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# ON THE NUMBER OF EIGENVALUES OF THE EVEN-ANTISYMMETRIC PART OF A TWO-PARTICLE DISCRETE SCHRÖDINGER OPERATOR

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***Abstract.*** *We consider the two-particle discrete Schrödinger operator $H_{\lambda\mu}(0)$ on the three-dimensional lattice $\mathbb{Z}^3$ with on-site and nearest-neighbor interactions. We study the restriction of $H_{\lambda\mu}(0)$ to the subspace of even functions that are antisymmetric with respect to the first two momentum variables. On this subspace the on-site interaction vanishes, and the restricted operator $H_\mu^{ea}(0)$ becomes a rank-one perturbation depending only on $\mu$. The eigenvalue problem outside the essential spectrum is reduced to the scalar equation $\Delta_\mu(z) = 1 + \mu a(z) = 0$. A positive critical value $\mu_*$ is found such that $H_\mu^{ea}(0)$ has exactly one eigenvalue below $[0,24]$ for $\mu < -\mu_*$, no eigenvalues outside $[0,24]$ for $|\mu| \le \mu_*$, and exactly one eigenvalue above $[0,24]$ for $\mu > \mu_*$. At the critical values $\mu = \pm\mu_*$ the corresponding threshold is an eigenvalue.*

***Keywords:*** *two-particle system, lattice Schrödinger operator, essential spectrum, eigenvalues, rank-one perturbation, Fredholm determinant.*

## IKKI ZARRACHALI DISKRET SHRYODINGER OPERATORI JUFT-ANTISIMMETRIK QISMI XOS QIYMATLARI SONI HAQIDA

***Annotatsiya.*** *Biz uch o'lchamli $\mathbb{Z}^3$ panjarada nuqtada va eng yaqin qo'shni tugunlarda o'zaro ta'sirlashuvchi ikki zarrachali diskret Shryodinger operatori $H_{\lambda\mu}(0)$ni qaraymiz. Bu operatorning birinchi ikkita impuls o'zgaruvchilariga nisbatan antisimmetrik bo'lgan juft funksiyalar qism fazosidagi qismi faqat $\mu$ ga bog'liq, qo'zg'alishining rangi bir bo'lgan $H_\mu^{ea}(0)$ operatorga keltiriladi. Ushbu operatorning muhim spektrdan tashqaridagi xos qiymatlari masalasi $\Delta_\mu(z) = 1 + \mu a(z) = 0$ skalyar tenglamaga teng kuchli ekanligi ko'rsatiladi. Bu holda shunday $\mu_* > 0$ kritik qiymat topiladiki, $H_\mu^{ea}(0)$ operatorning $\mu < -\mu_*$ bo'lganda muhim spektrdan pastda va $\mu > \mu_*$ bo'lganda muhim spektrdan yuqorida aniq bitta xos qiymatga egaligi, $|\mu| \le \mu_*$ bo'lganda esa muhim spektr tashqarisida xos qiymatlarga ega emasligi isbotlanadi. Shuningdek, $\mu = \pm\mu_*$ kritik qiymatlarda mos spektr chegarasi bo'sag'a xos qiymat bo'lishi ko'rsatiladi.*

***Kalit so'zlar:*** *ikki zarrachali sistema, panjaradagi diskret Shryodinger operatori, muhim spektr, xos qiymatlar, rangi birga teng qo'zg'alish, Fredgolm determinanti.*

## О ЧИСЛЕ СОБСТВЕННЫХ ЗНАЧЕНИЙ ЧЁТНО-АНТИСИММЕТРИЧНОЙ ЧАСТИ ДВУХЧАСТИЧНОГО ДИСКРЕТНОГО ОПЕРАТОРА ШРЁДИНГЕРА

***Аннотация.*** *Мы рассматриваем двухчастичный дискретный оператор Шрёдингера $H_{\lambda\mu}(0)$ на решётке $\mathbb{Z}^3$ с взаимодействиями на расстояниях ноль и один. Ограничение $H_{\lambda\mu}(0)$ на подпространство чётных функций, антисимметричных по первым двум импульсным переменным, сводит его к возмущению ранга один $H_\mu^{ea}(0)$, зависящему только от $\mu$. Показано, что задача на собственные значения вне существенного спектра эквивалентна скалярному уравнению $\Delta_\mu(z) = 1 + \mu a(z) = 0$. Найдено критическое значение $\mu_* > 0$: оператор $H_\mu^{ea}(0)$ имеет ровно одно собственное значение ниже существенного спектра при $\mu < -\mu_*$, выше существенного спектра при $\mu > \mu_*$ и не имеет собственных значений вне [0,24] при $|\mu| \le \mu_*$. При $\mu = \pm\mu_*$ край спектра является пороговым собственным значением.*

***Ключевые слова:*** *двухчастичная система, дискретный оператор Шрёдингера на решётке, существенный спектр, связанные состояния, возмущение ранга один, определитель Фредгольма.*

**Introduction.** Lattice Schrödinger operators are widely used in the spectral theory of few-particle quantum systems and in mathematical models of particles moving in periodic structures [1, 2, 5]. In contrast to continuous models, lattice systems may possess bound states both below and above the essential spectrum. This feature is of particular interest in models with short-range interactions and is closely related to spectral threshold phenomena [3, 4, 6].

In the present paper we consider the two-particle Schrödinger operator $H_{\lambda\mu}(0)$ on the three-dimensional lattice $\mathbb{Z}^3$. The parameter $\lambda$ describes the on-site interaction, whereas $\mu$ describes the nearest-neighbor interaction. Our purpose is to study the part of this operator acting in the subspace of functions that are even and antisymmetric with respect to interchange of the first two momentum coordinates.

The main observation is that the on-site term disappears after this restriction. Consequently, the resulting operator $H_{\mu}^{ea}(0)$ depends only on one parameter and is a rank-one perturbation of the free operator. This makes it possible to reduce the eigenvalue problem to a single Fredholm determinant. We determine its behavior at the lower and upper thresholds and obtain the exact number and location of all eigenvalues outside the essential spectrum.

The paper is organized as follows. In Section 2 we define $H_{\lambda\mu}(0)$ and its essential spectrum. In Section 3 we derive the even–antisymmetric restriction. Section 4 contains the scalar reduction and the required properties of the Fredholm determinant. The exact eigenvalue count is given in Section 5.

**The operator $H_{\lambda\mu}(0)$**

Let $\mathbb{T}^3 = (-\pi, \pi]^3$ and let

$$L^{2,\mathrm{e}}(\mathbb{T}^3) = \{f \in L^2(\mathbb{T}^3): f(-p) = f(p)\}$$

be the Hilbert space of even square-integrable functions on $\mathbb{T}^3$. For $\lambda, \mu \in \mathbb{R}$, consider the bounded self-adjoint operator

$$H_{\lambda\mu}(0) = H_0(0) + V_{\lambda\mu}$$

in $L^{2,\mathrm{e}}(\mathbb{T}^3)$.

The free operator $H_0(0)$ is the multiplication operator

$$(H_0(0)f)(p) = \mathcal{E}_0(p)f(p),$$

where

$$\mathcal{E}_0(p) = 4\sum_{i=1}^{3}(1 - \cos p_i), \qquad p = (p_1, p_2, p_3) \in \mathbb{T}^3.$$

The interaction potential is defined by

$$(V_{\lambda\mu}f)(p) = \frac{\lambda}{8\pi^3}\int_{\mathbb{T}^3} f(q)\,\mathrm{d}q + \frac{\mu}{4\pi^3}\sum_{i=1}^{3}\cos p_i \int_{\mathbb{T}^3}\cos q_i\, f(q)\,\mathrm{d}q. \qquad (1)$$

Thus, $\lambda$ and $\mu$ are the coupling constants of the on-site and nearest-neighbor interactions, respectively. The perturbation $V_{\lambda\mu}$ has finite rank. Hence, by Weyl's theorem,

$$\sigma_{\mathrm{ess}}\left(H_{\lambda\mu}(0)\right) = \sigma\big(H_0(0)\big) = [0{,}24].$$

**The even–antisymmetric restriction**

We introduce the closed subspace

$$L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3) := \{f \in L^{2,\mathrm{e}}(\mathbb{T}^3): \quad f(p_1, p_2, p_3) = -f(p_2, p_1, p_3)\}.$$

Since $\mathcal{E}_0(p)$ is symmetric in $p_1$ and $p_2$, the subspace $L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$ is invariant for $H_0(0)$.

Using

$$2\cos A\cos B + 2\cos C\cos D = (\cos A + \cos C)(\cos B + \cos D) + (\cos A - \cos C)(\cos B - \cos D),$$

we rewrite the first two nearest-neighbor terms in (1). For $f \in L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$, antisymmetry gives

$$\int_{\mathbb{T}^3} f(q)\,\mathrm{d}q = 0,$$

$$\int_{\mathbb{T}^3} (\cos q_1 + \cos q_2) f(q)\, dq = 0, \qquad \int_{\mathbb{T}^3} \cos q_3 f(q)\, dq = 0.$$

Therefore only the antisymmetric nearest-neighbor component remains:

$$(V_\mu^{\mathrm{ea}} f)(p) = \frac{\mu}{8\pi^3}(\cos p_1 - \cos p_2) \times \int_{\mathbb{T}^3} (\cos q_1 - \cos q_2) f(q)\, dq.$$

In particular, the on-site parameter $\lambda$ does not enter the restricted operator. We denote this restriction by

$$H_\mu^{\mathrm{ea}}(0) = H_0(0) + V_\mu^{\mathrm{ea}}.$$

Set

$$\alpha(p) = \frac{\cos p_1 - \cos p_2}{\sqrt{8\pi^3}}.$$

Then $\| \alpha \| = 1$ and

$$V_\mu^{\mathrm{ea}} f = \mu(f, \alpha)\alpha. \tag{2}$$

Thus,

$$rank V_\mu^{\mathrm{ea}} \le 1.$$

Consequently,

$$\sigma_{\mathrm{ess}}(H_\mu^{\mathrm{ea}}(0)) = [0,24].$$

Moreover, $H_\mu^{\mathrm{ea}}(0)$ can have at most one isolated eigenvalue below 0 and at most one isolated eigenvalue above 24.

**Fredholm determinant**

For $z \in \mathbb{C}\backslash[0,24]$, let $R_0(0,z) = [H_0(0) - zI]^{-1}$.

The eigenvalue equation

$$H_\mu^{\mathrm{ea}}(0) f = zf$$

is equivalent to the Lippmann–Schwinger equation

$$f = -R_0(0,z) V_\mu^{\mathrm{ea}} f. \tag{3}$$

By (2), a nonzero solution of (3) exists if and only if

$$1 + \mu a(z) = 0,$$

where

$$a(z) := (\alpha, R_0(0,z)\alpha) = \frac{1}{8\pi^3} \int_{\mathbb{T}^3} \frac{(\cos p_1 - \cos p_2)^2}{\varepsilon_0(p) - z}\, dp. \tag{4}$$

We therefore define the Fredholm determinant

$$\Delta_\mu(z) = 1 + \mu a(z).$$

**Lemma 1.** *Let $\mu \in \mathbb{R}$ and $z \in \mathbb{R}\backslash[0,24]$. Then $z$ is an eigenvalue of $H_\mu^{ea}(0)$ if and only if*

$$\Delta_\mu(z) = 0.$$

*Proof.* The equation $H_\mu^{\mathrm{ea}}(0) f = zf$ is equivalent to (3). Using (2), we obtain

$$f = -\mu(f, \alpha) R_0(0,z)\alpha.$$

Taking the inner product with $\alpha$ shows that a nonzero solution exists exactly when $1 + \mu a(z) = 0$.

The next lemma contains the properties of $a(z)$ needed below .

**Lemma 2.** *The function $a(z)$ has the following properties:*

1. $a(z) > 0$ for $z < 0$ and $a(z) < 0$ for $z > 24$;
2. $a(z)$ is strictly increasing on $(-\infty, 0)$ and on $(24, +\infty)$;
3. $\lim\limits_{z \to \pm\infty} a(z) = 0$;
4. The finite limits

$$a(0) := \lim_{z \nearrow 0} a(z), \qquad a(24) := \lim_{z \searrow 24} a(z)$$

exist and satisfy

$$a(0) > 0, \qquad a(24) = -a(0).$$

*Proof.* The sign of $a(z)$ follows immediately from (4). Moreover,

$$a'(z) = \frac{1}{8\pi^3}\int_{\mathbb{T}^3} \frac{(\cos p_1 - \cos p_2)^2}{(\mathcal{E}_0(p)-z)^2}\, dp > 0,$$

which proves strict monotonicity. The limits at infinity follow by dominated convergence.

Near $p = 0$,

$$\mathcal{E}_0(p)|p|^2, \qquad \cos p_1 - \cos p_2 = O(|p|^2),$$

so the integrand defining $a(0)$ is locally integrable. The same argument applies at the upper threshold. Finally,

$$\mathcal{E}_0(\pi - p) = 24 - \mathcal{E}_0(p), \qquad \alpha(\pi - p) = -\alpha(p),$$

and therefore

$$a(24 - z) = -a(z).$$

Passing to the threshold limits yields $a(24) = -a(0)$.

Set

$$\mu_* := \frac{1}{a(0)} = \left[\frac{1}{8\pi^3}\int_{\mathbb{T}^3} \frac{(\cos p_1 - \cos p_2)^2}{\mathcal{E}_0(p)}\, dp\right]^{-1} > 0. \tag{5}$$

Then

$$\lim_{z\to\pm\infty} \Delta_\mu(z) = 1,$$

while

$$\lim_{z\nearrow 0} \Delta_\mu(z) = 1 + \frac{\mu}{\mu_*}, \qquad \lim_{z\searrow 24} \Delta_\mu(z) = 1 - \frac{\mu}{\mu_*}. \tag{6}$$

**Main result.**

Let $n_-(H_\mu^{\mathrm{ea}}(0))$ and $n_+(H_\mu^{\mathrm{ea}}(0))$ denote the numbers of eigenvalues of $H_\mu^{\mathrm{ea}}(0)$ in $(-\infty, 0)$ and $(24, +\infty)$, respectively.

**Theorem 1.** *Let $\mu \in \mathbb{R}$ and let $\mu_* > 0$ be defined by (5). Then:*

1) if $\mu < -\mu_*$, then
$$n_-(H_\mu^{\mathrm{ea}}(0)) = 1, \qquad n_+(H_\mu^{\mathrm{ea}}(0)) = 0;$$
2) if $-\mu_* \le \mu \le \mu_*$, then
$$n_-(H_\mu^{\mathrm{ea}}(0)) = n_+(H_\mu^{\mathrm{ea}}(0)) = 0;$$
3) if $\mu > \mu_*$, then
$$n_-(H_\mu^{\mathrm{ea}}(0)) = 0, \qquad n_+(H_\mu^{\mathrm{ea}}(0)) = 1.$$

*Proof.* By Lemma 1, the eigenvalues outside $[0,24]$ are exactly the zeros of $\Delta_\mu(z) = 1 + \mu a(z)$.

Suppose first that $\mu < -\mu_*$. For $z < 0$, Lemma 2 and (6) give

$$\lim_{z\to-\infty} \Delta_\mu(z) = 1 > 0, \qquad \lim_{z\nearrow 0} \Delta_\mu(z) = 1 + \frac{\mu}{\mu_*} < 0.$$

Hence $\Delta_\mu$ has a zero in $(-\infty, 0)$. Since $a(z)$ is strictly increasing and $\mu < 0$, the function $\Delta_\mu(z)$ is strictly decreasing there, so this zero is unique. For $z > 24$ we have $a(z) < 0$ and $\mu < 0$, hence $\Delta_\mu(z) > 1$. Therefore

$$n_-(H_\mu^{\mathrm{ea}}(0)) = 1, \qquad n_+(H_\mu^{\mathrm{ea}}(0)) = 0.$$

Now let $-\mu_* \le \mu \le \mu_*$. If $z < 0$, then $0 < a(z) < a(0)$. For $\mu \ge 0$ one has $\Delta_\mu(z) \ge 1$, while for $-\mu_* \le \mu < 0$,

$$\Delta_\mu(z) > 1 + \mu a(0) = 1 + \frac{\mu}{\mu_*} \ge 0.$$

Thus there are no zeros below 0. Similarly, for $z > 24$,

$$-a(0) < a(z) < 0.$$

If $\mu \le 0$, then $\Delta_\mu(z) \ge 1$, whereas for $0 < \mu \le \mu_*$,

$$\Delta_\mu(z) > 1 - \mu a(0) = 1 - \frac{\mu}{\mu_*} \ge 0.$$

Hence there are no zeros above 24 either.

Finally, let $\mu > \mu_*$. For $z < 0$ we have $a(z) > 0$, so $\Delta_\mu(z) > 1$. Thus there are no eigenvalues below 0. On $(24, +\infty)$,

$$\lim_{z \searrow 24} \Delta_\mu(z) = 1 - \frac{\mu}{\mu_*} < 0, \qquad \lim_{z \to +\infty} \Delta_\mu(z) = 1 > 0.$$

Therefore $\Delta_\mu$ has a zero in $(24, +\infty)$. Since $\mu > 0$ and $a(z)$ is strictly increasing, this zero is unique. Hence

$$n_-(H_\mu^{\mathrm{ea}}(0)) = 0, \qquad n_+(H_\mu^{\mathrm{ea}}(0)) = 1.$$

The proof is complete.

The critical values also have a direct threshold interpretation.

**Corollary 1.** *For $\mu = -\mu_*$, the lower threshold $z = 0$ is an eigenvalue of $H_\mu^{ea}(0)$. For $\mu = \mu_*$, the upper threshold $z = 24$ is an eigenvalue of $H_\mu^{ea}(0)$.*

*Proof.* At $\mu = -\mu_*$ set

$$f_0(p) = \frac{\alpha(p)}{\varepsilon_0(p)}.$$

The behavior used in the proof of Lemma 2 shows that $f_0 \in L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$. Since $(f_0, \alpha) = a(0)$ and $\mu = -1/a(0)$, we obtain

$$H_{-\mu_*}^{\mathrm{ea}}(0) f_0 = 0.$$

The proof at $z = 24$ is analogous, with

$$f_{24}(p) = \frac{\alpha(p)}{\varepsilon_0(p) - 24}.$$

**Conclusion.** For the two-particle lattice Schrödinger operator $\boldsymbol{H_{\lambda\mu}(0)}$ with on-site and nearest-neighbor interactions, we have isolated the even–antisymmetric invariant subspace. The on-site term vanishes on this subspace, and the resulting operator $\boldsymbol{H_\mu^{\mathrm{ea}}(0)}$ is a rank-one perturbation depending only on the nearest-neighbor coupling constant $\boldsymbol{\mu}$. Its discrete spectrum is completely determined by the single critical value $\boldsymbol{\mu_* = 1/a(0)}$. There is one eigenvalue below the essential spectrum for $\boldsymbol{\mu < -\mu_*}$, one eigenvalue above it for $\boldsymbol{\mu > \mu_*}$, and no eigenvalues outside $\boldsymbol{[0, 24]}$ for $\boldsymbol{|\mu| < \mu_*}$. At the critical couplings the corresponding spectral threshold is itself an eigenvalue.

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