0$ bo‘lgani uchun, xarakteristik ildizlar $r_{1,2}^{(n)} = -\gamma_n \pm i\omega_n$ manfiy haqiqiy qismga ega ($\gamma_n > 0$), bu esa tebranishlarning vaqt o‘tishi bilan so‘nishini ko‘rsatadi. (11)-(12) masalaning analitik yechimi (14) ko‘rinishda bo‘ladi.

$$T_n(t) = e^{-\gamma_n t}\left(\phi_n \cos\omega_n t + \frac{\psi_n + \gamma_n\phi_n}{\omega_n}\sin\omega_n t\right) \tag{14}$$

Teorema 1. shartiga ko‘ra $\phi(x) \in C^3[0,l]$ va $\psi(x) \in C^2[0,l]$ bo‘lgani uchun, bo‘laklab integrallash usulini qo‘llab, Furye koeffitsiyentlarini topa olamiz.

$$\phi_n = \frac{1}{\lambda_n^{3/2}}\phi_n^{(3)}, \quad \psi_n = \frac{1}{\lambda_n}\psi_n^{(2)} \tag{15}$$

bu yerda $\sum_{n=1}^{\infty}\left(\phi_n^{(3)}\right)^2 < \infty$ va $\sum_{n=1}^{\infty}\left(\psi_n^{(2)}\right)^2 < \infty$. Veyershtrass va Koshi-Bunyakovskiy tengsizliklariga ko‘ra:

$$|u(x,t)| \leq \sum_{n=1}^{\infty} |T_n(t)| \leq M\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{3/2}}\left(|\phi_n^{(3)}| + |\psi_n^{(2)}|\right) \leq M_1\sum_{n=1}^{\infty}\frac{1}{n^3} < \infty \tag{16}$$

Xuddi shunday, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial t^2}$ va $\frac{\partial^4 u}{\partial x^2 \partial t^2}$ hosilalardan tuzilgan qatorlar tekis yaqinlashishi va $u(x,t) \in C^{2,2}(Q_T)$ ekanligi isbotlanadi.

Endi yechimni yagonaligi haqidagi qismi ko‘rib chiqamiz. Faraz qilaylik, masalaning ikkita $u_1(x,t)$ va $u_2(x,t)$ klassik yechimlari mavjud bo‘lsin. Ularning farqini $w(x,t)=u_1(x,t)-u_2(x,t)$ deb belgilaymiz. $w(x,t)$ funksiya bir jinsli (4) tenglamani hamda nolga teng chegaraviy va boshlang‘ich shartlarni qanoatlantiradi.

Masala uchun quyidagi integralni qaraymiz:

$$E(t)=\frac{1}{2}\int_0^l\left[\left(\frac{\partial w}{\partial t}\right)^2+a^2\left(\frac{\partial w}{\partial x}\right)^2+a^2\left(t_{ret}^{Kh2}\right)^2\left(\frac{\partial^2 w}{\partial x\partial t}\right)^2\right]dx \tag{17}$$

Ushbu $E(t)$ integralining vaqt bo‘yicha hosilasini olib, (4) tenglamani o‘rniga qo‘yamiz va bo‘laklab integrallaymiz:

$$\frac{dE(t)}{dt}=-a^2t_{ret}^{K}\int_0^l\left(\frac{\partial^2 w}{\partial x\partial t}\right)^2dx\le 0 \tag{18}$$

Bu $E(t)$ funksiyaning o‘smaydigan ekanligini ko‘rsatadi ($0\le E(t)\le E(0)$). Nol boshlang‘ich shartlarga ko‘ra $E(0)=0$, demak barcha $t\ge 0$ uchun $E(t)\equiv 0$. Bunga ko‘ra $\int_0^l\left(\frac{\partial w}{\partial x}\right)^2dx=0$ va $w(0,t)=0$ shartdan $w(x,t)\equiv 0$, ya’ni $u_1(x,t)\equiv u_2(x,t)$ kelib chiqadi. Teorema to‘liq isbotlandi.

Natija va muhokama.

1-Teoremaga muvofiq, uch fazali inersional-viskoelastik muhit tebranishini ifodalovchi (4)-(6) chegaraviy masalaning analitik yechimi quyidagi cheksiz trigonometric Furye qatori (8) ko‘rinishida shakllantiriladi. Bu yerda vaqtinchalik $T_n(t)$ funksiya (11) tenglamaning analitik yechimi bo‘lib, u muhitning reologik va inersional xarakteristikalariga bevosita bog‘liq bo‘ladi. (11) tenglamaning D_n diskriminantining ishorasi tebranishlarning fizik xarakterini belgilab beradi:

$$D_n=a^4\lambda_n^2(t_{ret}^{K})^2-4a^2\lambda_n\left(1+a^2\lambda_n\left(t_{ret}^{Kh2}\right)^2\right) \tag{19}$$

Parametrlarning nisbatiga qarab uch xil fizik-mexanik rejim bo‘lishi mumkin: ($D_n<0$) Ko‘pgina amaliy diskriminant manfiy bo‘ladi. Xarakteristik ildizlar kompleks qo‘shma ko‘rinishga ega ($r_{1,2}^{(n)}=-\gamma_n\pm i\omega_n$):

$$\gamma_n=\frac{a^2\lambda_n t_{ret}^{K}}{2\left(1+a^2\lambda_n\left(t_{ret}^{Kh2}\right)^2\right)},$$
$$\omega_n=\frac{\sqrt{4a^2\lambda_n\left(1+a^2\lambda_n\left(t_{ret}^{Kh2}\right)^2\right)-a^4\lambda_n^2(t_{ret}^{K})^2}}{2\left(1+a^2\lambda_n\left(t_{ret}^{Kh2}\right)^2\right)} \tag{20}$$

Ushbu holdagi yechim (22) ko‘rinishda bo‘lib o‘zgarmas koeffitsentli, bir jinsli differensial tenglamani yechimi ko‘rinishda ifodalanadi.

$$T_n(t)=e^{-\gamma_n t}\left(\phi_n\cos\omega_n t+\frac{\psi_n+\gamma_n\phi_n}{\omega_n}\sin\omega_n t\right) \tag{21}$$

($D_n>0$) Nisbatan yuqori yopishqoqlik va kichik inersiyada xarakteristik ildizlar haqiqiy va manfiy bo‘ladi. Tizim tebranmasdan, eksponentasimon ravishda muvozanat holatiga qaytadi. ($D_n=0$) Ikkala ildiz ham bir-biriga teng va manfiy bo‘lganda, kretik holat yuz beradi.

Klassik Kelvin-Voigt modelida ($t_{ret}^{Kh2}=0$) so‘nish koeffitsiyenti $\gamma_n^{(KV)}=\frac{1}{2}a^2\lambda_n t_{ret}^{K}$ bo‘lib, garmoniklik tartibi n ortishi bilan so‘nish tezligi kvadratik ravishda o‘sib ketadi. Uch fazali inersional-viskoelastik modelda esa inersion parametr $\left(t_{ret}^{Kh2}\right)^2$ mahrajda qatnashgani sababli, yuqori chastotali tebranishlarda asimptotik cheklanishlar yuz beradi.

$$\lim_{n\to\infty}\gamma_n=\lim_{n\to\infty}\frac{a^2\lambda_n t_{ret}^{K}}{2\left(1+a^2\lambda_n\left(t_{ret}^{Kh2}\right)^2\right)}=\frac{t_{ret}^{K}}{2\left(t_{ret}^{Kh2}\right)^2}=\text{const} \tag{22}$$

Xulosa. Ushbu maqolada uch fazali (elastik, viskoz va inersional) reologik model asosida quriladigan, tebranish jarayonlarini ifodalovchi to‘rtinchi tartibli gibrid xususiy hosilali differensial tenglama uchun birinchi tur chegaraviy va boshlang‘ich masala qo‘yildi hamda o‘zgaruvchilarni ajratish usuli yordamida to‘liq analitik tadqiq qilindi. Olib borilgan tadqiqot natijasida quyidagi asosiy xulosalarga kelindi: Muhitning holat tenglamasiga inersion tashkil etuvchi - f_2m_{li} elementining kiritilishi klassik Kelvin-Voigt modelini umumlashtirib, muhitning reologik xatti-harakatini ikkita retardatsiya vaqti - viskoz t_{ret}^{K} va inersional-viskoelastik t_{ret}^{Kh2} orqali tavsiflashga imkon berdi. Vaqtga bog‘liq (11) oddiy differensial tenglamaning

xarakteristik ildizlari tahlili natijasida tebranish jarayonining uchta fizik-mexanik rejimi ($D_n < 0$ so‘nuvchi tebranma rejim, $D_n > 0$ aperiodik so‘nish, $D_n = 0$ kritik holat) reologik parametrlarning o‘zaro nisbatiga bog‘liq holda aniq belgilandi. Masalaning (8) ko‘rinishidagi Furye qatori shaklidagi yechimi qurilib, Veyershtrass va Koshi-Bunyakovskiy tengsizliklari asosida uning tekis yaqinlashishi, shuningdek integral usuli ((17)-(18)) yordamida yechimning yagonaligi isbotlandi va Teorema 1. orqali klassik yechimning mavjudligini ham kafolatladi. Eng muhim natija sifatida, inersion parametr $\left(t_{ret}^{Kh2}\right)^2$ning so‘nish koeffitsiyenti γ_n ifodasining maxrajida qatnashishi aniqlandi (22) bu klassik Kelvin-Voigt modelidan farqli o‘laroq, garmonikalar tartibi $n \to \infty$ da so‘nish koeffitsiyentining cheksiz o‘sishining oldini olib, uni chekli qiymatga $(t_{ret}^{K}/2\left(t_{ret}^{Kh2}\right)^2$ intiltiradi. Demak, inersion elementning modelga kiritilishi yuqori chastotali tebranishlarda jarayonning fizik jihatdan reallikka yaqinroq tasvirlanishini ta'minlaydi.

Olingan analitik natijalar kon-metallurgiya sanoatidagi quyuq pulpa va shlam quvurlari, gidrotizimlar, neft-gaz muhandisligidagi burg‘ilash eritmalari hamda seysmik ta‘sir ostidagi nam g‘ovak grunt qatlamlaridagi nostatsionar tebranish jarayonlarini sonli modellashtirish va muhandislik hisob-kitoblarida amaliy qo‘llanilishi mumkin. Kelgusi tadqiqotlarda ushbu model ikki va uch o‘lchamli sohalarga, shuningdek nochiziqli reologik bog‘lanishlar va o‘zgaruvchan kesim yuzali quvurlar holatiga umumlashtirilishi maqsadga muvofiqdir.

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EXACT SOLUTIONS FOR THE NONISOSPECTRAL LOADED KORTEWEG-DE VRIES EQUATION VIA INVERSE SCATTERING TRANSFORM

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Abstract. *In this paper, soliton-like solutions for the Korteweg–de Vries (KdV) equation with variable coefficients are investigated using the Inverse Scattering Transform (IST) method. By applying appropriate transformations, the original equation is reduced to an integrable form. Subsequently, new general soliton solutions of the resulting equation are derived via the IST method, and explicit analytical solutions are obtained in closed form. The proposed approach can be effectively extended to analyze other classes of integrable nonlinear systems.*

Keywords: *variable-coefficient KdV equation, nonisospectral KdV equation, inverse scattering transform, Jost solutions, Lax pair, Cauchy problem.*

TESKARI SOCHILISH TRANSFORMATSIYASI YORDAMIDA NOIZOSPEKTRAL YUKLANGAN KORTEWEG-DE VRIES TENGLAMASI UCHUN ANIQ YECHIMLAR

Annotatsiya. *Ushbu ishda sochilish nazariyasining teskari masalalari usulidan foydalanib, o‘zgaruvchan koeffitsiyentli Korteveg–de Friz (KdF) tenglamasining solitonsimon yechimlari o‘rganiladi. Munosib almashtirishlar yordamida dastlabki tenglama integrallanuvchi ko‘rinishga keltiriladi. So‘ngra sochilish nazariyasining teskari masalalari usuli asosida ushbu tenglamaning yangi umumiy soliton yechimlari olinadi hamda ularning ochiq analitik ko‘rinishdagi aniq yechimlari hosil qilinadi. Maqolada taqdim etilgan yondashuv boshqa turdagi integrallanuvchi nochiziqli tenglamalarni tadqiq etishda ham keng qo‘llanilishi mumkin.*

Kalit so‘zlar: *o‘zgaruvchan koeffitsiyentli KdF tenglamasi, nonisospektral KdF tenglamasi, sochilish nazariyasining teskari masalalari, Yost yechimlari, Laks juftligi, Koshi masalasi.*

ТОЧНЫЕ РЕШЕНИЯ НЕИЗОСПЕКТРАЛЬНОГО НАГРУЖЕННОГО УРАВНЕНИЯ КОРТЕВЕГА-ДЕ ФРИЗА МЕТОДОМ ОБРАТНОЙ ЗАДАЧИ РАССЕЯНИЯ

Аннотация. *В данной работе методом обратной задачи рассеяния исследуются солитоноподобные решения уравнения Кортевега-де Фриза (КдФ) с переменными коэффициентами. С помощью соответствующих преобразований исходное уравнение сводится к интегрируемому виду. На основе метода обратной задачи рассеяния выводятся новые общие солитонные решения полученного уравнения, и строятся их точные аналитические решения в явном виде. Предложенный подход может быть успешно применён к исследованию других классов интегрируемых нелинейных систем.*

Ключевые слова: *уравнение КдФ с переменными коэффициентами, неизоспектральное уравнение КдФ, метод обратной задачи рассеяния, решения Йоста, пара Лакса, задача Коши.*

Introduction. Nonlinear evolution equations play an important role in the mathematical modeling of a wide range of physical phenomena. Among them, integrable nonlinear evolution equations are of particular

interest because their analytical properties allow exact solutions to be constructed and their dynamical behavior to be studied in detail. The KdV equation is one of the fundamental integrable nonlinear evolution equations and has been extensively studied in connection with the theory of solitons and nonlinear wave propagation.

The well-known IST [1] has been widely used to solve nonlinear evolution equations [2–4]. One of the advantages of the IST is that it can be applied to a broad set of evolution equations associated with a given spectral problem. Evolution equations associated with a given spectral problem can generally be classified into isospectral and non-isospectral types, depending on whether the spectral parameter remains constant or evolves with time. Isospectral equations often describe solitary waves in lossless and homogeneous media. Non-isospectral equations, which arise from spectral problems with a time-dependent spectral parameter η [5–6], can also be solved using the IST, and their solutions demonstrate the existence of solitary waves in certain types of nonuniform media [7–10].

In this paper, using the IST method, we solve the initial-value problem for the following non-isospectral loaded KdV equation

$$u_t - \gamma(t)u(x_0,t)(u_{xxx} - 6uu_x) + 4u_x - (xu_x + 2u) = 0, \tag{1}$$

where $\gamma(t)$ is a given continuous function, and $x_0 \in \mathbb{R}$. Equation (1) is considered under the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{2}$$

where the function $u_0(x)$ satisfies the following criteria:

$$\int_{-\infty}^{\infty} (1+|x|)(|u_0(x)| + |u_{0xx}(x)|)dx < \infty. \tag{3}$$

The solution $u(x,t)$ is assumed to exhibit sufficient smoothness and sufficiently rapidly decreasing as $x \to \pm\infty$, quantified by the condition:

$$\int_{-\infty}^{\infty} (1+|x|)\left(|u(x,t)| + \sum_{k=1}^{3}\left|\frac{\partial^k u(x,t)}{\partial x^k}\right|\right)dx < \infty. \tag{4}$$

The aim of this work is to derive explicit representations for the solution $u(x,t)$ of the problem (1)-(4) by means of the IST method.

Preliminaries

In this section, we provide some basic information on the scattering theory for the Sturm–Liouville problem on the whole real axis [4]. We temporarily omit the dependence of the function $u(x,t)$ on the variable t.

Consider the equation:

$$-\psi_{xx} + u\psi = \lambda\psi, \quad x \in \mathbb{R}, \tag{5}$$

where the potential $u(x)$ satisfies the condition

$$\int_{-\infty}^{\infty} (1+|x|)(|u(x)| + |u''(x)|)dx < \infty. \tag{6}$$

Under the assumption (6), equation (5) possesses the Jost solutions with the following asymptotics:

$$\psi_1(x,k) = e^{ikx} + o(1), \quad x \to +\infty,$$
$$\varphi_1(x,k) = e^{-ikx} + o(1), \quad x \to -\infty,$$

The solution $\psi_1(x,k)$ admits the representation

$$\psi_1(x,k) = e^{ikx} + \int_x^{\infty} K_+(x,z)e^{ikz}\,dz, \tag{7}$$

where the kernel $K_+(x,z)$ is a real-valued function related to the potential $u(x)$ by the relations:

$$u(x) = -2\frac{d}{dx}K_+(x,x).$$

For real k, the pairs $\{\varphi_1(x,k), \varphi_1(x,-k)\}$ and $\{\psi_1(x,k), \psi_1(x,-k)\}$ are pairs of linearly independent solutions for equation (5). Therefore,

$$\varphi_1(x,k) = a(k)\psi_1(x,-k) + b(k)\psi_1(x,k). \tag{8}$$

One can readily see that

$$a(k) = -\frac{1}{2ik} W\{\psi_1(x,k), \varphi_1(x,k)\}.$$

The function $a(k)$ admits an analytic continuation into the upper half-plane and has a finite number of zeroes $k = ik_n$, $k_n > 0$. Meanwhile,

$$\varphi_1(x, ik_n) = b_n \psi_1(x, ik_n), \qquad n = 1, 2, ..., N.$$

The scattering data for the spectral problem (5) are defined by

$$\left\{ R(k) = \frac{b(k)}{a(k)},\ k \in \mathbb{R},\ k_n,\ c_n,\ n = 1, 2, ..., N \right\},$$

where $c_n^2 = -\frac{b_n}{\dot{a}(ik_n)}$, $\dot{a}(ik_n) = \left.\frac{da(k)}{dk}\right|_{k=ik_n}$ and each c_n is so-called normalization constant of eigenfunction $\psi_1(x, ik_n)$.

The kernel $K_+(x,y)$ in representation (7) is a solution to the Gelfand–Levitan–Marchenko integral equation:

$$K_+(x,y) + F_+(x+y) + \int_x^\infty K_+(x,z) F_+(z+y) dz = 0, \quad (y > x),$$

where

$$F_+(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k) e^{ikx} dk + \sum_{j=1}^{N} c_j^2 e^{-k_j x}. \tag{9}$$

The time evolution of scattering data

In this section, we derive the time evolution of the scattering data associated with the non-isospectral loaded KdV equation (1). To this end, consider the following Lax pair:

$$-\psi_{xx} + u\psi = \lambda\psi,$$
$$\psi_t = A\psi + B\psi_x, \tag{10}$$

where $A = -\frac{1}{2} B_x$, $B = x - 2\gamma(t) u(x_0, t)(u + 2\lambda) - 4$, $\lambda_t = 2\lambda$. The compatibility condition $\psi_{xxt} = \psi_{txx}$ of the Lax pair (10) yields the non-isospectral loaded KdV equation (1).

Lemma 1. Suppose that $\psi(x,k,t)$ is a solution of (5), A and B satisfy the compatibility condition $\psi_{xxt} = \psi_{txx}$, i.e.,

$$2A_x + B_{xx} = 0,$$
$$u_t = A_{xx} + 2(u - \lambda) B_x + u_x B + \lambda_t.$$

Then

$$P(x,k,t) = \psi_t(x,k,t) - A\psi(x,k,t) - B\psi_x(x,k,t) \tag{11}$$

is also a solution of the spectral problem (5).

Proof. We show that $P(x,k,t)$ is a solution of the spectral problem (5). Direct calculation of $-P_{xx} + uP - \lambda P$ yields:

$$-P_{xx} + uP - \lambda P = -(\psi_t - A\psi - B\psi_x)_{xx} + (u - \lambda)(\psi_t - A\psi - B\psi_x).$$

By using the relation $\psi_{xx} = (u - \lambda)\psi$, the second-order spatial derivatives are evaluated as follows:

$$(\psi_t)_{xx} = (\psi_{xx})_t = u_t\psi + (u - \lambda)\psi_t - \lambda_t\psi,$$
$$(A\psi)_{xx} = A_{xx}\psi + 2A_x\psi_x + A(u - \lambda)\psi,$$
$$(B\psi_x)_{xx} = B_{xx}\psi_x + 2B_x(u - \lambda)\psi + B(u_x\psi + (u - \lambda)\psi_x).$$

Substituting these derivatives into $-P_{xx} + uP - \lambda P$ and collecting terms with respect to ψ_t, ψ_x and ψ, we obtain

$$-P_{xx} + uP - \lambda P = (-(u - \lambda) + (u - \lambda))\psi_t + (2A_x + B_{xx})\psi_x + (-u_t + \lambda_t + A_{xx} + 2(u - \lambda)B_x + u_x B)\psi.$$

Applying the compatibility conditions $2A_x + B_{xx} = 0$ and $u_t = A_{xx} + 2(u - \lambda)B_x + u_x B + \lambda_t$, all the coefficients vanish identically. Therefore,

$$-P_{xx}+uP=\lambda P.$$

The lemma 1 is proved.

Theorem 1. If the function $u(x,t)$ is a solution to the problem (1)-(4), then the scattering data of the equation (5) with the potential $u(x,t)$ vary with t as follows:

$$\frac{dR}{dt}=-8ik\left(\gamma(t)u(x_0,t)k^2+1\right)R,\quad (\operatorname{Im}k=0), \tag{12}$$

$$\frac{dc_n}{dt}=\left(\frac{1}{2}+4k_n-\gamma(t)u(x_0,t)4k_n^3\right)c_n, \tag{13}$$

$$\frac{dk_n}{dt}=k_n. \tag{14}$$

Proof. It is clear that the systam (5) and (10) has a solution

$$f(x,k,t)=s_1(t)\psi_1(x,-k,t)+s_2(t)\psi_1(x,k,t), \tag{15}$$

where $s_1^2(t)+s_2^2(t)\neq 0$. Substituting (15) into (10), and as $x\to\infty$,

$$\frac{ds_1(t)}{dt}e^{-ikx}-ixs_1(t)\frac{dk}{dt}e^{-ikx}+\frac{ds_1(t)}{dt}e^{ikx}+ixs_2(t)\frac{dk}{dt}e^{ikx}=-\frac{1}{2}\left(s_1(t)e^{-ikx}+s_2(t)e^{ikx}\right)-$$
$$-ixk\,s_1(t)\,e^{-ikx}+ixk\,s_2(t)e^{ikx}+\left(-4\gamma(t)u(x_0,t)k^2-4\right)\left(ik\,s_1(t)e^{-ikx}+ik\,s_2(t)e^{ikx}\right), \tag{16}$$

Comparing the coefficients of the terms xe^{-ikx} on both sides of (16), we have

$$\frac{dk(t)}{dt}=k(t). \tag{17}$$

Consequently, using (17), Eq. (16) becomes

$$\frac{ds_1(t)}{dt}e^{-ikx}+\frac{ds_2(t)}{dt}e^{ikx}=\left(-\frac{1}{2}+4ik^3\gamma(t)u(x_0,t)+4ik\right)s_1(t)e^{-ikx}+$$
$$+\left(-\frac{1}{2}-4ik^3\gamma(t)u(x_0,t)-4ik\right)s_2(t)e^{ikx}. \tag{18}$$

Equating the coefficients of $\exp(-ikx)$ and $\exp(ikx)$ on both sides of (18), we obtain

$$\frac{ds_1(t)}{dt}=s_1(t)\left(-\frac{1}{2}+4ik^3\gamma(t)u(x_0,t)+4ik\right),$$
$$\frac{ds_2(t)}{dt}=s_2(t)\left(-\frac{1}{2}-4ik^3\gamma(t)u(x_0,t)-4ik\right).$$

Solving for $s_1(t)$ and $s_2(t)$, we deduce

$$s_1(t)=s_1(0)\exp\left\{-\frac{t}{2}\right\}\exp\left\{\int_0^t\left(4ik^3\gamma(\tau)u(x_0,\tau)+4ik\right)d\tau\right\}, \tag{19}$$

$$s_2(t)=s_2(0)\exp\left\{-\frac{t}{2}\right\}\exp\left\{-\int_0^t\left(4ik^3\gamma(\tau)u(x_0,\tau)+4ik\right)d\tau\right\}. \tag{20}$$

In particular, taking

$$f(x,k,t)=\varphi_1(x,k,t)\exp\left\{-\frac{t}{2}\right\}\exp\left\{\int_0^t\left(4ik^3\gamma(\tau)u(x_0,\tau)+4ik\right)d\tau\right\}.$$

From (8), we have

$$f(x,k,t)=a(k,t)\psi_1(x,-k,t)W_0(t)+b(k,t)\psi_1(x,k,t)W_0(t),$$

where

$$W_0(t)=\exp\left\{-\frac{t}{2}\right\}W(t),\ W(t)=\exp\left\{\int_0^t\left(4ik^3\gamma(\tau)u(x_0,\tau)+4ik\right)d\tau\right\}.$$

In this case, $s_1(t)$ and $s_2(t)$ given by (19) and (20) become $s_1(t)=a(k,t)W_0(t)$, $s_2(t)=b(k,t)W_0(t)$ and $s_1(0)=a(k,0)$, $s_2(0)=b(k,0)$. Hence we obtain

$$a(k,t)=a(k,0),$$
$$b(k,t)=b(k,0)W^{-2}(t).$$

Now, using the definition of the function $R(k,t)$, we get (12).

Further, we calculate the dependence of $c_n(t)$ and $k_n(t)$, $n=1,2,...,N$, on time t.

Consider the following three solutions of (5) with $k=ik_n$, $(k_n>0)$,

$$P(x,ik_n,t)=\psi_t(x,ik_n,t)-A\psi(x,ik_n,t)-B\psi_x(x,ik_n,t),$$

$\psi(x,ik_n,t)$ and $\bar{\psi}(x,ik_n,t)$, where $\psi(x,ik_n,t)=c_n(t)\psi_1(x,ik_n,t)$ is a normalized Jost function, $c_n=c_n(t)$ is the related normalization constant, and $\bar{\psi}(x,ik_n,t)$ is another Jost function linearly independent of $\psi(x,ik_n,t)$.

There exist two constants α, β such that

$$\psi_t(x,ik_n,t)-A\psi(x,ik_n,t)-B\psi_x(x,ik_n,t)=\alpha\psi(x,ik_n,t)+\beta\bar{\psi}(x,ik_n,t),$$

Noting the fact of $\psi(x,ik_n,t)\to 0$ as $x\to+\infty$, it is easy to see $\beta=0$, i.e.,

$$\psi_t(x,ik_n,t)-A\psi(x,ik_n,t)-B\psi_x(x,ik_n,t)=\alpha\psi(x,ik_n,t), \tag{21}$$

To derive the value of α, multiplying (21) by $2\psi(x,ik_n,t)$, and noting that $A=-\frac{1}{2}B_x$, yields

$$2\psi(x,ik_n)\psi_t(x,ik_n,t)+B_x\psi^2(x,ik_n,t)-2B\psi(x,ik_n,t)\psi_x(x,ik_n,t)=2\alpha\psi^2(x,ik_n,t).$$

Further we have

$$\frac{d}{dt}\int_{-\infty}^{\infty}\psi^2(x,ik_n)dx+\int_{-\infty}^{\infty}\left[B_x\psi^2(x,ik_n,t)-B\left(\psi^2(x,ik_n,t)\right)_x\right]dx=2\alpha\int_{-\infty}^{\infty}\psi^2(x,ik_n,t)dx.$$

Since $\int_{-\infty}^{\infty}\psi^2(x,ik_n,t)dx=1$ and $B\psi^2(x,ik_n,t)\to 0$ as $x\to\pm\infty$, we obtain

$$\alpha=-\int_{-\infty}^{\infty}B(\psi^2(x,ik_n,t))_x\,dx=1.$$

Thus (21) becomes

$$\psi_t(x,ik_n,t)-A\psi(x,ik_n,t)-B\psi_x(x,ik_n,t)=\psi(x,ik_n,t). \tag{22}$$

Setting $x\to+\infty$ in (22) immediately yields

$$\frac{dc_n}{dt}-x\frac{dk_n}{dt}c_n+\frac{1}{2}c_n+c_n k_n\left(x+\gamma(t)u(x_0,t)4k_n^2-4\right)=c_n,$$

which further means

$$\frac{dk_n}{dt}=k_n,\qquad \frac{dc_n}{dt}=c_n\left(\frac{1}{2}+4k_n-\gamma(t)u(x_0,t)4k_n^3\right).$$

The theorem is proved.

An Illustrative example

Let us consider the following Cauchy problem:

$$u_t-\gamma(t)u(0,t)(u_{xxx}-6uu_x)+4u_x-(xu_x+2u)=0, \tag{23}$$

$$u_0(x)=-\frac{1}{2}\operatorname{sech}^2\frac{x}{2}. \tag{24}$$

Here

$$\gamma(t)=-\frac{16e^{-5t}}{\operatorname{sech}^2\left(4t-2\left(e^t-1\right)\right)}.$$

Solving the direct problem for the equation

$$-\psi_{xx}+u_0\psi=\lambda\psi\ ,$$

we find the scattering data for $t=0$:

$$c_1(0)=1,\ k_1(0)=\frac{1}{2},\ R(k,0)=0.$$

By theorem 1, we have

$$R(k,t)=0,\ k_1(t)=\frac{e^t}{2},\ c_1(t)=\exp\left\{\frac{t}{2}+2\left(e^t-1\right)-\frac{1}{2}\mu(t)\right\},$$

where

$$\mu(t)=\int_0^t e^{3\tau}\gamma(\tau)u(0,\tau)d\tau.$$

Substituting these data into Eq. (9), we obtain the following:

$$F_+(x,t)=\exp\left\{-\frac{e^t}{2}x+t+4(e^t-1)-\mu(t)\right\}.$$

Now, solving the Gelfand–Levitan–Marchenko integral equation, we obtain:

$$K_+(x,x,t)=-\frac{e^t}{2}\left[1-\tanh\left(\frac{e^t x-4(e^t-1)+\mu(t)}{2}\right)\right].$$

According to the formula $u(x,t)=-2\frac{d}{dx}K_+(x,x,t)$, it follows that the solution to the Cauchy problem (23)–(24) takes the following form:

$$u(x,t)=-\frac{1}{2}e^{2t}\operatorname{sech}^2\left(\frac{xe^t}{2}-2\left(e^t-1\right)+4t\right).$$

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A MIXED LOCAL-NONLOCAL PROBLEM FOR A DEGENERATE HIGH EVEN-ORDER EQUATION

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Abstract. *Within a rectangular region we formulate and study an initial-boundary value problem, furnished with a combination of local and paired (linked) boundary conditions, for a high even-order partial differential equation whose type is lost at once in the time variable - at each end of the time interval - and in the spatial variable, and which additionally carries a subordinate term. Resting on an energy-integral identity attuned to the temporal degeneracy and rounded out by a Gronwall-type bound, we settle the uniqueness of the solution. Merging spectral analysis with the Green's-function apparatus, we express the solution as a Fourier series in the eigenfunctions of the attendant spectral problem, whose Green's function is written out explicitly and shown to be symmetric. Weighted ℓ_2-type bouns for the Fourier coefficients are supplied, and an estimate of the solution is deduced, from which its continuous dependence on the prescribed data follows.*

Keywords: *initial-boundary value problem, degenerate equation, time degeneracy, local and paired boundary conditions, spectral problem, Green's function method, integral equation, Fourier series.*

BUZILADIGAN YUQORI JUFT TARTIBLI TENGLAMA UCHUN ARALASH CHEGARAVIY SHARTLI MASALA

Annotatsiya. *To'rtburchak sohada yuqori juft tartibli xususiy hosilali tenglama uchun lokal va nolakal chegaraviy shartlar kombinatsiyasi bilan berilgan boshlang'ich-chegaraviy masala qo'yilgan va o'rganilgan. Tenglamaning tipi bir vaqtning o'zida ham vaqt o'zgaruvchisi bo'yicha-vaqt oralig'ining har ikkala uchida-ham fazoviy o'zgaruvchi bo'yicha buziladi, shuningdek, tenglama qo'shimcha kichik yig'indini o'z ichiga oladi. Vaqt bo'yicha buzilishga moslashtirilgan va Gronuoll tipli baholash bilan to'ldirilgan energiya integrali ayniyatiga tayanib, yechimning yagonaligi isbotlangan. Spektral tahlil hamda Grin funksiyasi bilan birgalikda qo'llanilib, yechim tegishli spektral masalaning xos funksiyalari bo'yicha Furye qatori ko'rinishida tasvirlangan. Bunda spektral masalaning Grin funksiyasi oshkor ko'rinishda yozilgan va uning simmetrikligi ko'rsatilgan. Furye koeffitsiyentlari uchun vaznli ℓ_2 tipli baholar keltirilgan va yechimning bahosi olingan bo'lib, undan yechimning berilgan ma'lumotlarga uzluksiz bog'liqligi kelib chiqadi.*

Kalit so'zlar: *boshlang'ich-chegaraviy masala, buziladigan tenglama, vaqt bo'yicha buzilish, lokal va juftlangan chegaraviy shartlar, spektral masala, Grin funksiyasi usuli, integral tenglama, Furye qatori.*

СМЕШАННАЯ ЗАДАЧА ДЛЯ ВЫРОЖДАЮЩЕГОСЯ УРАВНЕНИЯ ВЫСОКОГО ЧЁТНОГО ПОРЯДКА

Аннотация. *В прямоугольной области сформулирована и исследована начально-краевая задача с комбинацией локальных и парных (связанных) краевых условий для уравнения в частных производных высокого чётного порядка. Тип данного уравнения теряется одновременно по временной переменной на каждом из концов временного интервала и по пространственной переменной, причём уравнение дополнительно содержит младший член. На основе тождества энергетического интеграла, согласованного с вырождением по времени и дополненного оценкой типа Гронуолла, установлена единственность решения. Сочетая спектральный анализ с аппаратом функций Грина, решение представлено в виде ряда Фурье по собственным функциям соответствующей спектральной задачи, функция Грина которой выписана в явном виде и показана её симметричность. Приведены весовые оценки типа ℓ_2 для коэффициентов Фурье, и получена оценка решения, из которой следует его непрерывная зависимость от заданных данных.*

Ключевые слова: *начально-краевая задача, вырождающееся уравнение, вырождение по времени, локальные и парные краевые условия, спектральная задача, метод функций Грина, интегральное уравнение, ряд Фурье.*

Introduction. The theory of partial differential equations finds broad application across numerous branches of contemporary science and technology, among them hydrodynamics, the theory of elasticity, and quantum mechanics. At present, this theory is advancing rapidly along a variety of lines. Among these, boundary value problems formulated for even-order partial differential equations of higher order hold a particularly prominent position. A survey of the studies carried out in this direction indicates that initial-boundary value problems for equations involving derivatives of integer order have received considerable attention. For instance, in [1] the author examined the problem of determining a solution of the equation

$$\frac{\partial^{2k}u}{\partial x^{2k}}-\frac{\partial^2 u}{\partial t^2}=f(x,t) \tag{1}$$

in the domain $\Delta=\{(x,t):0<x<p,\,0<t<T\}$. Here the sought solution is required to obey the boundary conditions

$$\frac{\partial^{2m}u}{\partial x^{2m}}(0,t)=\frac{\partial^{2m}u}{\partial x^{2m}}(p,t)=0,\quad m=\overline{0,k-1},\ 0\le t\le T,\ k\ge 2, \tag{2}$$

together with one of the following pairs of initial conditions:

$$\begin{cases} u(x,0)=u(x,T)=0, & 0\le x\le p,\\ u(x,0)=u_t(x,T)=0, & 0\le x\le p,\\ u(x,0)=u(x,T),\ u_t(x,0)=u_t(x,T), & 0\le x\le p. \end{cases} \tag{3}$$

The problem stated above was therein subjected to a thorough investigation. Moreover, in [1] a further problem was posed for equation (1) under the prescribed data and the boundary conditions

$$\frac{\partial^{2m+1}u}{\partial x^{2m+1}}(0,t)=\frac{\partial^{2m+1}u}{\partial x^{2m+1}}(p,t)=0,\quad m=\overline{0,k-1},\ 0<t<T,\ k\ge 2.$$

In [2], the problem for the equation in the domain Δ, endowed with particular boundary conditions and the initial data $u(x,0)=u_t(x,0)=0,\ 0\le x\le p$, was analyzed. In [3], an ill-posed problem for the equation in the domain $D=\{(x,t):0<x<\pi;\,0<t<2\pi\}$ with $f(x,t)\equiv 0$ and $k=2n+1,\ n\in\mathbb{N}$ was treated under the boundary and initial conditions

$$\frac{\partial^{2m}u}{\partial x^{2m}}(0,t)=\frac{\partial^{2m}u}{\partial x^{2m}}(\pi,t)=0,\ m=0,1,\ldots,k-1;\ u(\alpha\pi,t)=f(t),\quad 0<t<2\pi.$$

In [4], a problem was considered for the quasilinear equation

$$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^{2k}u}{\partial x^{2k}}=f(t,x,u),\quad k\in\mathbb{N},$$

in the domain $D_T=\{(t,x):0<t<T;\,0<x<\pi\}$, supplemented by the initial conditions $u(0,x)=\varphi(x)$, $u_t(T,x)=\psi(x)$, $0<x<\pi$, and by boundary conditions of periodic type in x. The Fourier method was employed to solve the problem, while the existence and uniqueness of the solution were established by means of the method of successive approximations in a Banach space. In [5], problems for the equation

$$\frac{\partial^{2k}u}{\partial x^{2k}}+\frac{\partial^2 u}{\partial t^2}=f(x,t)$$

in the domain Δ, subject to specific conditions and $u(x,0)=u_t(x,0)=0$, were investigated, it was demonstrated therein that the solvability of the problem is governed by the parity of the number k. Finally, in [6], initial-boundary value problems were studied for the equation $u_{tt}+A(x,D)u(x,t)=f(x,t)$, where $A(x,D)=\sum_{|\alpha|\le m}a_\alpha(x)D^\alpha$ is an arbitrary positive, formally self-adjoint elliptic differential operator of order $m=2l$ with sufficiently smooth coefficients, the problem is resolved in the form of a series by invoking the classical Fourier method together with the associated spectral problem, and the existence, uniqueness, and stability of the solution were proved.

The works cited so far concern equations whose leading spatial part is non-degenerate. A separate and more delicate line of research addresses equations in which the higher-order spatial operator degenerates

through a weight that vanishes on the boundary. Thus, in [7] a local initial-boundary value problem was analysed for an equation combining the Bessel operator with the operator $(-1)^k \partial^k / \partial x^k [x^\alpha \partial^k u / \partial x^k]$.

In [8], for a high even-order equation of order $2k$ carrying a Bessel operator in a rectangular domain,

$$B_t^{\gamma-1/2} u + (-1)^k \frac{\partial^{2k} u}{\partial x^{2k}} = f(x,t), \qquad B_t^{\gamma-1/2} \equiv \frac{\partial^2}{\partial t^2} + \frac{2\gamma}{t}\frac{\partial}{\partial t},$$

a pair of nonlocal initial-boundary value problems was set with requirements linking the values and the higher-order derivatives at the two endpoints $x=0$ and $x=1$, for one of them the existence, the uniqueness, and the continuous dependence of the solution upon the data were proved through the Fourier method, the Green's function, and spectral analysis.

A degenerate high even-order operator of the form $\partial^{2n} / \partial x^{2n} \left(x^\alpha \partial^{2n} u / \partial x^{2n} \right)$ underlies a further group of studies. For the equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^{2n}}{\partial x^{2n}} \left(x^\alpha \frac{\partial^{2n} u}{\partial x^{2n}} \right) = f(x,t), \tag{4}$$

the study [9] posed a problem under local boundary requirements imposing, at each endpoint $x=0$ and $x=1$, the vanishing of the even-order derivatives of u and of $x^\alpha \partial^{2n} u / \partial x^{2n}$. For the same equation (4), the work [11] set and investigated a problem under local conditions prescribed separately at the two endpoints (values of u at $x=0$ and of its $(n+j)$-th derivatives at $x=1$), while [10] treated, for the same family of degenerate equations, an initial-boundary value problem carrying yet another set of local conditions and justified the solvability of the linear system governing the Green's-function coefficients. The study [12], again for equation (4), examined a problem in which the boundary conditions link, at each endpoint, the even-order derivatives of u with its odd-order ones (and likewise for $x^\alpha \partial^{2n} u / \partial x^{2n}$), for $j = \overline{0, n-1}$ and $t \in [0,T]$.

A further, doubly degenerate operator,

$$(-1)^n \frac{\partial^n}{\partial x^n} \left[x^\alpha (1-x)^\beta \frac{\partial^n u(x,t)}{\partial x^n} \right],$$

whose degeneracy at both endpoints $x=0$ and $x=1$ is produced by the weight $x^\alpha (1-x)^\beta$, underlies the initial-boundary value problems treated in [13, 14]. The equation examined in [13] places a Caputo derivative alongside a Riemann—Liouville integral,

$${}^C D_{0t}^{\alpha_1} u + a_1 I_{0t}^{\beta_1} u + b_1 u + (-1)^n \frac{\partial^n}{\partial x^n} \left[x^\alpha (1-x)^\beta \frac{\partial^n u}{\partial x^n} \right] = 0,$$

whereas the one considered in [14] attaches a further factor t^γ to the Caputo derivative,

$$t^\gamma {}^C D_{0t}^{\delta} u + b_1 u + (-1)^n \frac{\partial^n}{\partial x^n} \left[x^\alpha (1-x)^\beta \frac{\partial^n u}{\partial x^n} \right] = 0, \quad 0 \le \gamma < \delta < 1,$$

whereby the degeneracy extends beyond x to the line $t=0$ as well. Both studies impose boundary and initial data of one and the same shape,

$$\left. \frac{\partial^j u}{\partial x^j} \right|_{x=0} = 0, \quad \left. \frac{\partial^j}{\partial x^j} \left[x^\alpha (1-x)^\beta \frac{\partial^n u}{\partial x^n} \right] \right|_{x=1} = 0, \quad j = 0,1,\ldots,n-1, \qquad u(x,0) = \varphi(x),$$

and in each of them the solution is shown to be unique through energy identities and is built by the Fourier method, the accompanying spectral problem being reduced, via its Green's function, to a Fredholm integral equation with a symmetric kernel. Consequently, the solution of [13] is written out by means of the two-variable Mittag-Leffler-type function $E_2(\cdot,\cdot)$, and that of [14] through the three-parameter Kilbas-Saigo function. In either setting, the uniform convergence of the corresponding Fourier series and the continuous dependence of the solution upon the prescribed data are secured.

A boundary value problem on a rectangle for an equation in which a third-order derivative is combined with a degenerate high even-order operator-the accompanying weight vanishing on the boundary of the domain-was treated in [15]. In [16] a Dirichlet-type problem was addressed for a degenerate high-order equation of fractional order, there Mittag-Leffler functions were brought into use, and both the convergence of the solution series and the estimates for the Green's function were secured by asymptotic

expansions. Furthermore, an equation of the form $t^m u_{tt} + at^{m-1}u_t + bu + [x^\alpha (1-x)^\beta u_{xx}]_{xx} = 0$, which possesses three lines of degeneracy, was examined within an initial-boundary value problem in [17], while in [18] the fractional-order equation $t^b\, {}^C D_{0t}^\gamma u - (x^\alpha (1-x)^\beta u_x)_x = f$ was investigated, a direct as well as an inverse problem being considered under suitable restrictions on the parameters.

To the authors' knowledge, however, no previous work has treated an equation that simultaneously brings together a first-order derivative in time, a two-sided temporal degeneracy expressed by the weight $t^a(1-t)^b$, a lower-order term cu, and a spatial degeneracy of high even order. In contrast to [17], where a *second*-order time derivative together with three lines of degeneracy is treated, and to [14], where the time derivative is of *fractional* order, the equation considered here retains an ordinary first-order time derivative alongside the two-sided weight $t^a(1-t)^b$. The novelty of the present study lies precisely in setting up and analyzing such a problem, supplied with boundary conditions of a mixed type that combine local requirements with paired (linked) ones relating the values at $x=0$ to those at $x=1$.

In the rectangle $\Omega = \{(x,t): 0<x<1;\ 0<t<1\}$ we consider the equation

$$t^a(1-t)^b \frac{\partial u}{\partial t} + cu + \frac{\partial^{2n}}{\partial x^{2n}}\left(x^\alpha \frac{\partial^{2n} u(x,t)}{\partial x^{2n}} \right) = f(x,t), \tag{5}$$

where a, b, c, α are prescribed real numbers, $n \in \mathbb{N}$, $0 \le a < 1$, $0 \le b < 1$, $c \ge 0$, $0 < \alpha < 1$, $u = u(x,t)$ is the unknown function, and $f(x,t)$ is a given sufficiently smooth function.

Statement of the problem. *Find a function* $u(x,t)$ *that meets the following requirements:*

1) the following continuity requirements hold: $u(x,t), \partial u/\partial x, \partial^2 u/\partial x^2, \ldots, \partial^{2n-1}u/\partial x^{2n-1} \in C(\bar{\Omega})$, $x^\alpha \partial^{2n}u/\partial x^{2n}, (\partial/\partial x)(x^\alpha \partial^{2n}u/\partial x^{2n}), \ldots, (\partial^{2n-1}/\partial x^{2n-1})(x^\alpha \partial^{2n}u/\partial x^{2n}) \in C(\bar{\Omega})$, $t^a(1-t)^b \partial u/\partial t \in C(\Omega)$ and $(\partial^{2n}/\partial x^{2n})(x^\alpha \partial^{2n}u/\partial x^{2n}) \in C(\Omega)$;

2) the function $u(x,t)$ satisfies equation (5) in Ω;

3) it fulfils the initial condition

$$u(x,0) = \varphi(x), \qquad x \in [0,1], \tag{6}$$

where $\varphi(x)$ is a prescribed continuous function, together with the boundary conditions

$$\begin{cases} \dfrac{\partial^j u(0,t)}{\partial x^j} = 0,\ \dfrac{\partial^{n+j} u(0,t)}{\partial x^{n+j}} + \dfrac{\partial^{n+j} u(1,t)}{\partial x^{n+j}} = 0,\ j = \overline{0,n-1},\ t \in [0,1]; \\ \left(x^\alpha \dfrac{\partial^{2n} u}{\partial x^{2n}} \right)^{(j)}\Bigg|_{x=0} + \left(x^\alpha \dfrac{\partial^{2n} u}{\partial x^{2n}} \right)^{(j)}\Bigg|_{x=1} = 0,\ \left(x^\alpha \dfrac{\partial^{2n} u}{\partial x^{2n}} \right)^{(n+j)}\Bigg|_{x=1} = 0,\ j = \overline{0,n-1}, t \in [0,1]. \end{cases} \tag{7}$$

Investigation of the spectral problem

Carrying out the separation of variables in the problem under consideration, one is brought to the spectral problem

$$M[v] \equiv \left(x^\alpha v^{(2n)}(x) \right)^{(2n)} = \lambda v(x), \qquad 0 < x < 1, \tag{8}$$

$$\begin{cases} v^{(j)}(x),\ \left(x^\alpha v^{(2n)}(x) \right)^{(j)} \in C[0,1],\ j = \overline{0,2n-1}; \\ v^{(j)}(0) = 0,\ v^{(n+j)}(0) + v^{(n+j)}(1) = 0,\ j = \overline{0,n-1}; \\ \left(x^\alpha v^{(2n)}(x) \right)^{(j)}\Big|_{x=0} + \left(x^\alpha v^{(2n)}(x) \right)^{(j)}\Big|_{x=1} = 0,\ \left(x^\alpha v^{(2n)}(x) \right)^{(n+j)}\Big|_{x=1} = 0,\ j = \overline{0,n-1}. \end{cases} \tag{9}$$

Throughout, $M[v]$ denotes the differential expression $\left(x^\alpha (v^{(2n)}(x) \right)^{(2n)}$ applied to any sufficiently smooth function, not only to eigenfunctions, in particular Mg in the lemmas below is this same expression applied to g.

Self-adjointness of the operator. Let $v(x)$ and $w(x)$ be functions satisfying conditions (9), and $Mv(x), Mw(x) \in L_2(0,1)$. Then, applying integration by parts $4n$ times, we have

$$\int_0^1 w(x) Mv(x)\,dx = \Big[w(x) \left(x^\alpha v^{(2n)}(x) \right)^{(2n-1)} - w'(x) \left(x^\alpha v^{(2n)}(x) \right)^{(2n-2)} + w''(x) \left(x^\alpha v^{(2n)}(x) \right)^{(2n-3)}$$

$$-\ldots-w^{(2n-1)}(x)\left(x^{\alpha}v^{(2n)}(x)\right)+\left(x^{\alpha}w^{(2n)}(x)\right)v^{(2n-1)}(x)-\left(x^{\alpha}w^{(2n)}(x)\right)'v^{(2n-2)}(x)$$

$$+\left(x^{\alpha}w^{(2n)}(x)\right)''v^{(2n-3)}(x)-\ldots-\left(x^{\alpha}w^{(2n)}(x)\right)^{(2n-1)}v(x)\Big]_{x=0}^{x=1}+\int_0^1 v(x)Mw(x)dx.$$

Hence, by virtue of the properties of the functions $v(x)$ and $w(x)$, the following equality holds:

$$\int_0^1 w(x)Mv(x)dx=\int_0^1 v(x)Mw(x)dx.$$

Consequently, the problem $\{Mv=0,\ (9)\}$ is *self-adjoint*.

Let $v(x)\not\equiv 0$, $x\in[0,1]$ be a solution to the problem {(8), (9)}. Then, multiplying both sides of equality (8) by $v(x)$ and integrating from 0 to 1, we have

$$\lambda\int_0^1 v^2(x)dx=\int_0^1 v(x)\left(x^{\alpha}v^{(2n)}(x)\right)^{(2n)}dx=\Big[v(x)\left(x^{\alpha}v^{(2n)}(x)\right)^{(2n-1)}-v'(x)\left(x^{\alpha}v^{(2n)}(x)\right)^{(2n-2)}+\ldots$$

$$\ldots-v^{(2n-1)}(x)\left(x^{\alpha}v^{(2n)}(x)\right)\Big]\Big|_{x=0}^{x=1}+\int_0^1 x^{\alpha}\left[v^{(2n)}(x)\right]^2dx=\int_0^1 x^{\alpha}\left[v^{(2n)}(x)\right]^2dx,$$

i.e.,

$$\lambda\int_0^1 v^2(x)dx=\int_0^1 x^{\alpha}\left[v^{(2n)}(x)\right]^2dx. \tag{10}$$

By condition (9), $x^{\alpha}v^{(2n)}(x)\in C[0,1]$, so $\left|x^{\alpha}v^{(2n)}(x)\right|\le C_0$ on $[0,1]$ for some constant C_0, and therefore

$$\int_0^1 x^{\alpha}\left[v^{(2n)}(x)\right]^2dx=\int_0^1 x^{-\alpha}\left[x^{\alpha}v^{(2n)}(x)\right]^2dx,$$

since $0<\alpha<1$. Thus the right-hand integral in (10) remains finite despite the degeneracy at $x=0$, and consequently the eigenvalue λ must satisfy $\lambda\ge 0$. If $\lambda=0$, then from the last equality it follows that $v^{(2n)}(x)=0$, $0<x<1$. Then

$$v(x)=c_1\frac{x^{2n-1}}{(2n-1)!}+c_2\frac{x^{2n-2}}{(2n-2)!}+\ldots+c_{2n-1}\frac{x}{1!}+c_{2n},\quad x\in(0,1),$$

where c_j are arbitrary real numbers. Subjecting this function to the conditions $v^{(j)}(0)=0,\quad v^{(n+j)}(0)+v^{(n+j)}(1)=0,\quad j=\overline{0,n-1}$, we obtain $c_j=0,\ j=\overline{1,2n}$. Then $v(x)\equiv 0$, $0\le x\le 1$. Consequently, problem {(8), (9)} can have nontrivial solutions only when $\lambda>0$.

Construction of the Green's function.

Let $\lambda>0$. We demonstrate that {(8), (9)} possesses eigenvalues by resorting to the method of the Green's function. The Green's function $G(x,s)$ attached to the problem is required to enjoy the following properties:

1) the functions $(\partial^j/\partial x^j)G(x,s)$, $j=\overline{0,2n-1}$, and $(\partial^j/\partial x^j)[x^{\alpha}(\partial^{2n}/\partial x^{2n})G(x,s)]$, $j=\overline{0,2n-2}$, are continuous for all $x,s\in[0,1]$;

2) on each of $[0,s)$ and $(s,1]$ the derivative $(\partial^{2n-1}/\partial x^{2n-1})[x^{\alpha}(\partial^{2n}/\partial x^{2n})G(x,s)]$ is continuous, while at $x=s$ it has the jump

$$\left(\partial^{2n-1}/\partial x^{2n-1}\right)\left[x^{\alpha}\left(\partial^{2n}/\partial x^{2n}\right)G(x,s)\right]\Big|_{x=s-0}^{x=s+0}=1; \tag{11}$$

3) on $(0,s)$ and $(s,1)$ the derivative $MG(x,s)$ exists (in x) and $MG(x,s)=0$;

4) for $s\in(0,1)$, as a function of x, it satisfies

$$\frac{\partial^j G(0,s)}{\partial x^j}=0,\quad \frac{\partial^{n+j}G(0,s)}{\partial x^{n+j}}+\frac{\partial^{n+j}G(1,s)}{\partial x^{n+j}}=0\quad j=\overline{0,n-1}; \tag{12}$$

$$\left(x^{\alpha}\frac{\partial^{2n}G(x,s)}{\partial x^{2n}}\right)^{(j)}\Big|_{x=0}+\left(x^{\alpha}\frac{\partial^{2n}G(x,s)}{\partial x^{2n}}\right)^{(j)}\Big|_{x=1}=0,\quad \left(x^{\alpha}\frac{\partial^{2n}G(x,s)}{\partial x^{2n}}\right)^{(n+j)}\Big|_{x=1}=0,\quad j=\overline{0,n-1}. \tag{13}$$

Guided by the structure of the general solution of $MG(x,s)=0$ on the two subintervals $(0,s)$ and $(s,1)$, we look for $G(x,s)$ in the shape

$$G(x,s)=\begin{cases}\displaystyle\sum_{j=1}^{2n}\frac{a_j x^{4n-\alpha-j}}{(2n-j)!(2n-\alpha-j+1)_{2n}}+\sum_{j=1}^{2n}\frac{a_{2n+j}x^{2n-j}}{(2n-j)!}, & 0\le x\le s,\\ \displaystyle\sum_{j=1}^{2n}\frac{b_j x^{4n-\alpha-j}}{(2n-j)!(2n-\alpha-j+1)_{2n}}+\sum_{j=1}^{2n}\frac{b_{2n+j}x^{2n-j}}{(2n-j)!}, & s\le x\le 1,\end{cases} \tag{14}$$

where a_j and b_j, $j=\overline{1,4n}$, are unknown functions of s, and $(z)_n=z(z+1)\cdots(z+n-1)$ is the Pochhammer symbol [19]. Requiring that properties 1) and 2) hold and that the jump relation (11) be satisfied, we are led to the following system governing the differences b_j-a_j:

$$\begin{cases} b_1-a_1=1,\\ \displaystyle\sum_{j=1}^{k}\frac{(b_j-a_j)s^{k-j}}{(k-j)!}=0, \qquad k=\overline{2,2n},\\ \displaystyle\sum_{j=1}^{2n}\frac{(b_j-a_j)s^{2n+m-j-\alpha}}{(2n-j)!(2n-j+1-\alpha)_m}+\sum_{j=1}^{m}\frac{(b_{2n+j}-a_{2n+j})s^{m-j}}{(m-j)!}=0, \quad m=\overline{1,2n}\end{cases} \tag{15}$$

This system admits the unique solution [9]:

$$b_j-a_j=\frac{(-1)^{j-1}s^{j-1}}{(j-1)!}, \quad b_{2n+j}-a_{2n+j}=\frac{(-1)^{j-1}s^{2n+j-1-\alpha}}{(j-1)!(j-\alpha)_{2n}}, \quad j=\overline{1,2n}. \tag{16}$$

Substituting the second part of condition (13) into (14), we obtain

$$b_j=0,\ j=\overline{1,n}, \tag{17}$$

and, taking (17) into account, it follows from (16) that

$$a_j=\frac{(-1)^j s^{j-1}}{(j-1)!}, \qquad j=\overline{1,n}. \tag{18}$$

Substituting the first part of (13) into (14), we correspondingly arrive at the equality

$$a_j+b_j=-\sum_{i=1}^{j-1}\frac{b_i}{(j-i)!}, \qquad j=\overline{n+1,2n}. \tag{19}$$

Now, combining b_j-a_j, b_j+a_j $(j=\overline{n+1,2n})$ from (16) together with (19) into a single system, we obtain, with respect to a_j and b_j, the solutions

$$a_j=-\frac{1}{2}\left(\sum_{i=1}^{j-1}\frac{b_i}{(j-i)!}+\frac{(-1)^{j-1}s^{j-1}}{(j-1)!}\right), \quad b_j=-\frac{1}{2}\left(\sum_{i=1}^{j-1}\frac{b_i}{(j-i)!}-\frac{(-1)^{j-1}s^{j-1}}{(j-1)!}\right), \quad j=\overline{n+1,2n} \tag{20}$$

Subjecting the first part of equality (12) to (14), we obtain

$$a_{4n-j}=0, \qquad j=\overline{0,n-1}; \tag{21}$$

taking (21) and (16) into account, the following equality is obtained:

$$b_{4n-j}=\frac{(-1)^{2n-j-1}s^{4n-j-1-\alpha}}{(2n-j-1)!(2n-j-\alpha)_{2n}}, \qquad j=\overline{0,n-1}. \tag{22}$$

Subjecting the second part of equality (12) to (14) accordingly, we get

$$a_{2n+j}+b_{2n+j}=-\left(\sum_{i=1}^{2n}\frac{b_i}{(2n-i)!(2n-\alpha-i+1)_j}+\sum_{k=1}^{j-1}\frac{b_{2n+k}}{(j-k)!}\right), \qquad j=\overline{1,n}, \tag{23}$$

where, for $j=1$ in (23),

$$a_{2n+1}+b_{2n+1}=-\sum_{i=1}^{2n}\frac{b_i}{(2n-i)!(2n-\alpha-i+1)}.$$

Reducing the corresponding parts of (23) and (16) to a single system, we obtain, for a_{2n+j} and b_{2n+j}, the following equalities:

$$a_{2n+j}=\frac{1}{2}\left(a_{2n+j}+b_{2n+j}-\frac{(-1)^{j-1}s^{2n+j-1-\alpha}}{(j-1)!(j-\alpha)_{2n}}\right),\quad b_{2n+j}=\frac{1}{2}\left(a_{2n+j}+b_{2n+j}+\frac{(-1)^{j-1}s^{2n+j-1-\alpha}}{(j-1)!(j-\alpha)_{2n}}\right). \tag{24}$$

Thus, the Green's function satisfying properties 1)-4) exists and is unique, having the form (14), where the coefficients a_j and b_j, $j=\overline{1,4n}$, are determined by formulas (17), (18), (20), (21), (22), (24).

The case $n=1$**.**

$$G(x,s)=\begin{cases}-\dfrac{x^{3-\alpha}}{(2-\alpha)_2}+\dfrac{sx^{2-\alpha}}{2(1-\alpha)_2}+\dfrac{sx}{4(1-\alpha)}-\dfrac{s^{2-\alpha}x}{2(1-\alpha)_2}, & x<s,\\ -\dfrac{sx^{2-\alpha}}{2(1-\alpha)_2}+\dfrac{sx}{4(1-\alpha)}+\dfrac{s^{2-\alpha}x}{2(1-\alpha)_2}-\dfrac{s^{3-\alpha}}{(2-\alpha)_2}, & x>s.\end{cases}$$

The case $n=2$**.**

$$G(x,s)=\begin{cases}-\dfrac{x^{7-\alpha}}{3!(4-\alpha)_4}+\dfrac{sx^{6-\alpha}}{2!(3-\alpha)_4}-\dfrac{s^2x^{5-\alpha}}{4(2-\alpha)_4}-\dfrac{s^2x^{4-\alpha}}{8(1-\alpha)_4}+\dfrac{s^3x^{4-\alpha}}{12(1-\alpha)_4}+\dfrac{\alpha s^2x^3}{16\cdot 3!(1-\alpha)_2}+\dfrac{s^3x^3}{24\cdot 3!(1-\alpha)}-\\ -\dfrac{s^{4-\alpha}x^3}{2\cdot 3!(1-\alpha)_4}+\dfrac{\left(\alpha^2-\alpha+2\right)s^2x^2}{32\cdot 2!(1-\alpha)_3}+\dfrac{\alpha s^3x^2}{48\cdot 2!(1-\alpha)_2}-\dfrac{s^{4-\alpha}x^2}{4\cdot 2!(1-\alpha)_4}+\dfrac{s^{5-\alpha}x^2}{2\cdot 2!(2-\alpha)_4}, & x<s,\\ -\dfrac{s^{7-\alpha}}{3!(4-\alpha)_4}+\dfrac{s^{6-\alpha}x}{2(3-\alpha)_4}-\dfrac{s^{5-\alpha}x^2}{2\cdot 2!(2-\alpha)_4}-\dfrac{s^{4-\alpha}x^2}{4\cdot 2!(1-\alpha)_4}+\dfrac{s^{4-\alpha}x^3}{2\cdot 3!(1-\alpha)_4}+\dfrac{\alpha s^3x^2}{48\cdot 2!(1-\alpha)_2}+\\ +\dfrac{s^3x^3}{24\cdot 3!(1-\alpha)}-\dfrac{s^3x^{4-\alpha}}{12(1-\alpha)_4}+\dfrac{\left(\alpha^2-\alpha+2\right)s^2x^2}{32\cdot 2!(1-\alpha)_3}+\dfrac{\alpha s^2x^3}{16\cdot 3!(1-\alpha)_2}-\dfrac{s^2x^{4-\alpha}}{8(1-\alpha)_4}+\dfrac{s^2x^{5-\alpha}}{4(2-\alpha)_4}, & x>s.\end{cases}$$

Symmetry of the Green's function.

We now prove that $G(x,s)$ is symmetric in the general case. It is known that the problem $\{Mv=0$, (9)$\}$ is self-adjoint, therefore, its Green's function (14) is symmetric with respect to its arguments x and s.

Let us prove this assertion. Suppose $v(x),h(x)\in C^{2n-1}[0,1]$ and $x^\alpha v^{(2n)}(x)$, $x^\alpha h^{(2n)}(x)\in C^{2n-1}[0,1]\cap C^{2n}(0,1)$. We introduce the following notations:

$$Mv(x)\equiv\left(x^\alpha v^{(2n)}(x)\right)^{(2n)}=f(x),\ \ Mh(x)\equiv\left(x^\alpha h^{(2n)}(x)\right)^{(2n)}=g(x).$$

Then the following equality holds:

$$h(x)Mv(x)-v(x)Mh(x)\ =h(x)\left(x^\alpha v^{(2n)}(x)\right)^{(2n)}-v(x)\left(x^\alpha h^{(2n)}(x)\right)^{(2n)}=$$

$$=\sum_{j=0}^{2n-1}\frac{d}{dx}\left\{(-1)^j\left[h^{(j)}(x)\left(x^\alpha v^{(2n)}(x)\right)^{(2n-1-j)}-v^{(j)}(x)\left(x^\alpha h^{(2n)}(x)\right)^{(2n-1-j)}\right]\right\}=f(x)h(x)-g(x)v(x),\ \ 0<x<1 \tag{25}$$

.

Assuming $v(x)=G(x,s)$ and $h(x)=G(x,\xi)$, at all points of the interval $(0,1)$ except for $x\neq\xi$ and $x\neq s$, the relations $Mv(x)=0$ and $Mh(x)=0$ hold. Then equality (25) takes the form:

$$\sum_{j=0}^{2n-1}\frac{d}{dx}\left\{(-1)^j\left[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)-\right.\right.$$

$$\left.\left.-\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\right]\right\}=0,\quad x\in(0,1)\setminus\{s,\xi\} \tag{26}$$

Without loss of generality, let us assume $s<\xi$. Then the interval $[0,1]$ is divided into three subintervals: $[0,s]$, $[s,\xi]$, and $[\xi,1]$. Integrating equality (26) over these subintervals yields:

$$\sum_{j=0}^{2n-1}\left\{(-1)^j\left[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)-\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\right]\right\}_{x=0}^{x=s-0}+$$

$$+\sum_{j=0}^{2n-1}\left\{(-1)^j\left[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)-\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\right]\right\}_{x=s+0}^{x=\xi-0}+$$

$$+\sum_{j=0}^{2n-1}\left\{(-1)^j\left[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)-\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\right]\right\}_{x=\xi+0}^{x=1}=0.$$

Taking into account properties 1) and 4) of Green's function $G(x,s)$, the last equality reduces to:

$$-\left[G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)\right]\Bigg|_{x=s-0}^{x=s+0}+\left[G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\right]\Bigg|_{x=s-0}^{x=s+0}-$$

$$-\left[G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)\right]\Bigg|_{x=\xi-0}^{x=\xi+0}+\left[G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\right]\Bigg|_{x=\xi-0}^{x=\xi+0}=0.$$

According to property 2) of Green's function, the derivative $\left(\partial^{2n-1}/\partial x^{2n-1}\right)\left[x^\alpha\left(\partial^{2n}/\partial x^{2n}\right)G(x,\eta)\right]$ is continuous for $x\neq\eta$. Accounting for this continuity, we obtain:

$$\left[G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)\Bigg|_{x=s-0}-G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)\Bigg|_{x=s+0}\right]+$$

$$+\left[G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\Bigg|_{x=\xi+0}-G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\Bigg|_{x=\xi-0}\right]=0.$$

Hence, by virtue of equality (11), we arrive at $-G(s,\xi)+G(\xi,s)=0$, which completes the proof.

Reduction to an integral equation. Following the technique set out in [20, §3, pp. 40-42], one shows that problem {(8), (9)} is equivalent to the integral equation

$$v(x)=\lambda\int_0^1 G(x,s)v(s)ds. \tag{27}$$

The proof follows the same line of reasoning as that of Theorem 2 in (21 §3, pp. 40-42). Although the boundary conditions considered there differ from (9), the argument remains valid in the present case, since it uses only Properties 1)-4) of the Green function established above. Repeating the proof of Theorem 2 with the Green function constructed here, one verifies that the function

$$v(x)=\int_0^1 G(x,s)\,f(s)ds$$

satisfies $M[v]=f$ together with the boundary conditions (9). Therefore, setting $f(x)=\lambda v(x)$, we arrive at the homogeneous Fredholm integral equation (27).

This equivalence is established because $G(x,s)$ is the Green's function of the differential operator M under the specified boundary conditions, which allows the differential operator to be represented as an inverse integral operator. Since the kernel $G(x,s)$ is real, symmetric (as shown above) and continuous, by the theory of symmetric integral equations the corresponding integral operator possesses a countable set of characteristic numbers $|\lambda_1|\leqslant|\lambda_2|\leqslant\ldots\leqslant|\lambda_k|\leqslant\ldots$, (see [20], §26, pp. 140-145), and, by the Hilbert-Schmidt theorem (see [20], §27, Theorem 1, p. 150), the corresponding system of eigenfunctions is complete in $L_2(0,1)$. Recall that, as shown above, every eigenvalue satisfies $\lambda>0$; hence the eigenvalues can be arranged as

$$0<\lambda_1\leqslant\lambda_2\leqslant\lambda_3\leqslant\ldots\leqslant\lambda_k<\ldots,\quad \lambda_k\to+\infty,$$

and the corresponding system of eigenfunctions $v_1(x),v_2(x),v_3(x),\ldots,v_k(x)\ldots$ forms an orthonormal system in space $L_2(0,1)$ [20].

In addition, it is not difficult to verify that the system of functions $s^{\alpha/2}v_k^{(n)}(s)/\sqrt{\lambda}_k$ also $k=1,2,\ldots$ forms an orthonormal system in $L_2(0,1)$.

The following lemma demonstrates which type of functions can be expanded into series by the system of eigenfunctions of the problem {(8), (9)}.

Auxiliary lemmas. In all that follows, g_k stands for the Fourier coefficient of the function $g(x)$,

$$g_k = \int_0^1 g(x)v_k(x)dx, \qquad k \in \mathbb{N}. \tag{28}$$

Lemma 2.1. Let $g(x)$ comply with conditions (9) and let $Mg(x) \in C(0,1) \cap L_2(0,1)$. Then $g(x)$ can be expanded on $[0,1]$ into an absolutely and uniformly convergent series over the eigenfunctions of problem {(8), (9)}

Proof. By integrating by parts and drawing on the properties of $G(x,s)$ along with the hypotheses placed on g, one confirms the identity

$$\int_0^1 G(x,s)Mg(s)ds = \int_0^1 G(x,s)\left[s^\alpha g^{(2n)}(s)\right]^{(2n)} ds = g(x).$$

As $Mg(x) \in L_2(0,1)$, the identity establishes that g can be represented through the kernel $G(x,s)$. Since, moreover, $G(x,s)$ is continuous on $\bar{\Omega}$, the claim is a consequence of Theorem 2 of [20, p. 153]. Lemma 2.1 has been proved.

Lemma 2.2. The series

$$\sum_{k=1}^{+\infty} \frac{\left[v_k^{(j)}(x)\right]^2}{\lambda_k}, \qquad \sum_{k=1}^{+\infty} \frac{\left(\left[x^\alpha v_k^{(2n)}(x)\right]^{(j)}\right)^2}{\lambda_k^2}, \qquad j = \overline{0, 2n-1}, \tag{29}$$

converge uniformly on $[0,1]$.

Proof. Using (8) and the properties of the Green's function, from (27) with $v(x) \equiv v_k(x)$ one derives

$$v_k^{(j)}(x) = \lambda_k \int_0^1 \frac{\partial^j}{\partial x^j} G(x,s)v_k(s)ds = \int_0^1 \left[s^\alpha v_k^{(2n)}(s)\right]^{(2n)} \frac{\partial^j}{\partial x^j} G(x,s)ds.$$

After $2n$ integrations by parts and an appeal to (9),

$$v_k^{(j)}(x) = \int_0^1 s^\alpha v_k^{(2n)}(s) \frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s)ds,$$

whence, as $\lambda_k > 0$,

$$\frac{v_k^{(j)}(x)}{\sqrt{\lambda_k}} = \int_0^1 \left(s^{\alpha/2} \frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s) \right) \left(\frac{s^{\alpha/2} v_k^{(2n)}(s)}{\sqrt{\lambda_k}} \right) ds. \tag{30}$$

Thus $v_k^{(j)}(x)/\sqrt{\lambda_k}$ is the Fourier coefficient of $s^{\alpha/2}\partial^{2n+j}G(x,s)/(\partial x^j \partial s^{2n})$ in the orthonormal system $\{s^{\alpha/2}v_k^{(2n)}(s)/\sqrt{\lambda_k}\}$, and Bessel's inequality [see: [22], pp. 23-24)] gives

$$\sum_{k=1}^{+\infty} \frac{\left[v_k^{(j)}(x)\right]^2}{\lambda_k} \le \int_0^1 s^\alpha \left[\frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s) \right]^2 ds, \quad j = \overline{0, 2n-1}. \tag{31}$$

The integral on the right-hand side of (31) can be rewritten as

$$\int_0^1 s^\alpha \left[\frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s) \right]^2 ds = \int_0^1 s^{-\alpha} \left[\frac{\partial^j}{\partial x^j} \left(s^\alpha \frac{\partial^{2n}}{\partial s^{2n}} G(x,s) \right) \right]^2 ds, \quad j = \overline{0, 2n-1}.$$

Since

$$s^\alpha \frac{\partial^{2n} G(x,s)}{\partial s^{2n}}, \quad \frac{\partial^j G(x,s)}{\partial x^j} \in C(\bar{\Omega}), \quad j = \overline{0, 2n-1},$$

the function in the square bracket is continuous on $\bar{\Omega}$. Then, due to $0 < \alpha < 1$, the integral on the right-hand side, and therefore the integral in (31), is uniformly bounded at $j = \overline{0, 2n-1}$, which implies that the first series in (29) converges uniformly.

The convergence of the remaining series can be proved similarly. Lemma 2.2 has been proved.

Lemma 2.3. If $g^{(j)}(x) \in C[0,1]$, $j = \overline{0, 2n-1}$, $x^{\alpha/2} g^{(2n)}(x) \in C(0,1) \cap L_2(0,1)$, and g satisfies the corresponding boundary conditions in $g^{(j)}(0) = 0, \quad g^{(n+j)}(0) + g^{(n+j)}(1) = 0, \quad j = \overline{0, n-1}$, then

$$\sum_{k=1}^{+\infty} \lambda_k g_k^2 \le \int_0^1 x^\alpha \left[g^{(2n)}(x) \right]^2 dx \tag{32}$$

in particular, the series on the left converges.}

Proof. By (8),

$$\lambda_k^{1/2} g_k = \lambda_k^{-1/2} \int_0^1 g(x)[x^\alpha v_k^{(2n)}(x)]^{(2n)}\,dx.$$

Carrying out $2n$ integrations by parts and using the properties of g and v_k,

$$\lambda_k^{1/2} g_k = \int_0^1 \left\{x^{\alpha/2} g^{(2n)}(x)\right\}\left\{\lambda_k^{-1/2} x^{\alpha/2} v_k^{(2n)}(x)\right\}dx.$$

Thus $\lambda_k^{1/2} g_k$ is the Fourier coefficient of $x^{\alpha/2} g^{(2n)}(x)$ with respect to the orthonormal family $\{x^{\alpha/2} v_k^{(2n)}(x)/\sqrt{\lambda_k}\}$, and (32) issues from Bessel's inequality. Lemma 2.3 has been proved.

Lemma 2.4. If $g(x)$ obeys (9) and $Mg(x) \in C(0,1) \cap L_2(0,1)$, then

$$\sum_{k=1}^{+\infty} \lambda_k^2 g_k^2 \le \int_0^1 \left[Mg(x)\right]^2 dx; \tag{33}$$

in particular, the series on the left converges.

Proof. From (8), $\lambda_k g_k = \int_0^1 g(x)[x^\alpha v_k^{(2n)}(x)]^{(2n)}\,dx$. Performing $4n$ integrations by parts and accounting for the properties of g gives $\lambda_k g_k = \int\limits_0^1 \left[Mg\left(x\right)\right] v_k\left(x\right) dx$. Accordingly, $\lambda_k g_k$ is the Fourier coefficient of $Mg(x)$ in the system $\{v_k\}$, and (33) results from Bessel's inequality. Lemma 2.4 has been proved.

Lemma 2.5. If $g(x)$ satisfies (9) and, in addition, $[Mg(x)]^{(j)} \in C[0,1]$, $j = \overline{0,2n-1}$, $x^{\alpha/2}[Mg(x)]^{(2n)} \in C(0,1) \cap L_2(0,1)$, $\left[Mg\left(x\right)\right]^{(j)} = 0, \left[Mg\left(x\right)\right]^{(n+j)}\Big|_{x=0} + \left[Mg\left(x\right)\right]^{(n+j)}\Big|_{x=1} = 0,\ \overline{j=0,n-1},$ then

$$\sum_{k=1}^{+\infty} \lambda_k^3 g_k^2 \le \int_0^1 x^\alpha \left\{[Mg(x)]^{(2n)}\right\}^2 dx. \tag{34}$$

Proof. The function Mg satisfies the assumptions of Lemma 2.3, and, by what was shown in Lemma 2.4, the quantities $\lambda_k g_k$ are the Fourier coefficients of Mg in $\{v_k\}$. Invoking Lemma 2.3 for Mg (its Fourier coefficients being $\lambda_k g_k$) produces $\sum\limits_{k=1}^{+\infty} \lambda_k (\lambda_k g_k)^2 \le \int_0^1 x^\alpha \{[Mg]^{(2n)}\}^2 dx$, which is precisely (34). Lemma 2.5 has been proved.

Existence, uniqueness and stability of the solution

We look for the solution in the form

$$u(x,t) = \sum_{k=1}^{+\infty} u_k(t) v_k(x), \tag{35}$$

in which $v_k(x)$ denote the eigenfunctions of {(8), (9)} and the functions $u_k(t)$ remain to be found. Inserting (35) into (5) and (6) and exploiting the orthonormality of $\{v_k\}$, we obtain, for every $k \in \mathbb{N}$, the degenerate Cauchy problem

$$t^a (1-t)^b u_{k'}(t) + (c+\lambda_k) u_k(t) = f_k(t), \qquad t \in (0,1), \tag{36}$$

$$u_k(0) = \varphi_k, \tag{37}$$

with $\varphi_k = \int_0^1 \varphi(x) v_k(x)\,dx$ and $f_k(t) = \int_0^1 f(x,t) v_k(x)\,dx$. Relation (36) is a linear ordinary differential equation of the first order that degenerates at the endpoints $t=0$ and $t=1$, because $0 \le a < 1$ and $0 \le b < 1$, the weight $1/[t^a(1-t)^b]$ is integrable over $(0,1)$. Dividing (36) through by $t^a(1-t)^b$ and first treating the homogeneous equation, we evaluate

$$\int_0^t \frac{dz}{z^a (1-z)^b} = \frac{t^{1-a}}{1-a}\, {}_2F_1\left(1-a, b; 2-a; t\right),$$

with ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ the Gauss hypergeometric function. Setting $\Phi(t)=\dfrac{t^{1-a}}{1-a}{}_2F_1(1-a,b;2-a;t)$, the homogeneous equation has the solution $u_k(t)=A_k e^{-(c+\lambda_k)\Phi(t)}$, and (37) forces $A_k=\varphi_k$. Varying the constant in the inhomogeneous equation, we find the unique solution of (36)-(37) to be

$$u_k(t)=\varphi_k e^{-(c+\lambda_k)\Phi(t)}+\int_0^t \frac{f_k(z)}{z^a(1-z)^b}e^{-(c+\lambda_k)[\Phi(t)-\Phi(z)]}dz. \tag{38}$$

As $c+\lambda_k>0$ and, for $t>z$, $\Phi(t)-\Phi(z)=\int_z^t s^{-a}(1-s)^{-b}ds\ge 0$, both $e^{-(c+\lambda_k)[\Phi(t)-\Phi(z)]}\le 1$ and $e^{-(c+\lambda_k)\Phi(t)}\le 1$ hold, consequently

$$|u_k(t)|\le|\varphi_k|+\int_0^t\frac{|f_k(z)|}{z^a(1-z)^b}dz,\qquad 0\le t\le 1. \tag{39}$$

Denoting $\rho(t)=1/[t^a(1-t)^b]$, $B_0=(\int_0^1\rho(z)dz)^{1/2}<\infty$ and $\|f_k\|^2_{L_{2,\rho}(0,1)}=\int_0^1\rho(z)f_k^2(z)dz$, appealing to the Cauchy-Schwarz inequality one obtains

$$|u_k(t)|\le|\varphi_k|+B_0\|f_k\|_{L_{2,\rho}(0,1)},\qquad 0\le t\le 1. \tag{40}$$

Theorem 3.1. Let $\varphi(x)$ satisfy the hypotheses of Lemma 2.5. Let $f(x,t)$ be continuous on $\bar{\Omega}$ and, for each fixed $t\in[0,1]$, satisfy with respect to x the hypotheses of Lemma 2.5 uniformly in t, with $x^{\alpha/2}[Mf]^{(2n)}\in L_{2,\rho}(\Omega)$, assume moreover that $f(x,\cdot)$ is Hölder continuous in t uniformly in x, so that $f_k(t)$ is continuous and $f_k(z)/[z^a(1-z)^b]$ is integrable on $(0,1)$, as required in (38). Then the series (35), whose coefficients are given by (38), determines a solution of the problem.

Proof. It has to be shown that (35) and the series obtained from it by formal term-by-term differentiation,

$$\partial^j u/\partial x^j=\sum_{k=1}^{+\infty}u_k(t)v_k^{(j)}(x),\quad j=\overline{1,2n-1},\quad\text{and}\quad(\partial^j/\partial x^j)(x^\alpha\partial^{2n}u/\partial x^{2n})=\sum_{k=1}^{+\infty}u_k(t)(x^\alpha v_k^{(2n)}(x))^{(j)},$$

$j=\overline{0,2n-1}$, converge uniformly on $\bar{\Omega}$, and that the series $(\partial^{2n}/\partial x^{2n})(x^\alpha\partial^{2n}u/\partial x^{2n})=\sum_{k=1}^{+\infty}u_k(t)(x^\alpha v_k^{(2n)}(x))^{(2n)}$ and $t^a(1-t)^b u_t=\sum_{k=1}^{+\infty}t^a(1-t)^b u_{k'}(t)v_k(x)$. By (40), for $(x,t)\in\bar{\Omega}$, the Cauchy—Schwarz inequality gives

$$|u(x,t)|\le\sqrt{\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}}\left(\sqrt{\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2}+B_0\sqrt{\int_0^1\rho(z)\sum_{k=1}^{+\infty}\lambda_k f_k^2(z)dz}\right). \tag{41}$$

By Lemmas 2.2 and 2.3 the right-hand series converge, and so (35) is uniformly convergent on $\bar{\Omega}$. As for the series $\sum_{k=1}^{+\infty}\lambda_k u_k(t)v_k(x)$ - which, by (8), stands for $(\partial^{2n}/\partial x^{2n})(x^\alpha\partial^{2n}u/\partial x^{2n})$ - it is enough, in light of (40), to secure the convergence of $\sum_{k=1}^{+\infty}\lambda_k\varphi_k v_k(x)$ and $\sum_{k=1}^{+\infty}\lambda_k\|f_k\|_{L_{2,\rho}(0,1)}v_k(x)$, the Cauchy-Schwarz inequality majorizes them by

$$\sqrt{\sum_{k=1}^{+\infty}\lambda_k^3\varphi_k^2}\sqrt{\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}},\qquad\sqrt{\int_0^1\rho(z)\sum_{k=1}^{+\infty}\lambda_k^3 f_k^2(z)dz}\sqrt{\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}},$$

both of which converge by virtue of Lemmas 2.2, 2.4 and 2.5. The series therefore converges uniformly on Ω_0, uniform convergence of the final series is a consequence of (5). The series left over are treated in an analogous fashion. The proof of Theorem 3.1 is complete.

Theorem 3.2. The problem cannot possess more than one solution.

Proof. Assume the problem has two solutions $u_1(x,t)$ and $u_2(x,t)$, and put $u=u_1-u_2$. Then u obeys (5) with $f\equiv 0$, the initial datum (6) with $\varphi\equiv 0$, and (7). Choose $T_0\in(0,1)$ and write $\Omega_{T_0}=\{(x,t):0<x<1,0<t<T_0\}$. Multiply (5) (with $f\equiv 0$) by u and integrate over Ω_{T_0}, using

$uu_t = \frac{1}{2}(u^2)_t$ in the first term and integrating by parts $2n$ times in the spatial one - where the boundary form disappears thanks to the combined local-and-paired relations (7), just as in Section 2 - we arrive at

$$\frac{1}{2}\int_0^1 T_0^a(1-T_0)^b u^2(x,T_0)dx + c\int_0^{T_0}\int_0^1 u^2\,dxdt + \int_0^{T_0}\int_0^1 x^\alpha\left[\frac{\partial^{2n}u}{\partial x^{2n}}\right]^2 dxdt = \frac{1}{2}\int_0^{T_0}\int_0^1 \frac{\partial}{\partial t}\left[t^a(1-t)^b\right]u^2(x,t)dxdt.$$

As $\partial_t[t^a(1-t)^b] = at^{a-1}(1-t)^b - bt^a(1-t)^{b-1}$, one can find $M_0 = M_0(a,b) > 0$ with $|\partial_t[t^a(1-t)^b]| \le M_0\rho_1(t)$, where we set $\rho_1(t) = t^{a-1} + (1-t)^{b-1}$, since $a,b<1$, the integrals $\int_0^1 t^{a-1}dt$ and $\int_0^1 (1-t)^{b-1}dt$ are finite, so $\rho_1 \in L_1(0,1)$ and the right-hand member does not exceed $K\int_0^{T_0}\int_0^1 u^2\,dxdt$ for a suitable $K>0$. We stress that ρ_1 is the Gronwall coefficient appearing here and differs from the weight $\rho(t) = t^{-a}(1-t)^{-b}$ used earlier. Because the two integral terms on the left are nonnegative, $\frac{1}{2}T_0^a(1-T_0)^b\int_0^1 u^2(x,T_0)dx \le K\Psi(T_0)$, $\Psi(T_0) = \int_0^{T_0}\int_0^1 u^2\,dxdt$, and, as $\Psi'(T_0) = \int_0^1 u^2(x,T_0)dx$, this becomes $\Psi'(T_0) \le 2K\rho(T_0)\Psi(T_0)$, $\Psi(0)=0$, the factor $2K\rho(T_0)$ being integrable over $(0,1)$. Gronwall's inequality now forces $\Psi(T_0)\equiv 0$, so that $u\equiv 0$ on $\bar{\Omega}$ that is, $u_1 \equiv u_2$. The proof of Theorem 3.2 is complete.

Theorem 3.3. Let $\varphi(x)$ and $f(x,t)$ satisfy the hypotheses of Theorem 3.1. Then the solution obeys the estimates

$$\| u(x,t)\|^2_{L_2(0,1)} \le K_0\left[\| \varphi(x)\|^2_{L_2(0,1)} + \| f(x,t)\|^2_{L_{2,\rho}(\Omega)}\right], \tag{42}$$

$$\| u(x,t)\|_{C(\bar{\Omega})} \le K_1\left[\| \varphi^{(2n)}(x)\|_{L_{2,r}(0,1)} + \| f(x,t)\|_{L_{2,\rho}(\Omega)}\right], \tag{43}$$

Where $\| \varphi(x)\|_{L_{2,r}(0,1)} = [\int_0^1 x^\alpha \varphi^2(x)dx]^{1/2}$, $r(x) = x^\alpha$, and K_0, K_1 are positive constants. These entail the continuous dependence of the solution on the data φ and f .}

Proof. By the orthonormality of $\{v_k\}$ and (40), in conjunction with $(p+q)^2 \le 2(p^2+q^2)$,

$$\| u(x,t)\|^2_{L_2(0,1)} = \sum_{k=1}^{+\infty} u_k^2(t) \le 2\sum_{k=1}^{+\infty}\left[\varphi_k^2 + B_0^2\| f_k\|^2_{L_{2,\rho}(0,1)}\right].$$

Bessel's inequality, combined with the identity $\| f\|^2_{L_{2,\rho}(\Omega)} = \sum_{k=1}^{+\infty}\| f_k\|^2_{L_{2,\rho}(0,1)}$ (which holds because ρ depends on t alone while $\{v_k\}$ is orthonormal in x), delivers (42) with $K_0 = 2\max(1, B_0^2)$. As for (43), drawing on Lemmas 2.2, 2.3 and on (41) we get

$$\| u(x,t)\|_{C(\bar{\Omega})} \le K_1\left\{\sqrt{\int_0^1 x^\alpha\left[\varphi^{(2n)}(x)\right]^2 dx} + \sqrt{\int_0^1 \rho(z)\sum_{k=1}^{+\infty} f_k^2(z)dz}\right\},$$

with $K_1 = \sup_{x\in[0,1]}(\sum_{k=1}^{+\infty} v_k^2(x)/\lambda_k)^{1/2}$, which is (43). The proof of Theorem 3.3 is complete.

Conclusion. Over a rectangular region we have set up and studied an initial-boundary value problem carrying a blend of local and paired (linked) boundary conditions, for a high even-order partial differential equation that loses its type in space at $x=0$ and in time at $t=0$ and $t=1$, and that additionally involves a subordinate term. Uniqueness of the solution was secured through an energy-integral technique tailored to the temporal degeneracy and rounded off with an inequality of Gronwall type. The solution itself was constructed as a Fourier expansion in the eigenfunctions of the attendant spectral problem, whose Green's function was produced in explicit form - displayed in closed form for $n=1$ and, for $n=2$, through the explicit determination of all its coefficients – shown to be symmetric, and converted into a Fredholm integral equation of the second kind with a continuous and symmetric kernel that generates a positive-definite operator. We isolated sufficient conditions on the data guaranteeing solvability and derived stability bounds in the spaces $L_2(0,1)$ and $C(\bar{\Omega})$.

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YUKLANGAN HADLI NOCHIZIQLI DEFOKUSLANGAN SHREDINGER TENGLAMASIGA QO‘YILGAN KOSHI MASALASINI YECHISH

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Annotatsiya. *Ushbu maqolada teskari spektral masalalar usuli yordamida cheksiz zonali davriy funksiyalar sinfida yuklangan hadli nochiziqli Shredinger tenglamasining integrallanuvchanligi tadqiq etiladi. Bir zonali potensial holati uchun Koshi masalasining aniq yechimi topilgan. Ishda Dirak operatori, spektral ma’lumotlar, Dubrovin tenglamalar sistemasi va iz formulalaridan foydalangan holda tenglamaning spektral tahlili amalga oshiriladi.*

Kalit so‘zlar: *nochiziqli Shredinger tenglamasi, Dirak operatori, teskari spektral masala, spektral berilganlar, Dubrovin tenglamalar sistemasi, izlar formulasi.*

РЕШЕНИЕ ЗАДАЧИ КОШИ ДЛЯ ДЕФОКУСИРУЮЩЕГО НЕЛИНЕЙНОГО УРАВНЕНИЯ ШРЁДИНГЕРА С НАГРУЖЕННЫМ ЧЛЕНОМ

Аннотация. *В данной статье с помощью метода обратных спектральных задач исследуется интегрируемость нелинейного уравнения Шрёдингера с нагруженным членом в классе бесконечнозонных периодических функций. Для случая однозонного потенциала найдено точное решение задачи Коши. В работе проводится спектральный анализ уравнения с использованием оператора Дирака, спектральных данных, системы уравнений Дубровина и формул следов.*

Ключевые слова: *нелинейное уравнение Шрёдингера, оператор Дирака, обратная спектральная задача, спектральные данные, система уравнений Дубровина, формула следов.*

SOLUTION OF THE CAUCHY PROBLEM FOR THE DEFOCUSING NONLINEAR SCHRÖDINGER EQUATION WITH A LOADED TERM

Abstract. *In this paper, the integrability of the nonlinear Schrödinger equation with a loaded term is investigated in the class of finite-gap periodic functions using the inverse spectral problem method. For the one-gap potential case, an exact solution to the Cauchy problem is obtained. The spectral analysis of the equation is carried out using the Dirac operator, spectral data, the Dubrovin system of equations, and trace formulas.*

Keywords: *nonlinear Schrödinger equation, Dirac operator, inverse spectral problem, spectral data system of Dubrovin equations, trace formula.*

Kirish. Nochiziqli optika, kvant mexanikasi, gidrodinamika masalalarini tadqiqotlarida nochiqli Shredinger tenglamasi uchun qo‘yilgan Koshi masalasi yechimini topish muhim ahamiyatga ega. Bundan yuklangan hadlarga ega nochiziqli Shredinger tenglamasini integrallash, Dubrovin differensial tenglamalar sistemasini keltirib chiqarish dolzarb masala hisoblanadi.

Mazkur ishda o‘rganilgan yuklangan hadlarga ega nochiziqli defokuslangan Shredinger tenglamasiga qo‘yilgan Koshi masalasini davriy cheksiz zonali funksiyalar sinfida yechimga egaligini o‘rganish bo‘yicha olingan natijalar zamonaviy matematik fizika, kvant mexanikasida, nochiziqli optikada va davriy to‘lqinlarning nochiziqli muhitdagi tarqalish jarayonlarida qo‘llaniladi.

Ushbu ishda quyidagi

$$\begin{cases} p_t = -q_{xx} + 2q\left(p^2 + q^2\right) + p(x_0,t)q, \\ q_t = p_{xx} - 2p\left(p^2 + q^2\right) - p(x_0,t)p. \end{cases} \tag{1}$$

ko‘rinishdagi yuklangan hadli nochiziqli defokuslangan Shredinger tenglamasini

$$p(x,t)\big|_{t=0} = p_0(x), q(x,t)\big|_{t=0} = q_0(x),\ x_0, x \in R, t > 0 \tag{2}$$

boshlang‘ich shartlarini qanoatlantiruvchi $p(x,t), q(x,t)$ yechimini davriy cheksiz zonali funksiyalar sinfida topish masalasini qaraymiz. Bu yerda $p(x,t)$ va $p(x,t)$ lar ushbu

$$\begin{aligned} &p(x,t) \in C_x^2(t>0) \bigcap C_t^1(t>0) \bigcap C(t \ge 0), \\ &q(x,t) \in C_x^2(t>0) \bigcap C_t^1(t>0) \bigcap C(t \ge 0) \end{aligned} \tag{3}$$

silliqlik shartlarini qanoatlantiruvchi *x* o‘zgaruvchi bo‘yicha π - davrli funksiyalar.
(1) - (3) Koshi masalasining yechimini, quyidagi

$$L(\tau,t)y \equiv B\frac{dy}{dx} + \Omega(x+\tau,t)y = \lambda y,\ \tau \in R, t > 0 \tag{4}$$

Dirak operatoriga qo‘yilgan teskari spektral masala usulidan foydalanib topamiz.
Bu yerda

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \Omega(x,t) = \begin{pmatrix} p(x,t) & q(x,t) \\ q(x,t) & -p(x,t) \end{pmatrix},\ y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

(4) ko‘rinishdagi Dirak tenglamalar sistemasining

$$c(0,\lambda,\tau,t) = (1,0)^T \text{ va } s(0,\lambda,\tau,t) = (0,1)^T \tag{5}$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimlarini mos ravishda

$c(x,\lambda,\tau,t) = (c_1(x,\lambda,\tau,t), c_2(x,\lambda,\tau,t))^T$ va $s(x,\lambda,\tau,t) = (s_1(x,\lambda,\tau,t), s_2(x,\lambda,\tau,t))^T$

orqali belgilaymiz. Bundan tashqari (4) tenglamalar sistemasiga qo‘yilgan davriy va yarimdavriy $y(0,\lambda,\tau,t) = \pm y(\pi,\lambda,\tau,t)$ chegaraviy masalaning xos qiymatlarini $\lambda_n(\tau,t), n \in Z$ orqali, (4) tenglamaga qo‘yilgan $y_1(0,\lambda,\tau,t) = 0, y_1(\pi,\lambda,\tau,t) = 0$ Dirixle chegaraviy masalaning xos qiymatlarini $\xi_n(\tau,t), n \in Z$ orqali belgilaymiz.

1-teorema. Aytaylik $p(x,t), q(x,t), x \in R, t > 0$ funksiyalar (1) - (3) Koshi masalasining yechimi bo‘lsin. U holda (4) Dirak operatorining

$\{\lambda_n(\tau,t), \xi_n(\tau,t), \sigma_n(\tau,t) = \pm 1, n \in Z\}$ spektral berilganlari quyidagi Dubrovin differensial tenglamalar sistemasining analogini qanoatlantiradi:

$$1)\ \frac{\partial \lambda_n(\tau,t)}{\partial t} = 0,\ \lambda_n(\tau,0) = \lambda_n, n \in Z;$$

$$2)\ \frac{\partial \xi_n(\tau,t)}{\partial t} = 2(-1)^n \sigma_n(\tau,t) h_n(\xi(\tau,t)) \Big\{ q^2(\tau,t) + q_\tau(\tau,t) + \Big[p(\tau,t) + \xi_n(\tau,t) \Big]^2 + \\ + \xi_n^2(\tau,t) + \frac{1}{2} p(x_0,t) \Big\}, n \in Z. \tag{5}$$

Bundan tashqari quyidagi

$$\xi_n(\tau,t)\big|_{t=0} = \xi_n^0(\tau),\ \sigma_n(\tau,t)\big|_{t=0} = \sigma_n^0(\tau), n \in Z \tag{6}$$

boshlang‘ich shartlar ham bajariladi. Bu yerda $\xi_n^0(\tau), \sigma_n^0(\tau) = \pm 1, n \in Z$ lar $L(\tau,0)$ Dirak operatorining spektral parametrlari. (5) tenglamadagi $h_n(\xi(\tau,t)), n \in Z$ ketma-ketlik

$$h_n(\xi(\tau,t)) = f_n(\xi(\tau,t)) \sqrt{(\xi_n(\tau,t) - \lambda_{2n-1})(\lambda_{2n} - \xi_n(\tau,t))},$$

$$f_n(\xi(\tau,t)) = \sqrt{\prod_{k=-\infty, k \ne n}^{\infty} \frac{(\lambda_{2k-1} - \xi_n(\tau,t))(\lambda_{2k} - \xi_n(\tau,t))}{(\xi_k(\tau,t) - \xi_n(\tau,t))^2}}$$

formula bilan aniqlanadi.

Isbot. $L(\tau,t)y = B\frac{dy}{dx} + \Omega(x+\tau,t)y = \lambda y, \tau \in \mathbb{R}$, tenglamalar sistemasiga qo‘yilgan Dirixle chegaraviy $y_1(0,\tau,t) = 0, y_1(\pi,\tau,t) = 0$ masalasining xos qiymatlari $\xi_n(\tau,t)$ va ularga mos ortonormal xos vektor funksiyalar $y_n(x,\tau,t) = \left(y_{n,1}(x,\tau,t),\ y_{n,2}(x,\tau,t)\right)^T, n \in \mathbb{Z}$ bo‘lsin, u holda $L(\tau,t)y_n(x,\tau,t) = \xi_n(\tau,t)y_n(x,\tau,t)$

tenglik bajariladi. Bu tenglikning ikki tomonini $y_n(x,\tau,t)$ ortonormal vektor funksiyaga skalyar ko‘paytirib, quyidagi

$$\xi_n(\tau,t = (L(\tau,t)y_n, y_n) \tag{7}$$

tenglikni hosil qilamiz. (3.1.6) tenglikning ikki tomonini t bo‘yicha differensiallab,$L(\tau,t)$operatorning simmetrikligidan foydalansak u holda ushbu

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = \left(\dot{\Omega}(x+\tau,t)y_n, y_n\right), n \in \mathbb{Z}$$

munosabat hosil bo‘ladi.Skalyar ko‘paytmaning ta’rifidan foydalanib quyidagi

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = \int_0^\pi \left[\left(y_{n,1}^2 - y_{n,2}^2\right)p_t + 2y_{n,1}y_{n,2}q_t\right]dx \tag{8}$$

tenglikni hosil qilamiz. (1)dagi p_t,q_t larni (8) ga qo‘yamiz:

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = \int_0^\pi \left[\left(y_{n,1}^2 - y_{n,2}^2\right)(-q_{\text{xx}} + 2q(p^2+q^2) + 2y_{n,1}y_{n,2}(p_{\text{xx}} - 2p(p^2+q^2)\right]dx +$$
$$[p(x_0,t) + q(x_0,t)]\times\int_0^\pi \left[q\left(y_{n,1}^2 - y_{n,2}^2\right) + 2py_{n,1}y_{n,2}\right]dx.$$

Endi quyidagi belgilashlarni kiritib olamiz.

$$I_1(\tau,t) = \int_0^\pi \{[-q_{xx} + 2q(p^2+q^2) + q]\left(y_{n,1}^2 - y_{n,2}^2\right) +$$
$$+2[p_{xx} - 2p(p^2+q^2) - p]y_{n,1}y_{n,2}\}dx$$
$$I_2(\tau,t) = \int_0^\pi \left[p\left(y_{n,1}^2 - y_{n,2}^2\right) + 2qy_{n,1}y_{n,2}\right]dx$$

Natijada ushbu

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = I_1(\tau,t) + [p(x_0,t) + q(x_0,t)]I_2(\tau,t) \tag{9}$$

tenglikka ega bo‘lamiz.

$I_1(\tau,t)$, $I_2(\tau,t)$ integrallarni hisoblaymiz.

Buning uchun integrallar ostidagi ifodalarning boshlang‘ich funksiyalarini $a_i(x,\xi_n,t)y_{n,1}^2 + b_i(x,\xi_n,t)y_{n,1}y_{n,2} + c_i(x,\xi_n,t)y_{n,2}^2, a(t), \quad i = 1,2,3..$ kvadratik formalarning hosilalari ko‘rinishida izlaymiz va quydagi tenglikdan foydalanamiz:

$$\left\{\left(y_{n,1}^2 - y_{n,2}^2\right)\left[-q_{xx} + 2q(p^2+q^2) + 2y_{n,1}y_{n,2}[p_{xx} - 2p(p^2+q^2)]\right]\right\}' = \left(a_1(x,\xi_n,t)y_{n,1}^2 +\right.$$
$$\left. + b_1(x,\xi_n,t)y_{n,1}y_{n,2} + c_1(x,\xi_n,t)y_{n,2}^2\right)',$$
$$q\left(y_{n,1}^2 - y_{n,2}^2\right) + 2py_{n,1}y_{n,2} = (a_2(x,\xi_n,t)y_{n,1}^2 + b_2(x,\xi_n,t)y_{n,1}y_{n,2} + c_2(x,\xi_n,t)y_{n,2}^2)'$$

Ma’lumki,

$$\begin{cases} y'_{n,1} = qy_{n,1} - py_{n,2} - \xi_n y_{n,2} \\ y'_{n,2} = \xi_n y_{n,1} - py_{n,1} - qy_{n,2} \end{cases} \tag{10}$$

o‘rinli. Bundan esa

$$\left(y_{n,1}^2 - y_{n,2}^2\right)\left[-q_{xx} + 2q(p^2+q^2) + 2y_{n,1}y_{n,2}[p_{xx} - 2p(p^2+q^2)]\right] = \{[-q_x + q^2 + (p -$$
$$\xi_n)^2 + \xi_n^2]y_{n,1}^2 + (p_x - 4q\xi_n)y_{n,1}y_{n,2} + [q_x + q^2 + (p+\xi_n)^2 + \xi_n^2]y_{n,2}^2\}',$$
$$q(y_{n,1}^2 - y_{n,2}^2) + 2py_{n,1}y_{n,2} = \frac{1}{2}\left(y_{n,1}^2 + y_{n,2}^2\right)'$$

ekanligini ko‘rsatish qiyinchilik tug‘dirmaydi.

U holda, $I_1(\tau,t)$, $I_2(\tau,t)$ integrallar natijasi quyidagicha bo‘ladi:

$$\begin{cases} I_1(\tau,t) = q_x + q^2 + p^2 + 2p\xi_n + 2\xi_n^2 \\ I_2(\tau,t) = \frac{1}{2}(y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t)) \end{cases} \tag{11}$$

(9) va (11) tengliklardan foydalanib, quyidagi tenglamani hosil qilamiz:

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = \left(q^2(\tau,t) + q_\tau(\tau,t) + \left(p(\tau,t) + \xi_n(\tau,t)\right)^2 + \xi_n^2(\tau,t)\right)\times$$

$$\times\left[y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t)\right] \tag{12}$$
$$y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t) = 2(-1)^n\sigma_n(\tau,t)h_n(\xi(\tau,t)) \tag{13}$$

Bu tenglikni (12)ga qo‘yib , (4)ni hosil qilamiz. Agar Dirixle chegaraviy shartlarini davriy $y(0,\tau,t)=y(\pi,\tau,t)$ yoki yarimdavriy $y(0,\tau,t)=-y(\pi,\tau,t)$ chegaraviy masalani qarasak, u holda uning $\lambda_n(\tau,t)$ xos qiymatlari uchun

$\frac{\partial\lambda_n(\tau,t)}{\partial t}=0, n\in\mathbb{Z}$, yoki $\lambda_n(\tau,t)=\lambda_n(\tau,0), n\in\mathbb{Z}$ tengliklar hosil bo‘ladi. Endi $L(\tau,t)v_n=\lambda_n(\tau,t)v_n, n\in\mathbb{Z}$ tenglamaga t=0 ni qo‘yamiz, u holda $\lambda_n(\tau,t)=\lambda_n(\tau,0), n\in\mathbb{Z}$ ga ega bo‘lamiz. Bundan $L(\tau,0)v_n=\lambda_n(\tau)v_n, n\in\mathbb{Z}$

davriy yoki yarimdavriy masalaning xos qiymatlari $\tau\in\mathbb{R}$ parametrga bog‘liq emasligi kelib chiqadi, ya’ni $\lambda_n(\tau,0)=\lambda_n(\tau)=\lambda_n, n\in\mathbb{Z}, \tau\in\mathbb{R}$ kelib chiqadi. Teorema isbotlandi.

1-natija. Ushbu

$$p(\tau,t)=\sum_{k=-\infty}^{\infty}\left(\frac{\lambda_{2k-1}+\lambda_{2k}}{2}-\xi_k(\tau,t)\right), \tag{14}$$

$$q(\tau,t)=\sum_{k=-\infty}^{\infty}(-1)^{k-1}\sigma_k(\tau,t)h_k(\xi(\tau,t)), \tag{15}$$

$$q^2(\tau,t)+q_\tau(\tau,t)=\sum_{k=-\infty}^{\infty}\left(\frac{\lambda_{2k-1}^2+\lambda_{2k}^2}{2}-\xi_k^2(\tau,t)\right). \tag{16}$$

izlar formulalaridan foydalanib Dubrovin differensial tenglamalar sistemasini yopiq ko‘rinishda yozish mumkin.

2-natija. 1- teorema (1) - (3) masalaning yechimini topish usulini beradi. Buning uchun dastlab $L(\tau,0)$ Dirak operatorining $\lambda_n, \xi_n^0(\tau), \sigma_n^0(\tau)=\pm1, n\in Z$ spektral berilganlarini aniqlaymiz, so‘nra $L(\tau,t)$ operatorning spektral berilganlarini $\lambda_n, \xi_n(\tau,t), \sigma_n(\tau,t)=\pm1, n\in Z$ orqali belgilaymiz. Keyin (5) tenglamalar sistemasi va (6) boshlang‘ich shartlar ketma-ketligida $\tau=x_0$ deb olamiz. Hosil bo‘lgan Koshi masalasini yechib, $\xi_n(x_0,t), \sigma_n(x_0,t), n\in Z$ ni topamiz. So‘ngra (8), (9) izlar formulalaridan $p(x_0,t)$ funksiyani aniqlaymiz. Shundan keyin, buni (5) tenglamalar sistemasiga qo‘yamiz va τ ning ixtiyoriy qiymatida (5)-(6) Koshi masalasini yechib $\xi_n(\tau,t), \sigma_n(\tau,t), n\in Z$ larni aniqlaymiz. Topilgan $\xi_n(\tau,t), \sigma_n(\tau,t), n\in Z$ spektral parametrlardan hamda (8) va (9) izlar formulalaridan foydalanib (1) - (3) Koshi masalasining $(p(\tau,t), q(\tau,t))$ yechimlarni topamiz.

3 - natija. Bir zonali potensial holida (1) tenglamani ushbu

$$p(x,t)\big|_{t=0}=\frac{\lambda_0-\lambda_{-1}}{2}\cos\{-(\lambda_{-1}+\lambda_0)x+2\sigma_0(0)\varphi(0)\}$$

$$q(x,t)\big|_{t=0}=-\frac{\lambda_0-\lambda_{-1}}{2}\sin\{-(\lambda_{-1}+\lambda_0)x+2\sigma_0(0)\varphi(0)\}$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimi 2-natijadagi usul bo‘yicha yechamiz, bu holat uchun Dubrovin differensial tenglamalar sistemasi ushbu ko‘rinishda bo‘ladi:

$$\frac{\partial\xi_n(\tau,t)}{\partial t}=2\sigma_0(\tau,t)\sqrt{(\xi_0(\tau,t)-\lambda_{-1})(\lambda_0-\xi_0(\tau,t))}\times$$

$$\times\left\{\frac{\lambda_{-1}^2+\lambda_0^2}{2}+\frac{(\lambda_{-1}+\lambda_0)^2}{4}+\frac{1}{2}\left[\frac{\lambda_{-1}+\lambda}{2}-\xi_0(x_0,t)\right]\right\}$$

$$\xi_0(\tau,t)\big|_{t=0}=\frac{\lambda_0+\lambda_{-1}}{2}-\frac{\lambda_0-\lambda_{-1}}{2}\cos\{-(\lambda_{-1}+\lambda_0)\tau+2\sigma_0(0)\varphi(0)\}\quad \sigma_0(\tau,t)\big|_{t=0}=\sigma_0(\tau)$$

$\lambda_{-1}=-\lambda_0$ bo ‘lganda yechim quyidagicha bo‘ladi:

$$p(\tau,t)=\lambda_0\cos\left\{2\lambda_0^2 t+\lambda_0\left[\frac{2M}{N(M^2-1)}arctg\left[M(tg(Nt+L))\right]-\frac{M^2+1}{M^2-1}t\right]+2C(\tau)\right\}$$

$$q(\tau,t)=-\lambda_0\sin\left\{2\lambda_0^2 t+\lambda_0\left[\frac{2M}{N(M^2-1)}arctg\left[M(tg(Nt+L))\right]-\frac{M^2+1}{M^2-1}t\right]+2C(\tau)\right\}$$

Bu yerda $M=\sqrt{\frac{2\lambda_0^2+\lambda_0}{2\lambda_0^2-\lambda_0}}$, $N=\lambda_0\sqrt{\lambda_0^2-\frac{1}{4}}$, $\lambda_0>\frac{1}{2}$, $L=arctg\left\{\frac{1}{M}tg\left[\sigma_0(0)\varphi_0(0)\right]\right\}$

$C(\tau)$ berilgan boshlang‘ich shartlardan topiladi.

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ON THE NUMBER OF EIGENVALUES OF THE EVEN-ANTISYMMETRIC PART OF A TWO-PARTICLE DISCRETE SCHRÖDINGER OPERATOR

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Abstract. *We consider the two-particle discrete Schrödinger operator $H_{\lambda\mu}(0)$ on the three-dimensional lattice $\mathbb{Z}^3$ with on-site and nearest-neighbor interactions. We study the restriction of $H_{\lambda\mu}(0)$ to the subspace of even functions that are antisymmetric with respect to the first two momentum variables. On this subspace the on-site interaction vanishes, and the restricted operator $H_\mu^{ea}(0)$ becomes a rank-one perturbation depending only on μ. The eigenvalue problem outside the essential spectrum is reduced to the scalar equation $\Delta_\mu(z) = 1 + \mu a(z) = 0$. A positive critical value μ_* is found such that $H_\mu^{ea}(0)$ has exactly one eigenvalue below $[0,24]$ for $\mu < -\mu_*$, no eigenvalues outside $[0,24]$ for $|\mu| \le \mu_*$, and exactly one eigenvalue above $[0,24]$ for $\mu > \mu_*$. At the critical values $\mu = \pm\mu_*$ the corresponding threshold is an eigenvalue.*

Keywords: *two-particle system, lattice Schrödinger operator, essential spectrum, eigenvalues, rank-one perturbation, Fredholm determinant.*

IKKI ZARRACHALI DISKRET SHRYODINGER OPERATORI JUFT-ANTISIMMETRIK QISMI XOS QIYMATLARI SONI HAQIDA

Annotatsiya. *Biz uch o'lchamli $\mathbb{Z}^3$ panjarada nuqtada va eng yaqin qo'shni tugunlarda o'zaro ta'sirlashuvchi ikki zarrachali diskret Shryodinger operatori $H_{\lambda\mu}(0)$ni qaraymiz. Bu operatorning birinchi ikkita impuls o'zgaruvchilariga nisbatan antisimmetrik bo'lgan juft funksiyalar qism fazosidagi qismi faqat μ ga bog'liq, qo'zg'alishining rangi bir bo'lgan $H_\mu^{ea}(0)$ operatorga keltiriladi. Ushbu operatorning muhim spektrdan tashqaridagi xos qiymatlari masalasi $\Delta_\mu(z) = 1 + \mu a(z) = 0$ skalyar tenglamaga teng kuchli ekanligi ko'rsatiladi. Bu holda shunday $\mu_* > 0$ kritik qiymat topiladiki, $H_\mu^{ea}(0)$ operatorning $\mu < -\mu_*$ bo'lganda muhim spektrdan pastda va $\mu > \mu_*$ bo'lganda muhim spektrdan yuqorida aniq bitta xos qiymatga egaligi, $|\mu| \le \mu_*$ bo'lganda esa muhim spektr tashqarisida xos qiymatlarga ega emasligi isbotlanadi. Shuningdek, $\mu = \pm\mu_*$ kritik qiymatlarda mos spektr chegarasi bo'sag'a xos qiymat bo'lishi ko'rsatiladi.*

Kalit so'zlar: *ikki zarrachali sistema, panjaradagi diskret Shryodinger operatori, muhim spektr, xos qiymatlar, rangi birga teng qo'zg'alish, Fredgolm determinanti.*

О ЧИСЛЕ СОБСТВЕННЫХ ЗНАЧЕНИЙ ЧЁТНО-АНТИСИММЕТРИЧНОЙ ЧАСТИ ДВУХЧАСТИЧНОГО ДИСКРЕТНОГО ОПЕРАТОРА ШРЁДИНГЕРА

Аннотация. *Мы рассматриваем двухчастичный дискретный оператор Шрёдингера $H_{\lambda\mu}(0)$ на решётке $\mathbb{Z}^3$ с взаимодействиями на расстояниях ноль и один. Ограничение $H_{\lambda\mu}(0)$ на подпространство чётных функций, антисимметричных по первым двум импульсным переменным, сводит его к возмущению ранга один $H_\mu^{ea}(0)$, зависящему только от μ. Показано, что задача на собственные значения вне существенного спектра эквивалентна скалярному уравнению $\Delta_\mu(z) = 1 + \mu a(z) = 0$. Найдено критическое значение $\mu_* > 0$: оператор $H_\mu^{ea}(0)$ имеет ровно одно собственное значение ниже существенного спектра при $\mu < -\mu_*$, выше существенного спектра при $\mu > \mu_*$ и не имеет собственных значений вне [0,24] при $|\mu| \le \mu_*$. При $\mu = \pm\mu_*$ край спектра является пороговым собственным значением.*

Ключевые слова: *двухчастичная система, дискретный оператор Шрёдингера на решётке, существенный спектр, связанные состояния, возмущение ранга один, определитель Фредгольма.*

Introduction. Lattice Schrödinger operators are widely used in the spectral theory of few-particle quantum systems and in mathematical models of particles moving in periodic structures [1, 2, 5]. In contrast to continuous models, lattice systems may possess bound states both below and above the essential spectrum. This feature is of particular interest in models with short-range interactions and is closely related to spectral threshold phenomena [3, 4, 6].

In the present paper we consider the two-particle Schrödinger operator $H_{\lambda\mu}(0)$ on the three-dimensional lattice $\mathbb{Z}^3$. The parameter λ describes the on-site interaction, whereas μ describes the nearest-neighbor interaction. Our purpose is to study the part of this operator acting in the subspace of functions that are even and antisymmetric with respect to interchange of the first two momentum coordinates.

The main observation is that the on-site term disappears after this restriction. Consequently, the resulting operator $H_{\mu}^{ea}(0)$ depends only on one parameter and is a rank-one perturbation of the free operator. This makes it possible to reduce the eigenvalue problem to a single Fredholm determinant. We determine its behavior at the lower and upper thresholds and obtain the exact number and location of all eigenvalues outside the essential spectrum.

The paper is organized as follows. In Section 2 we define $H_{\lambda\mu}(0)$ and its essential spectrum. In Section 3 we derive the even–antisymmetric restriction. Section 4 contains the scalar reduction and the required properties of the Fredholm determinant. The exact eigenvalue count is given in Section 5.

The operator $H_{\lambda\mu}(0)$

Let $\mathbb{T}^3 = (-\pi, \pi]^3$ and let

$$L^{2,\mathrm{e}}(\mathbb{T}^3) = \{f \in L^2(\mathbb{T}^3): f(-p) = f(p)\}$$

be the Hilbert space of even square-integrable functions on $\mathbb{T}^3$. For $\lambda, \mu \in \mathbb{R}$, consider the bounded self-adjoint operator

$$H_{\lambda\mu}(0) = H_0(0) + V_{\lambda\mu}$$

in $L^{2,\mathrm{e}}(\mathbb{T}^3)$.

The free operator $H_0(0)$ is the multiplication operator

$$(H_0(0)f)(p) = \mathcal{E}_0(p)f(p),$$

where

$$\mathcal{E}_0(p) = 4\sum_{i=1}^{3}(1 - \cos p_i), \qquad p = (p_1, p_2, p_3) \in \mathbb{T}^3.$$

The interaction potential is defined by

$$(V_{\lambda\mu}f)(p) = \frac{\lambda}{8\pi^3}\int_{\mathbb{T}^3} f(q)\,\mathrm{d}q + \frac{\mu}{4\pi^3}\sum_{i=1}^{3}\cos p_i \int_{\mathbb{T}^3}\cos q_i\, f(q)\,\mathrm{d}q. \qquad (1)$$

Thus, λ and μ are the coupling constants of the on-site and nearest-neighbor interactions, respectively. The perturbation $V_{\lambda\mu}$ has finite rank. Hence, by Weyl's theorem,

$$\sigma_{\mathrm{ess}}\left(H_{\lambda\mu}(0)\right) = \sigma\big(H_0(0)\big) = [0{,}24].$$

The even–antisymmetric restriction

We introduce the closed subspace

$$L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3) := \{f \in L^{2,\mathrm{e}}(\mathbb{T}^3): \quad f(p_1, p_2, p_3) = -f(p_2, p_1, p_3)\}.$$

Since $\mathcal{E}_0(p)$ is symmetric in p_1 and p_2, the subspace $L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$ is invariant for $H_0(0)$.

Using

$$2\cos A\cos B + 2\cos C\cos D = (\cos A + \cos C)(\cos B + \cos D) + (\cos A - \cos C)(\cos B - \cos D),$$

we rewrite the first two nearest-neighbor terms in (1). For $f \in L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$, antisymmetry gives

$$\int_{\mathbb{T}^3} f(q)\,\mathrm{d}q = 0,$$

$$\int_{\mathbb{T}^3} (\cos q_1 + \cos q_2) f(q)\, dq = 0, \qquad \int_{\mathbb{T}^3} \cos q_3 f(q)\, dq = 0.$$

Therefore only the antisymmetric nearest-neighbor component remains:

$$(V_\mu^{\mathrm{ea}} f)(p) = \frac{\mu}{8\pi^3}(\cos p_1 - \cos p_2) \times \int_{\mathbb{T}^3} (\cos q_1 - \cos q_2) f(q)\, dq.$$

In particular, the on-site parameter λ does not enter the restricted operator. We denote this restriction by

$$H_\mu^{\mathrm{ea}}(0) = H_0(0) + V_\mu^{\mathrm{ea}}.$$

Set

$$\alpha(p) = \frac{\cos p_1 - \cos p_2}{\sqrt{8\pi^3}}.$$

Then $\| \alpha \| = 1$ and

$$V_\mu^{\mathrm{ea}} f = \mu(f, \alpha)\alpha. \tag{2}$$

Thus,

$$rank V_\mu^{\mathrm{ea}} \le 1.$$

Consequently,

$$\sigma_{\mathrm{ess}}(H_\mu^{\mathrm{ea}}(0)) = [0,24].$$

Moreover, $H_\mu^{\mathrm{ea}}(0)$ can have at most one isolated eigenvalue below 0 and at most one isolated eigenvalue above 24.

Fredholm determinant

For $z \in \mathbb{C}\backslash[0,24]$, let $R_0(0,z) = [H_0(0) - zI]^{-1}$.

The eigenvalue equation

$$H_\mu^{\mathrm{ea}}(0) f = zf$$

is equivalent to the Lippmann–Schwinger equation

$$f = -R_0(0,z) V_\mu^{\mathrm{ea}} f. \tag{3}$$

By (2), a nonzero solution of (3) exists if and only if

$$1 + \mu a(z) = 0,$$

where

$$a(z) := (\alpha, R_0(0,z)\alpha) = \frac{1}{8\pi^3} \int_{\mathbb{T}^3} \frac{(\cos p_1 - \cos p_2)^2}{\varepsilon_0(p) - z}\, dp. \tag{4}$$

We therefore define the Fredholm determinant

$$\Delta_\mu(z) = 1 + \mu a(z).$$

Lemma 1. *Let $\mu \in \mathbb{R}$ and $z \in \mathbb{R}\backslash[0,24]$. Then z is an eigenvalue of $H_\mu^{ea}(0)$ if and only if*

$$\Delta_\mu(z) = 0.$$

Proof. The equation $H_\mu^{\mathrm{ea}}(0) f = zf$ is equivalent to (3). Using (2), we obtain

$$f = -\mu(f, \alpha) R_0(0,z)\alpha.$$

Taking the inner product with α shows that a nonzero solution exists exactly when $1 + \mu a(z) = 0$.

The next lemma contains the properties of $a(z)$ needed below .

Lemma 2. *The function $a(z)$ has the following properties:*

1. $a(z) > 0$ for $z < 0$ and $a(z) < 0$ for $z > 24$;
2. $a(z)$ is strictly increasing on $(-\infty, 0)$ and on $(24, +\infty)$;
3. $\lim\limits_{z \to \pm\infty} a(z) = 0$;
4. The finite limits

$$a(0) := \lim_{z \nearrow 0} a(z), \qquad a(24) := \lim_{z \searrow 24} a(z)$$

exist and satisfy

$$a(0) > 0, \qquad a(24) = -a(0).$$

Proof. The sign of $a(z)$ follows immediately from (4). Moreover,

$$a'(z) = \frac{1}{8\pi^3}\int_{\mathbb{T}^3} \frac{(\cos p_1 - \cos p_2)^2}{(\mathcal{E}_0(p)-z)^2}\, dp > 0,$$

which proves strict monotonicity. The limits at infinity follow by dominated convergence.

Near $p = 0$,

$$\mathcal{E}_0(p)|p|^2, \qquad \cos p_1 - \cos p_2 = O(|p|^2),$$

so the integrand defining $a(0)$ is locally integrable. The same argument applies at the upper threshold. Finally,

$$\mathcal{E}_0(\pi - p) = 24 - \mathcal{E}_0(p), \qquad \alpha(\pi - p) = -\alpha(p),$$

and therefore

$$a(24 - z) = -a(z).$$

Passing to the threshold limits yields $a(24) = -a(0)$.

Set

$$\mu_* := \frac{1}{a(0)} = \left[\frac{1}{8\pi^3}\int_{\mathbb{T}^3} \frac{(\cos p_1 - \cos p_2)^2}{\mathcal{E}_0(p)}\, dp\right]^{-1} > 0. \tag{5}$$

Then

$$\lim_{z\to\pm\infty} \Delta_\mu(z) = 1,$$

while

$$\lim_{z\nearrow 0} \Delta_\mu(z) = 1 + \frac{\mu}{\mu_*}, \qquad \lim_{z\searrow 24} \Delta_\mu(z) = 1 - \frac{\mu}{\mu_*}. \tag{6}$$

Main result.

Let $n_-(H_\mu^{\mathrm{ea}}(0))$ and $n_+(H_\mu^{\mathrm{ea}}(0))$ denote the numbers of eigenvalues of $H_\mu^{\mathrm{ea}}(0)$ in $(-\infty, 0)$ and $(24, +\infty)$, respectively.

Theorem 1. *Let $\mu \in \mathbb{R}$ and let $\mu_* > 0$ be defined by (5). Then:*

1) if $\mu < -\mu_*$, then
$$n_-(H_\mu^{\mathrm{ea}}(0)) = 1, \qquad n_+(H_\mu^{\mathrm{ea}}(0)) = 0;$$
2) if $-\mu_* \le \mu \le \mu_*$, then
$$n_-(H_\mu^{\mathrm{ea}}(0)) = n_+(H_\mu^{\mathrm{ea}}(0)) = 0;$$
3) if $\mu > \mu_*$, then
$$n_-(H_\mu^{\mathrm{ea}}(0)) = 0, \qquad n_+(H_\mu^{\mathrm{ea}}(0)) = 1.$$

Proof. By Lemma 1, the eigenvalues outside $[0,24]$ are exactly the zeros of $\Delta_\mu(z) = 1 + \mu a(z)$.

Suppose first that $\mu < -\mu_*$. For $z < 0$, Lemma 2 and (6) give

$$\lim_{z\to-\infty} \Delta_\mu(z) = 1 > 0, \qquad \lim_{z\nearrow 0} \Delta_\mu(z) = 1 + \frac{\mu}{\mu_*} < 0.$$

Hence Δ_μ has a zero in $(-\infty, 0)$. Since $a(z)$ is strictly increasing and $\mu < 0$, the function $\Delta_\mu(z)$ is strictly decreasing there, so this zero is unique. For $z > 24$ we have $a(z) < 0$ and $\mu < 0$, hence $\Delta_\mu(z) > 1$. Therefore

$$n_-(H_\mu^{\mathrm{ea}}(0)) = 1, \qquad n_+(H_\mu^{\mathrm{ea}}(0)) = 0.$$

Now let $-\mu_* \le \mu \le \mu_*$. If $z < 0$, then $0 < a(z) < a(0)$. For $\mu \ge 0$ one has $\Delta_\mu(z) \ge 1$, while for $-\mu_* \le \mu < 0$,

$$\Delta_\mu(z) > 1 + \mu a(0) = 1 + \frac{\mu}{\mu_*} \ge 0.$$

Thus there are no zeros below 0. Similarly, for $z > 24$,

$$-a(0) < a(z) < 0.$$

If $\mu \le 0$, then $\Delta_\mu(z) \ge 1$, whereas for $0 < \mu \le \mu_*$,

$$\Delta_\mu(z) > 1 - \mu a(0) = 1 - \frac{\mu}{\mu_*} \ge 0.$$

Hence there are no zeros above 24 either.

Finally, let $\mu > \mu_*$. For $z < 0$ we have $a(z) > 0$, so $\Delta_\mu(z) > 1$. Thus there are no eigenvalues below 0. On $(24, +\infty)$,

$$\lim_{z \searrow 24} \Delta_\mu(z) = 1 - \frac{\mu}{\mu_*} < 0, \qquad \lim_{z \to +\infty} \Delta_\mu(z) = 1 > 0.$$

Therefore Δ_μ has a zero in $(24, +\infty)$. Since $\mu > 0$ and $a(z)$ is strictly increasing, this zero is unique. Hence

$$n_-(H_\mu^{\mathrm{ea}}(0)) = 0, \qquad n_+(H_\mu^{\mathrm{ea}}(0)) = 1.$$

The proof is complete.

The critical values also have a direct threshold interpretation.

Corollary 1. *For $\mu = -\mu_*$, the lower threshold $z = 0$ is an eigenvalue of $H_\mu^{ea}(0)$. For $\mu = \mu_*$, the upper threshold $z = 24$ is an eigenvalue of $H_\mu^{ea}(0)$.*

Proof. At $\mu = -\mu_*$ set

$$f_0(p) = \frac{\alpha(p)}{\varepsilon_0(p)}.$$

The behavior used in the proof of Lemma 2 shows that $f_0 \in L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$. Since $(f_0, \alpha) = a(0)$ and $\mu = -1/a(0)$, we obtain

$$H_{-\mu_*}^{\mathrm{ea}}(0) f_0 = 0.$$

The proof at $z = 24$ is analogous, with

$$f_{24}(p) = \frac{\alpha(p)}{\varepsilon_0(p) - 24}.$$

Conclusion. For the two-particle lattice Schrödinger operator $\boldsymbol{H_{\lambda\mu}(0)}$ with on-site and nearest-neighbor interactions, we have isolated the even–antisymmetric invariant subspace. The on-site term vanishes on this subspace, and the resulting operator $\boldsymbol{H_\mu^{\mathrm{ea}}(0)}$ is a rank-one perturbation depending only on the nearest-neighbor coupling constant $\boldsymbol{\mu}$. Its discrete spectrum is completely determined by the single critical value $\boldsymbol{\mu_* = 1/a(0)}$. There is one eigenvalue below the essential spectrum for $\boldsymbol{\mu < -\mu_*}$, one eigenvalue above it for $\boldsymbol{\mu > \mu_*}$, and no eigenvalues outside $\boldsymbol{[0, 24]}$ for $\boldsymbol{|\mu| < \mu_*}$. At the critical couplings the corresponding spectral threshold is itself an eigenvalue.

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УДК 539.3

СВОБОДНЫЕ ВОЛНЫ В СЛОЕ НА ВЯЗКОУПРУГОМ ПОЛУПРОСТРАНСТВЕ

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Аннотация. *В работе исследовано распространение свободных волн в механической системе, состоящей из вязкоупругого слоя, находящегося на вязкоупругом полупространстве. Рассмотрены два типа граничных условий на контактной границе: жёсткий и скользящий контакт. Поставленная задача сведена к системе линейных дифференциальных уравнений с комплексными коэффициентами. Для построения дисперсионных уравнений и расчёта комплексных собственных частот применён численный метод Мюллера в сочетании с методом Гаусса. Выявлены особенности влияния диссипативной однородности и неоднородности системы на комплексный спектр частот и коэффициенты демпфирования.*

Ключевые слова: *вязкоупругость, свободные волны, дисперсионное уравнение, комплексная частота, демпфирование, жесткий контакт, скользящий контакт.*

QOVUSHQOQ-ELASTIK YARIMFAZODAGI QATLAMDA ERKIN TO‘L QINLAR

Annotatsiya. *Ishda qovushqoq-elastik yarim fazoda joylashgan qovushqoq-elastik qatlamdan tashkil topgan mexanik sistemada erkin to‘lqinlarning tarqalishi tadqiq qilingan. Kontakt chegarasida ikki turdagi chegaraviy shartlar ko‘rib chiqilgan: qattiq va sirpanuvchi kontakt. Qo‘yilgan masala kompleks koeffitsiyentli chiziqli differensial tenglamalar sistemasiga keltirilgan. Dispersiya tenglamalarini tuzish va kompleks xos chastotalarni hisoblash uchun Myullerning sonli usuli Gauss usuli bilan birgalikda qo‘llanilgan. Sistemaning dissipativ bir jinsliligi va nobir jinsliligining kompleks chastotalar spektri va dempferlash koeffitsiyentlariga ta’sirining o‘ziga xos xususiyatlari aniqlangan.*

Kalit so‘zlar: *qovushqoq-elastiklik, erkin to‘lqinlar, dispersiya tenglamasi, kompleks chastota, dempferlash, qattiq kontakt, sirpanuvchi kontakt.*

FREE WAVES IN A LAYER ON A VISCOELASTIC HALF-SPACE

Abstract. *This study investigates the propagation of free waves in a mechanical system consisting of a viscoelastic layer resting on a viscoelastic half-space. Two types of boundary conditions at the contact interface are considered: rigid and sliding contact. The formulated problem is reduced to a system of linear differential equations with complex coefficients. To construct the dispersion equations and calculate the complex eigenfrequencies, Müller's numerical method combined with the Gauss method is applied. The specific features of the influence of the system's dissipative homogeneity and heterogeneity on the complex frequency spectrum and damping coefficients are identified.*

Keywords: *viscoelasticity, free waves, dispersion equation, complex frequency, damping, rigid contact, sliding contact.*

Введение. Изучение волновых процессов в многослойных вязкоупругих средах имеет важное значение для динамики деформируемого твёрдого тела, сейсмостойкости сооружений и геомеханики [1,2,3]. Наличие диссипативных свойств в элементах конструкции приводит к затуханию волн и изменению спектра собственных частот. В зависимости от реологических характеристик составных частей системы механические структуры можно разделить на диссипативно однородные (с идентичными вязкоупругими свойствами) и диссипативно неоднородные (с различными свойствами) [4,5]. В литературе также хорошо известны два типа упругих поверхностных волн, названные волнами Рэлея и Лава в честь их первооткрывателей. Волны Рэлея возникают в результате эллиптического движения, характеризующегося сочетанием волн давления и вертикального сдвига; тогда как волны Лава возникают только в результате горизонтального сдвига [6,7]. Волны Лава

нашли множество применений, превосходящих волны Рэлея, в различных областях науки, таких как биосенсоры в биологии и медицине, хемосенсоры в химии, неразрушающий контроль материалов в технике и так далее, помимо их большого значения в сейсмологии, геологии и геофизике [8,9,10]. В настоящей работе предполагалось, что верхняя поверхность свободна от напряжений, а смещения и напряжения на границе раздела являются непрерывными, а также соблюдалось условие излучения в полупространстве. При условии, что скорость сдвиговых волн в слое должна быть ниже скорости сдвиговых волн в полупространстве, было показано существование горизонтальной сдвиговой волны, амплитуда которой быстро убывает с толщиной в полупространстве. Целью данной работы является исследование влияния условий контакта (жёсткий и скользящий contact) и диссипативной неоднородности на волновые характеристики системы «слой – полупространство».

Постановка задачи и основные уравнения

Рассмотрим однородный изотропный линейно-вязкоупругий слой толщиной h и полупространство в прямоугольной системе координат (x, y, z). Составляющими перемещений по осям x, y, z являются u, ϑ, ω соответственно.

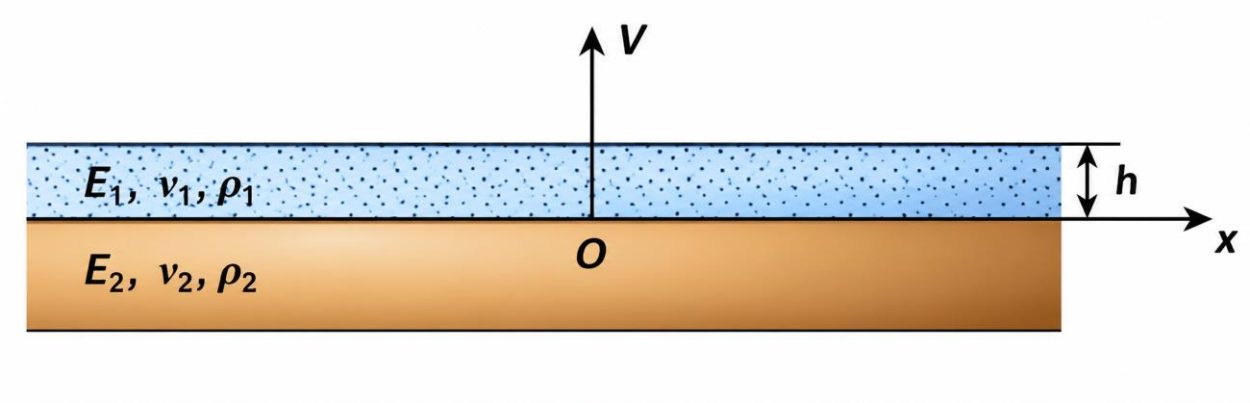


Рисунок 1. Расчетная схема системы «вязкоупругий слой – вязкоупругое полупространство».

В расчётной модели слой толщиной h контактирует с вязкоупругим полупространством; параметры материалов слоя и основания обозначены соответственно $E_1, \nu_1, \rho_1,$ и E_2, ν_2, ρ_2. Уравнения движения в напряжениях для каждой из областей имеют вид:

$$\nabla \bullet \sigma_n = \rho_n \frac{\partial^2 u_n}{\partial t^2}, \qquad (n = 1,2) \qquad (1)$$

где ρ_n - плотность материала слоя $(n = 1)$ и полупространства $(n = 2)$.

Граничные условия:

На свободной поверхности слоя $(y = h)$:

$$\sigma_{yy1} = 0, \qquad \sigma_{yx1} = 0 \qquad (2)$$

На границе раздела $(y = 0)$:

При жёстком контакте:

$$\sigma_{yy1} = \sigma_{yy2}, \qquad \sigma_{yx1} = \sigma_{yx2}, \quad u_1 = u_2, \qquad \vartheta_1 = \vartheta_2 \qquad (3a)$$

При скользящем контакте (без сдвиговых усилий):

$$\sigma_{yx1} = 0, \qquad \sigma_{yx2} = 0, \qquad \vartheta_1 = \vartheta_2 \qquad (3b)$$

Метод решения

Решение системы уравнений ищется в виде гармонических волн:

$$u_n = U_n(y) e^{i(kx - \omega t)}, \qquad \vartheta_n = V_n(y) e^{i(kx - \omega t)}, \qquad (4)$$

где $U_n(y)$ и $V_n(y)$ — амплитудные комплексные вектор-функции; k — волновое число; $\omega = \omega_R + i\omega_I$ — комплексная частота ($C = C_R + iC_I$ — комплексная фазовая скорость). Подстановка

решений (4) в уравнения движения (1) приводит к системе обыкновенных дифференциальных уравнений второго порядка относительно $U_n(y)$ и $V_n(y)$:

$$L_n(U_n, V_n) = 0 \qquad (5)$$

Частное решение системы (5) ищется в виде $e^{r_n y}$. Составляя определяющий характеристический определитель, находим корни r_n, отвечающие скоростям волн сжатия и сдвига. Вязкоупругие свойства материала учитываются через комплексный модуль Юнга $E(\omega)$:

$$E(\omega) = E(\omega)\left[1 + i\eta(\omega)\right] \qquad (6)$$

где функции $E(\omega)$ и $\eta(\omega)$ представляются аналитически:

$$E(\omega) = E_{\min} + (E_{\max} - E_{\min}) f(\omega), \qquad \eta(\omega) = \eta_0 \cdot g(\omega) \qquad (7)$$

Подстановка выражений для перемещений в граничные условия (3a) или (3b) приводит к однородной системе шести алгебраических уравнений относительно неизвестных постоянных C_{ij}. Условие существования нетривиального решения (равенство нулю определителя системы) даёт **дисперсионное уравнение**:

$$\Delta(\zeta, \omega) = 0 \qquad (8)$$

Где $\zeta = kh$ — безразмерное волновое число.

«Поиск корней частотного уравнения (8) осуществляется методом Мюллера, при этом на каждой итерации вычисление определителя производится методом Гаусса с выделением главного элемента, что исключает необходимость прямого раскрытия определителя **высокого порядка**».

Численные результаты и их обсуждение

При расчётах приняты следующие параметры материалов:

$E_{\min} = 6.9 \times 10^6 H/m^2$, $E_{\min} = 6.9 \times 10^5 H/m^2$, $\theta = 0.75$, $\alpha = 10^{-4}$, $n = 1$

Безразмерное волновое число ζ варьировалось в пределах $0 \le \zeta \le 4$.

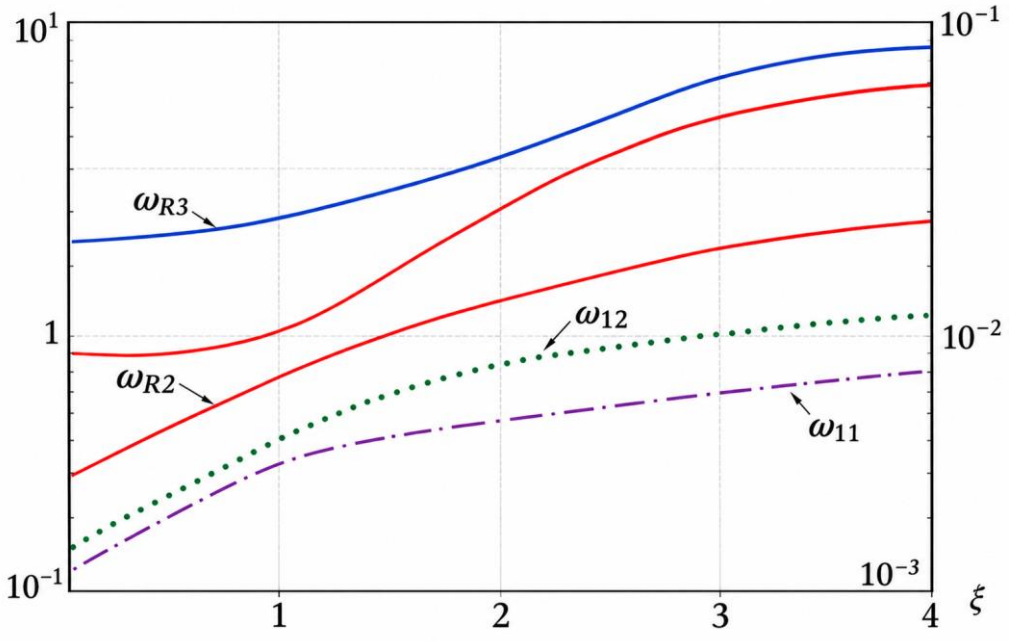


Рисунок 2. Зависимость комплексных собственных частот от безразмерного волнового числа ξ для диссипативно однородной системы

На рисунке 2 представлены действительные и мнимые составляющие комплексных собственных частот. При увеличении ξ действительные части частот изменяются монотонно, тогда как мнимые составляющие характеризуют интенсивность диссипативного затухания.

Диссипативно неоднородная система. ля системы с различными реологическими характеристиками слоя и полупространства:

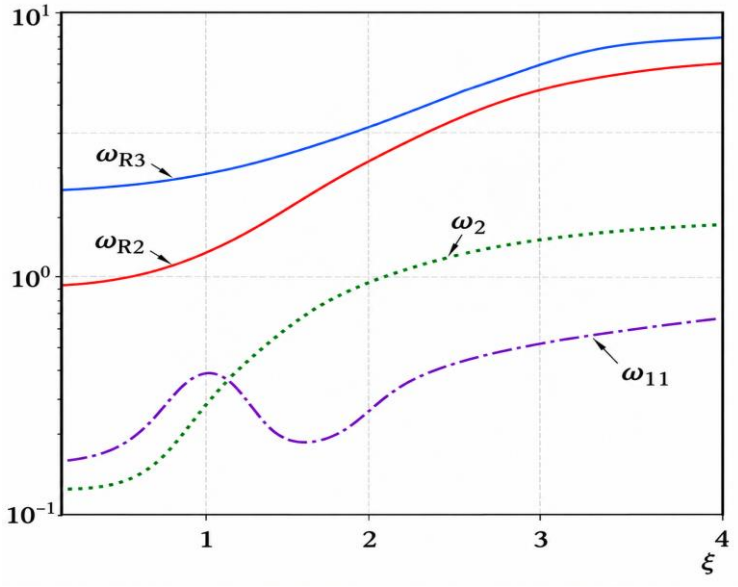


Рисунок 3. Комплексный спектр собственных частот диссипативно неоднородной системы

Сопоставление с рисунком 3 показывает, что диссипативная неоднородность оказывает существенно большее влияние на мнимые части частот и коэффициенты демпфирования, чем на их действительные части.

• Зависимость действительной части частоты ω_R от ζ практически совпадает с однородным случаем (отклонение не превышает 5%).

• Зависимости коэффициентов демпфирования ω_I от ζ становятся немонотонными.

Особый интерес представляет минимальный коэффициент демпфирования системы:

$$\delta = \min\left(-\omega_{Ik}\right)$$

Для неоднородной системы наблюдается эффект «смены ролей» мнимых частей первой и второй частот при некотором характерном значении ζ. В этой точке действительные части ω_{R1} и ω_{R2} максимально близки, а коэффициент демпфирования δ имеет ярко выраженный максимум.

Скользящий контакт. Для скользящего контакта дисперсионное уравнение сохраняет аналогичную структуру, однако условие отсутствия сдвигового усилия на границе раздела изменяет комплексный спектр. На рисунка 4 и 5 приведены соответствующие зависимости для диссипативно однородной и неоднородной систем.

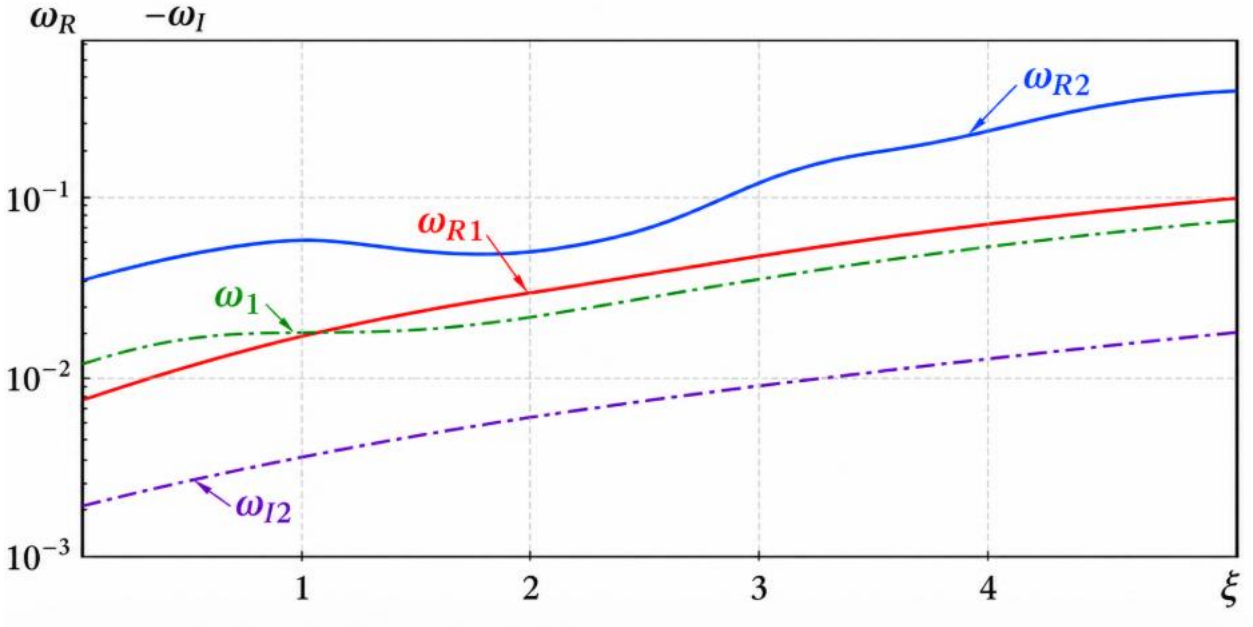


Рисунок 4. Комплексные собственные частоты от безразмерного волнового числа ξ при скользящем контакте: диссипативно однородная система

Для однородной системы кривые действительных частей частот изменяются плавно, а мнимые части определяют уровень волнового затухания при скользящем контакте.

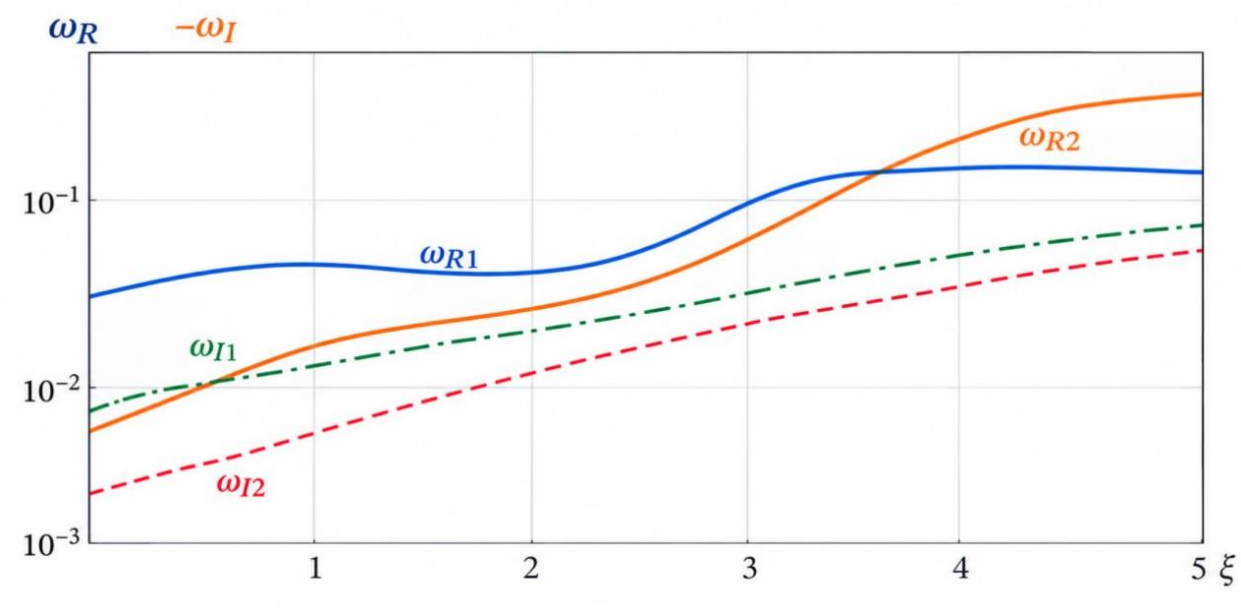


Рисунок 5. Комплексные собственные частоты от безразмерного волнового числа ξ при скользящем контакте: диссипативно неоднородная система

В неоднородной системе сохраняется эффект сближения действительных частей соседних мод, сопровождающийся выраженным максимумом коэффициента демпфирования. Это подтверждает возможность управления диссипативными характеристиками системы посредством изменения её структуры и условий контакта.

Заключение.

1. Разработан эффективный численный алгоритм на основе методов Мюллера и Гаусса для расчёта комплексных спектров частот вязкоупругих многослойных систем.

2. Установлено, что диссипативная неоднородность приводит к немонотонной зависимости коэффициентов демпфирования от волнового числа.

3. Появление локального максимума коэффициента демпфирования δ позволяет эффективно управлять вибропоглощающими характеристиками конструкций путем подбора геометрических параметров элементов без изменения их химического или механического состава.

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