UDC 517.95

A MIXED LOCAL-NONLOCAL PROBLEM FOR A DEGENERATE HIGH EVEN-ORDER EQUATION

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Abstract. *Within a rectangular region we formulate and study an initial-boundary value problem, furnished with a combination of local and paired (linked) boundary conditions, for a high even-order partial differential equation whose type is lost at once in the time variable - at each end of the time interval - and in the spatial variable, and which additionally carries a subordinate term. Resting on an energy-integral identity attuned to the temporal degeneracy and rounded out by a Gronwall-type bound, we settle the uniqueness of the solution. Merging spectral analysis with the Green's-function apparatus, we express the solution as a Fourier series in the eigenfunctions of the attendant spectral problem, whose Green's function is written out explicitly and shown to be symmetric. Weighted ℓ_2-type bouns for the Fourier coefficients are supplied, and an estimate of the solution is deduced, from which its continuous dependence on the prescribed data follows.*

Keywords: *initial-boundary value problem, degenerate equation, time degeneracy, local and paired boundary conditions, spectral problem, Green's function method, integral equation, Fourier series.*

BUZILADIGAN YUQORI JUFT TARTIBLI TENGLAMA UCHUN ARALASH CHEGARAVIY SHARTLI MASALA

Annotatsiya. *To'rtburchak sohada yuqori juft tartibli xususiy hosilali tenglama uchun lokal va nolakal chegaraviy shartlar kombinatsiyasi bilan berilgan boshlang'ich-chegaraviy masala qo'yilgan va o'rganilgan. Tenglamaning tipi bir vaqtning o'zida ham vaqt o'zgaruvchisi bo'yicha-vaqt oralig'ining har ikkala uchida-ham fazoviy o'zgaruvchi bo'yicha buziladi, shuningdek, tenglama qo'shimcha kichik yig'indini o'z ichiga oladi. Vaqt bo'yicha buzilishga moslashtirilgan va Gronuoll tipli baholash bilan to'ldirilgan energiya integrali ayniyatiga tayanib, yechimning yagonaligi isbotlangan. Spektral tahlil hamda Grin funksiyasi bilan birgalikda qo'llanilib, yechim tegishli spektral masalaning xos funksiyalari bo'yicha Furye qatori ko'rinishida tasvirlangan. Bunda spektral masalaning Grin funksiyasi oshkor ko'rinishda yozilgan va uning simmetrikligi ko'rsatilgan. Furye koeffitsiyentlari uchun vaznli ℓ_2 tipli baholar keltirilgan va yechimning bahosi olingan bo'lib, undan yechimning berilgan ma'lumotlarga uzluksiz bog'liqligi kelib chiqadi.*

Kalit so'zlar: *boshlang'ich-chegaraviy masala, buziladigan tenglama, vaqt bo'yicha buzilish, lokal va juftlangan chegaraviy shartlar, spektral masala, Grin funksiyasi usuli, integral tenglama, Furye qatori.*

СМЕШАННАЯ ЗАДАЧА ДЛЯ ВЫРОЖДАЮЩЕГОСЯ УРАВНЕНИЯ ВЫСОКОГО ЧЁТНОГО ПОРЯДКА

Аннотация. *В прямоугольной области сформулирована и исследована начально-краевая задача с комбинацией локальных и парных (связанных) краевых условий для уравнения в частных производных высокого чётного порядка. Тип данного уравнения теряется одновременно по временной переменной на каждом из концов временного интервала и по пространственной переменной, причём уравнение дополнительно содержит младший член. На основе тождества энергетического интеграла, согласованного с вырождением по времени и дополненного оценкой типа Гронуолла, установлена единственность решения. Сочетая спектральный анализ с аппаратом функций Грина, решение представлено в виде ряда Фурье по собственным функциям соответствующей спектральной задачи, функция Грина которой выписана в явном виде и показана её симметричность. Приведены весовые оценки типа ℓ_2 для коэффициентов Фурье, и получена оценка решения, из которой следует его непрерывная зависимость от заданных данных.*

Ключевые слова: *начально-краевая задача, вырождающееся уравнение, вырождение по времени, локальные и парные краевые условия, спектральная задача, метод функций Грина, интегральное уравнение, ряд Фурье.*

Introduction. The theory of partial differential equations finds broad application across numerous branches of contemporary science and technology, among them hydrodynamics, the theory of elasticity, and quantum mechanics. At present, this theory is advancing rapidly along a variety of lines. Among these, boundary value problems formulated for even-order partial differential equations of higher order hold a particularly prominent position. A survey of the studies carried out in this direction indicates that initial-boundary value problems for equations involving derivatives of integer order have received considerable attention. For instance, in [1] the author examined the problem of determining a solution of the equation

$$\frac{\partial^{2k}u}{\partial x^{2k}}-\frac{\partial^2 u}{\partial t^2}=f(x,t) \tag{1}$$

in the domain $\Delta=\{(x,t):0<x<p,\,0<t<T\}$. Here the sought solution is required to obey the boundary conditions

$$\frac{\partial^{2m}u}{\partial x^{2m}}(0,t)=\frac{\partial^{2m}u}{\partial x^{2m}}(p,t)=0,\quad m=\overline{0,k-1},\ 0\le t\le T,\ k\ge 2, \tag{2}$$

together with one of the following pairs of initial conditions:

$$\begin{cases} u(x,0)=u(x,T)=0, & 0\le x\le p,\\ u(x,0)=u_t(x,T)=0, & 0\le x\le p,\\ u(x,0)=u(x,T),\ u_t(x,0)=u_t(x,T), & 0\le x\le p. \end{cases} \tag{3}$$

The problem stated above was therein subjected to a thorough investigation. Moreover, in [1] a further problem was posed for equation (1) under the prescribed data and the boundary conditions

$$\frac{\partial^{2m+1}u}{\partial x^{2m+1}}(0,t)=\frac{\partial^{2m+1}u}{\partial x^{2m+1}}(p,t)=0,\quad m=\overline{0,k-1},\ 0<t<T,\ k\ge 2.$$

In [2], the problem for the equation in the domain Δ, endowed with particular boundary conditions and the initial data $u(x,0)=u_t(x,0)=0,\ 0\le x\le p$, was analyzed. In [3], an ill-posed problem for the equation in the domain $D=\{(x,t):0<x<\pi;\,0<t<2\pi\}$ with $f(x,t)\equiv 0$ and $k=2n+1,\ n\in\mathbb{N}$ was treated under the boundary and initial conditions

$$\frac{\partial^{2m}u}{\partial x^{2m}}(0,t)=\frac{\partial^{2m}u}{\partial x^{2m}}(\pi,t)=0,\ m=0,1,\ldots,k-1;\ u(\alpha\pi,t)=f(t),\quad 0<t<2\pi.$$

In [4], a problem was considered for the quasilinear equation

$$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^{2k}u}{\partial x^{2k}}=f(t,x,u),\quad k\in\mathbb{N},$$

in the domain $D_T=\{(t,x):0<t<T;\,0<x<\pi\}$, supplemented by the initial conditions $u(0,x)=\varphi(x)$, $u_t(T,x)=\psi(x)$, $0<x<\pi$, and by boundary conditions of periodic type in x. The Fourier method was employed to solve the problem, while the existence and uniqueness of the solution were established by means of the method of successive approximations in a Banach space. In [5], problems for the equation

$$\frac{\partial^{2k}u}{\partial x^{2k}}+\frac{\partial^2 u}{\partial t^2}=f(x,t)$$

in the domain Δ, subject to specific conditions and $u(x,0)=u_t(x,0)=0$, were investigated, it was demonstrated therein that the solvability of the problem is governed by the parity of the number k. Finally, in [6], initial-boundary value problems were studied for the equation $u_{tt}+A(x,D)u(x,t)=f(x,t)$, where $A(x,D)=\sum_{|\alpha|\le m}a_\alpha(x)D^\alpha$ is an arbitrary positive, formally self-adjoint elliptic differential operator of order $m=2l$ with sufficiently smooth coefficients, the problem is resolved in the form of a series by invoking the classical Fourier method together with the associated spectral problem, and the existence, uniqueness, and stability of the solution were proved.

The works cited so far concern equations whose leading spatial part is non-degenerate. A separate and more delicate line of research addresses equations in which the higher-order spatial operator degenerates

through a weight that vanishes on the boundary. Thus, in [7] a local initial-boundary value problem was analysed for an equation combining the Bessel operator with the operator $(-1)^k \partial^k / \partial x^k [x^\alpha \partial^k u / \partial x^k]$.

In [8], for a high even-order equation of order $2k$ carrying a Bessel operator in a rectangular domain,

$$B_t^{\gamma-1/2} u + (-1)^k \frac{\partial^{2k} u}{\partial x^{2k}} = f(x,t), \qquad B_t^{\gamma-1/2} \equiv \frac{\partial^2}{\partial t^2} + \frac{2\gamma}{t}\frac{\partial}{\partial t},$$

a pair of nonlocal initial-boundary value problems was set with requirements linking the values and the higher-order derivatives at the two endpoints $x=0$ and $x=1$, for one of them the existence, the uniqueness, and the continuous dependence of the solution upon the data were proved through the Fourier method, the Green's function, and spectral analysis.

A degenerate high even-order operator of the form $\partial^{2n} / \partial x^{2n} \left(x^\alpha \partial^{2n} u / \partial x^{2n} \right)$ underlies a further group of studies. For the equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^{2n}}{\partial x^{2n}} \left(x^\alpha \frac{\partial^{2n} u}{\partial x^{2n}} \right) = f(x,t), \tag{4}$$

the study [9] posed a problem under local boundary requirements imposing, at each endpoint $x=0$ and $x=1$, the vanishing of the even-order derivatives of u and of $x^\alpha \partial^{2n} u / \partial x^{2n}$. For the same equation (4), the work [11] set and investigated a problem under local conditions prescribed separately at the two endpoints (values of u at $x=0$ and of its $(n+j)$-th derivatives at $x=1$), while [10] treated, for the same family of degenerate equations, an initial-boundary value problem carrying yet another set of local conditions and justified the solvability of the linear system governing the Green's-function coefficients. The study [12], again for equation (4), examined a problem in which the boundary conditions link, at each endpoint, the even-order derivatives of u with its odd-order ones (and likewise for $x^\alpha \partial^{2n} u / \partial x^{2n}$), for $j = \overline{0, n-1}$ and $t \in [0,T]$.

A further, doubly degenerate operator,

$$(-1)^n \frac{\partial^n}{\partial x^n} \left[x^\alpha (1-x)^\beta \frac{\partial^n u(x,t)}{\partial x^n} \right],$$

whose degeneracy at both endpoints $x=0$ and $x=1$ is produced by the weight $x^\alpha (1-x)^\beta$, underlies the initial-boundary value problems treated in [13, 14]. The equation examined in [13] places a Caputo derivative alongside a Riemann—Liouville integral,

$${}^C D_{0t}^{\alpha_1} u + a_1 I_{0t}^{\beta_1} u + b_1 u + (-1)^n \frac{\partial^n}{\partial x^n} \left[x^\alpha (1-x)^\beta \frac{\partial^n u}{\partial x^n} \right] = 0,$$

whereas the one considered in [14] attaches a further factor t^γ to the Caputo derivative,

$$t^\gamma {}^C D_{0t}^{\delta} u + b_1 u + (-1)^n \frac{\partial^n}{\partial x^n} \left[x^\alpha (1-x)^\beta \frac{\partial^n u}{\partial x^n} \right] = 0, \quad 0 \le \gamma < \delta < 1,$$

whereby the degeneracy extends beyond x to the line $t=0$ as well. Both studies impose boundary and initial data of one and the same shape,

$$\left. \frac{\partial^j u}{\partial x^j} \right|_{x=0} = 0, \quad \left. \frac{\partial^j}{\partial x^j} \left[x^\alpha (1-x)^\beta \frac{\partial^n u}{\partial x^n} \right] \right|_{x=1} = 0, \quad j = 0,1,\ldots,n-1, \qquad u(x,0) = \varphi(x),$$

and in each of them the solution is shown to be unique through energy identities and is built by the Fourier method, the accompanying spectral problem being reduced, via its Green's function, to a Fredholm integral equation with a symmetric kernel. Consequently, the solution of [13] is written out by means of the two-variable Mittag-Leffler-type function $E_2(\cdot,\cdot)$, and that of [14] through the three-parameter Kilbas-Saigo function. In either setting, the uniform convergence of the corresponding Fourier series and the continuous dependence of the solution upon the prescribed data are secured.

A boundary value problem on a rectangle for an equation in which a third-order derivative is combined with a degenerate high even-order operator-the accompanying weight vanishing on the boundary of the domain-was treated in [15]. In [16] a Dirichlet-type problem was addressed for a degenerate high-order equation of fractional order, there Mittag-Leffler functions were brought into use, and both the convergence of the solution series and the estimates for the Green's function were secured by asymptotic

expansions. Furthermore, an equation of the form $t^m u_{tt} + at^{m-1}u_t + bu + [x^\alpha (1-x)^\beta u_{xx}]_{xx} = 0$, which possesses three lines of degeneracy, was examined within an initial-boundary value problem in [17], while in [18] the fractional-order equation $t^b\, {}^C D_{0t}^\gamma u - (x^\alpha (1-x)^\beta u_x)_x = f$ was investigated, a direct as well as an inverse problem being considered under suitable restrictions on the parameters.

To the authors' knowledge, however, no previous work has treated an equation that simultaneously brings together a first-order derivative in time, a two-sided temporal degeneracy expressed by the weight $t^a(1-t)^b$, a lower-order term cu, and a spatial degeneracy of high even order. In contrast to [17], where a *second*-order time derivative together with three lines of degeneracy is treated, and to [14], where the time derivative is of *fractional* order, the equation considered here retains an ordinary first-order time derivative alongside the two-sided weight $t^a(1-t)^b$. The novelty of the present study lies precisely in setting up and analyzing such a problem, supplied with boundary conditions of a mixed type that combine local requirements with paired (linked) ones relating the values at $x=0$ to those at $x=1$.

In the rectangle $\Omega = \{(x,t): 0<x<1;\ 0<t<1\}$ we consider the equation

$$t^a(1-t)^b \frac{\partial u}{\partial t} + cu + \frac{\partial^{2n}}{\partial x^{2n}}\left(x^\alpha \frac{\partial^{2n} u(x,t)}{\partial x^{2n}} \right) = f(x,t), \tag{5}$$

where a, b, c, α are prescribed real numbers, $n \in \mathbb{N}$, $0 \le a < 1$, $0 \le b < 1$, $c \ge 0$, $0 < \alpha < 1$, $u = u(x,t)$ is the unknown function, and $f(x,t)$ is a given sufficiently smooth function.

Statement of the problem. *Find a function* $u(x,t)$ *that meets the following requirements:*

1) the following continuity requirements hold: $u(x,t), \partial u/\partial x, \partial^2 u/\partial x^2, \ldots, \partial^{2n-1}u/\partial x^{2n-1} \in C(\bar{\Omega})$, $x^\alpha \partial^{2n}u/\partial x^{2n}, (\partial/\partial x)(x^\alpha \partial^{2n}u/\partial x^{2n}), \ldots, (\partial^{2n-1}/\partial x^{2n-1})(x^\alpha \partial^{2n}u/\partial x^{2n}) \in C(\bar{\Omega})$, $t^a(1-t)^b \partial u/\partial t \in C(\Omega)$ and $(\partial^{2n}/\partial x^{2n})(x^\alpha \partial^{2n}u/\partial x^{2n}) \in C(\Omega)$;

2) the function $u(x,t)$ satisfies equation (5) in Ω;

3) it fulfils the initial condition

$$u(x,0) = \varphi(x), \qquad x \in [0,1], \tag{6}$$

where $\varphi(x)$ is a prescribed continuous function, together with the boundary conditions

$$\begin{cases} \dfrac{\partial^j u(0,t)}{\partial x^j} = 0,\ \dfrac{\partial^{n+j} u(0,t)}{\partial x^{n+j}} + \dfrac{\partial^{n+j} u(1,t)}{\partial x^{n+j}} = 0,\ j = \overline{0,n-1},\ t \in [0,1]; \\ \left(x^\alpha \dfrac{\partial^{2n} u}{\partial x^{2n}} \right)^{(j)}\Bigg|_{x=0} + \left(x^\alpha \dfrac{\partial^{2n} u}{\partial x^{2n}} \right)^{(j)}\Bigg|_{x=1} = 0,\ \left(x^\alpha \dfrac{\partial^{2n} u}{\partial x^{2n}} \right)^{(n+j)}\Bigg|_{x=1} = 0,\ j = \overline{0,n-1}, t \in [0,1]. \end{cases} \tag{7}$$

Investigation of the spectral problem

Carrying out the separation of variables in the problem under consideration, one is brought to the spectral problem

$$M[v] \equiv \left(x^\alpha v^{(2n)}(x) \right)^{(2n)} = \lambda v(x), \qquad 0 < x < 1, \tag{8}$$

$$\begin{cases} v^{(j)}(x),\ \left(x^\alpha v^{(2n)}(x) \right)^{(j)} \in C[0,1],\ j = \overline{0,2n-1}; \\ v^{(j)}(0) = 0,\ v^{(n+j)}(0) + v^{(n+j)}(1) = 0,\ j = \overline{0,n-1}; \\ \left(x^\alpha v^{(2n)}(x) \right)^{(j)}\Big|_{x=0} + \left(x^\alpha v^{(2n)}(x) \right)^{(j)}\Big|_{x=1} = 0,\ \left(x^\alpha v^{(2n)}(x) \right)^{(n+j)}\Big|_{x=1} = 0,\ j = \overline{0,n-1}. \end{cases} \tag{9}$$

Throughout, $M[v]$ denotes the differential expression $\left(x^\alpha (v^{(2n)}(x) \right)^{(2n)}$ applied to any sufficiently smooth function, not only to eigenfunctions, in particular Mg in the lemmas below is this same expression applied to g.

Self-adjointness of the operator. Let $v(x)$ and $w(x)$ be functions satisfying conditions (9), and $Mv(x), Mw(x) \in L_2(0,1)$. Then, applying integration by parts $4n$ times, we have

$$\int_0^1 w(x) Mv(x)\,dx = \Big[w(x) \left(x^\alpha v^{(2n)}(x) \right)^{(2n-1)} - w'(x) \left(x^\alpha v^{(2n)}(x) \right)^{(2n-2)} + w''(x) \left(x^\alpha v^{(2n)}(x) \right)^{(2n-3)}$$

$$-\ldots-w^{(2n-1)}(x)\left(x^{\alpha}v^{(2n)}(x)\right)+\left(x^{\alpha}w^{(2n)}(x)\right)v^{(2n-1)}(x)-\left(x^{\alpha}w^{(2n)}(x)\right)'v^{(2n-2)}(x)$$

$$+\left(x^{\alpha}w^{(2n)}(x)\right)''v^{(2n-3)}(x)-\ldots-\left(x^{\alpha}w^{(2n)}(x)\right)^{(2n-1)}v(x)\Big]_{x=0}^{x=1}+\int_0^1 v(x)Mw(x)dx.$$

Hence, by virtue of the properties of the functions $v(x)$ and $w(x)$, the following equality holds:

$$\int_0^1 w(x)Mv(x)dx=\int_0^1 v(x)Mw(x)dx.$$

Consequently, the problem $\{Mv=0,\ (9)\}$ is *self-adjoint*.

Let $v(x)\not\equiv 0$, $x\in[0,1]$ be a solution to the problem {(8), (9)}. Then, multiplying both sides of equality (8) by $v(x)$ and integrating from 0 to 1, we have

$$\lambda\int_0^1 v^2(x)dx=\int_0^1 v(x)\left(x^{\alpha}v^{(2n)}(x)\right)^{(2n)}dx=\Big[v(x)\left(x^{\alpha}v^{(2n)}(x)\right)^{(2n-1)}-v'(x)\left(x^{\alpha}v^{(2n)}(x)\right)^{(2n-2)}+\ldots$$

$$\ldots-v^{(2n-1)}(x)\left(x^{\alpha}v^{(2n)}(x)\right)\Big]\Big|_{x=0}^{x=1}+\int_0^1 x^{\alpha}\left[v^{(2n)}(x)\right]^2dx=\int_0^1 x^{\alpha}\left[v^{(2n)}(x)\right]^2dx,$$

i.e.,

$$\lambda\int_0^1 v^2(x)dx=\int_0^1 x^{\alpha}\left[v^{(2n)}(x)\right]^2dx. \tag{10}$$

By condition (9), $x^{\alpha}v^{(2n)}(x)\in C[0,1]$, so $\left|x^{\alpha}v^{(2n)}(x)\right|\le C_0$ on $[0,1]$ for some constant C_0, and therefore

$$\int_0^1 x^{\alpha}\left[v^{(2n)}(x)\right]^2dx=\int_0^1 x^{-\alpha}\left[x^{\alpha}v^{(2n)}(x)\right]^2dx,$$

since $0<\alpha<1$. Thus the right-hand integral in (10) remains finite despite the degeneracy at $x=0$, and consequently the eigenvalue λ must satisfy $\lambda\ge 0$. If $\lambda=0$, then from the last equality it follows that $v^{(2n)}(x)=0$, $0<x<1$. Then

$$v(x)=c_1\frac{x^{2n-1}}{(2n-1)!}+c_2\frac{x^{2n-2}}{(2n-2)!}+\ldots+c_{2n-1}\frac{x}{1!}+c_{2n},\quad x\in(0,1),$$

where c_j are arbitrary real numbers. Subjecting this function to the conditions $v^{(j)}(0)=0,\quad v^{(n+j)}(0)+v^{(n+j)}(1)=0,\quad j=\overline{0,n-1}$, we obtain $c_j=0,\ j=\overline{1,2n}$. Then $v(x)\equiv 0$, $0\le x\le 1$. Consequently, problem {(8), (9)} can have nontrivial solutions only when $\lambda>0$.

Construction of the Green's function.

Let $\lambda>0$. We demonstrate that {(8), (9)} possesses eigenvalues by resorting to the method of the Green's function. The Green's function $G(x,s)$ attached to the problem is required to enjoy the following properties:

1) the functions $(\partial^j/\partial x^j)G(x,s)$, $j=\overline{0,2n-1}$, and $(\partial^j/\partial x^j)[x^{\alpha}(\partial^{2n}/\partial x^{2n})G(x,s)]$, $j=\overline{0,2n-2}$, are continuous for all $x,s\in[0,1]$;

2) on each of $[0,s)$ and $(s,1]$ the derivative $(\partial^{2n-1}/\partial x^{2n-1})[x^{\alpha}(\partial^{2n}/\partial x^{2n})G(x,s)]$ is continuous, while at $x=s$ it has the jump

$$\left(\partial^{2n-1}/\partial x^{2n-1}\right)\left[x^{\alpha}\left(\partial^{2n}/\partial x^{2n}\right)G(x,s)\right]\Big|_{x=s-0}^{x=s+0}=1; \tag{11}$$

3) on $(0,s)$ and $(s,1)$ the derivative $MG(x,s)$ exists (in x) and $MG(x,s)=0$;

4) for $s\in(0,1)$, as a function of x, it satisfies

$$\frac{\partial^j G(0,s)}{\partial x^j}=0,\quad \frac{\partial^{n+j}G(0,s)}{\partial x^{n+j}}+\frac{\partial^{n+j}G(1,s)}{\partial x^{n+j}}=0\quad j=\overline{0,n-1}; \tag{12}$$

$$\left(x^{\alpha}\frac{\partial^{2n}G(x,s)}{\partial x^{2n}}\right)^{(j)}\Big|_{x=0}+\left(x^{\alpha}\frac{\partial^{2n}G(x,s)}{\partial x^{2n}}\right)^{(j)}\Big|_{x=1}=0,\quad \left(x^{\alpha}\frac{\partial^{2n}G(x,s)}{\partial x^{2n}}\right)^{(n+j)}\Big|_{x=1}=0,\quad j=\overline{0,n-1}. \tag{13}$$

Guided by the structure of the general solution of $MG(x,s)=0$ on the two subintervals $(0,s)$ and $(s,1)$, we look for $G(x,s)$ in the shape

$$G(x,s)=\begin{cases}\displaystyle\sum_{j=1}^{2n}\frac{a_j x^{4n-\alpha-j}}{(2n-j)!(2n-\alpha-j+1)_{2n}}+\sum_{j=1}^{2n}\frac{a_{2n+j}x^{2n-j}}{(2n-j)!}, & 0\le x\le s,\\ \displaystyle\sum_{j=1}^{2n}\frac{b_j x^{4n-\alpha-j}}{(2n-j)!(2n-\alpha-j+1)_{2n}}+\sum_{j=1}^{2n}\frac{b_{2n+j}x^{2n-j}}{(2n-j)!}, & s\le x\le 1,\end{cases} \tag{14}$$

where a_j and b_j, $j=\overline{1,4n}$, are unknown functions of s, and $(z)_n=z(z+1)\cdots(z+n-1)$ is the Pochhammer symbol [19]. Requiring that properties 1) and 2) hold and that the jump relation (11) be satisfied, we are led to the following system governing the differences b_j-a_j:

$$\begin{cases} b_1-a_1=1,\\ \displaystyle\sum_{j=1}^{k}\frac{(b_j-a_j)s^{k-j}}{(k-j)!}=0, \qquad k=\overline{2,2n},\\ \displaystyle\sum_{j=1}^{2n}\frac{(b_j-a_j)s^{2n+m-j-\alpha}}{(2n-j)!(2n-j+1-\alpha)_m}+\sum_{j=1}^{m}\frac{(b_{2n+j}-a_{2n+j})s^{m-j}}{(m-j)!}=0, \quad m=\overline{1,2n}\end{cases} \tag{15}$$

This system admits the unique solution [9]:

$$b_j-a_j=\frac{(-1)^{j-1}s^{j-1}}{(j-1)!}, \quad b_{2n+j}-a_{2n+j}=\frac{(-1)^{j-1}s^{2n+j-1-\alpha}}{(j-1)!(j-\alpha)_{2n}}, \quad j=\overline{1,2n}. \tag{16}$$

Substituting the second part of condition (13) into (14), we obtain

$$b_j=0,\ j=\overline{1,n}, \tag{17}$$

and, taking (17) into account, it follows from (16) that

$$a_j=\frac{(-1)^j s^{j-1}}{(j-1)!}, \qquad j=\overline{1,n}. \tag{18}$$

Substituting the first part of (13) into (14), we correspondingly arrive at the equality

$$a_j+b_j=-\sum_{i=1}^{j-1}\frac{b_i}{(j-i)!}, \qquad j=\overline{n+1,2n}. \tag{19}$$

Now, combining b_j-a_j, b_j+a_j $(j=\overline{n+1,2n})$ from (16) together with (19) into a single system, we obtain, with respect to a_j and b_j, the solutions

$$a_j=-\frac{1}{2}\left(\sum_{i=1}^{j-1}\frac{b_i}{(j-i)!}+\frac{(-1)^{j-1}s^{j-1}}{(j-1)!}\right), \quad b_j=-\frac{1}{2}\left(\sum_{i=1}^{j-1}\frac{b_i}{(j-i)!}-\frac{(-1)^{j-1}s^{j-1}}{(j-1)!}\right), \quad j=\overline{n+1,2n} \tag{20}$$

Subjecting the first part of equality (12) to (14), we obtain

$$a_{4n-j}=0, \qquad j=\overline{0,n-1}; \tag{21}$$

taking (21) and (16) into account, the following equality is obtained:

$$b_{4n-j}=\frac{(-1)^{2n-j-1}s^{4n-j-1-\alpha}}{(2n-j-1)!(2n-j-\alpha)_{2n}}, \qquad j=\overline{0,n-1}. \tag{22}$$

Subjecting the second part of equality (12) to (14) accordingly, we get

$$a_{2n+j}+b_{2n+j}=-\left(\sum_{i=1}^{2n}\frac{b_i}{(2n-i)!(2n-\alpha-i+1)_j}+\sum_{k=1}^{j-1}\frac{b_{2n+k}}{(j-k)!}\right), \qquad j=\overline{1,n}, \tag{23}$$

where, for $j=1$ in (23),

$$a_{2n+1}+b_{2n+1}=-\sum_{i=1}^{2n}\frac{b_i}{(2n-i)!(2n-\alpha-i+1)}.$$

Reducing the corresponding parts of (23) and (16) to a single system, we obtain, for a_{2n+j} and b_{2n+j}, the following equalities:

$$a_{2n+j}=\frac{1}{2}\left(a_{2n+j}+b_{2n+j}-\frac{(-1)^{j-1}s^{2n+j-1-\alpha}}{(j-1)!(j-\alpha)_{2n}}\right),\quad b_{2n+j}=\frac{1}{2}\left(a_{2n+j}+b_{2n+j}+\frac{(-1)^{j-1}s^{2n+j-1-\alpha}}{(j-1)!(j-\alpha)_{2n}}\right). \tag{24}$$

Thus, the Green's function satisfying properties 1)-4) exists and is unique, having the form (14), where the coefficients a_j and b_j, $j=\overline{1,4n}$, are determined by formulas (17), (18), (20), (21), (22), (24).

The case $n=1$**.**

$$G(x,s)=\begin{cases}-\dfrac{x^{3-\alpha}}{(2-\alpha)_2}+\dfrac{sx^{2-\alpha}}{2(1-\alpha)_2}+\dfrac{sx}{4(1-\alpha)}-\dfrac{s^{2-\alpha}x}{2(1-\alpha)_2}, & x<s,\\ -\dfrac{sx^{2-\alpha}}{2(1-\alpha)_2}+\dfrac{sx}{4(1-\alpha)}+\dfrac{s^{2-\alpha}x}{2(1-\alpha)_2}-\dfrac{s^{3-\alpha}}{(2-\alpha)_2}, & x>s.\end{cases}$$

The case $n=2$**.**

$$G(x,s)=\begin{cases}-\dfrac{x^{7-\alpha}}{3!(4-\alpha)_4}+\dfrac{sx^{6-\alpha}}{2!(3-\alpha)_4}-\dfrac{s^2x^{5-\alpha}}{4(2-\alpha)_4}-\dfrac{s^2x^{4-\alpha}}{8(1-\alpha)_4}+\dfrac{s^3x^{4-\alpha}}{12(1-\alpha)_4}+\dfrac{\alpha s^2x^3}{16\cdot 3!(1-\alpha)_2}+\dfrac{s^3x^3}{24\cdot 3!(1-\alpha)}-\\ -\dfrac{s^{4-\alpha}x^3}{2\cdot 3!(1-\alpha)_4}+\dfrac{\left(\alpha^2-\alpha+2\right)s^2x^2}{32\cdot 2!(1-\alpha)_3}+\dfrac{\alpha s^3x^2}{48\cdot 2!(1-\alpha)_2}-\dfrac{s^{4-\alpha}x^2}{4\cdot 2!(1-\alpha)_4}+\dfrac{s^{5-\alpha}x^2}{2\cdot 2!(2-\alpha)_4}, & x<s,\\ -\dfrac{s^{7-\alpha}}{3!(4-\alpha)_4}+\dfrac{s^{6-\alpha}x}{2(3-\alpha)_4}-\dfrac{s^{5-\alpha}x^2}{2\cdot 2!(2-\alpha)_4}-\dfrac{s^{4-\alpha}x^2}{4\cdot 2!(1-\alpha)_4}+\dfrac{s^{4-\alpha}x^3}{2\cdot 3!(1-\alpha)_4}+\dfrac{\alpha s^3x^2}{48\cdot 2!(1-\alpha)_2}+\\ +\dfrac{s^3x^3}{24\cdot 3!(1-\alpha)}-\dfrac{s^3x^{4-\alpha}}{12(1-\alpha)_4}+\dfrac{\left(\alpha^2-\alpha+2\right)s^2x^2}{32\cdot 2!(1-\alpha)_3}+\dfrac{\alpha s^2x^3}{16\cdot 3!(1-\alpha)_2}-\dfrac{s^2x^{4-\alpha}}{8(1-\alpha)_4}+\dfrac{s^2x^{5-\alpha}}{4(2-\alpha)_4}, & x>s.\end{cases}$$

Symmetry of the Green's function.

We now prove that $G(x,s)$ is symmetric in the general case. It is known that the problem $\{Mv=0$, (9)$\}$ is self-adjoint, therefore, its Green's function (14) is symmetric with respect to its arguments x and s.

Let us prove this assertion. Suppose $v(x),h(x)\in C^{2n-1}[0,1]$ and $x^\alpha v^{(2n)}(x)$, $x^\alpha h^{(2n)}(x)\in C^{2n-1}[0,1]\cap C^{2n}(0,1)$. We introduce the following notations:

$$Mv(x)\equiv\left(x^\alpha v^{(2n)}(x)\right)^{(2n)}=f(x),\ \ Mh(x)\equiv\left(x^\alpha h^{(2n)}(x)\right)^{(2n)}=g(x).$$

Then the following equality holds:

$$h(x)Mv(x)-v(x)Mh(x)\ =h(x)\left(x^\alpha v^{(2n)}(x)\right)^{(2n)}-v(x)\left(x^\alpha h^{(2n)}(x)\right)^{(2n)}=$$

$$=\sum_{j=0}^{2n-1}\frac{d}{dx}\left\{(-1)^j\left[h^{(j)}(x)\left(x^\alpha v^{(2n)}(x)\right)^{(2n-1-j)}-v^{(j)}(x)\left(x^\alpha h^{(2n)}(x)\right)^{(2n-1-j)}\right]\right\}=f(x)h(x)-g(x)v(x),\ \ 0<x<1 \tag{25}$$

.

Assuming $v(x)=G(x,s)$ and $h(x)=G(x,\xi)$, at all points of the interval $(0,1)$ except for $x\neq\xi$ and $x\neq s$, the relations $Mv(x)=0$ and $Mh(x)=0$ hold. Then equality (25) takes the form:

$$\sum_{j=0}^{2n-1}\frac{d}{dx}\left\{(-1)^j\left[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)-\right.\right.$$

$$\left.\left.-\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\right]\right\}=0,\quad x\in(0,1)\setminus\{s,\xi\} \tag{26}$$

Without loss of generality, let us assume $s<\xi$. Then the interval $[0,1]$ is divided into three subintervals: $[0,s]$, $[s,\xi]$, and $[\xi,1]$. Integrating equality (26) over these subintervals yields:

$$\sum_{j=0}^{2n-1}\left\{(-1)^j\left[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)-\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\right]\right\}_{x=0}^{x=s-0}+$$

$$+\sum_{j=0}^{2n-1}\left\{(-1)^j\left[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)-\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\right]\right\}_{x=s+0}^{x=\xi-0}+$$

$$+\sum_{j=0}^{2n-1}\left\{(-1)^j\left[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)-\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\right]\right\}_{x=\xi+0}^{x=1}=0.$$

Taking into account properties 1) and 4) of Green's function $G(x,s)$, the last equality reduces to:

$$-\left[G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)\right]\Bigg|_{x=s-0}^{x=s+0}+\left[G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\right]\Bigg|_{x=s-0}^{x=s+0}-$$

$$-\left[G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)\right]\Bigg|_{x=\xi-0}^{x=\xi+0}+\left[G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\right]\Bigg|_{x=\xi-0}^{x=\xi+0}=0.$$

According to property 2) of Green's function, the derivative $\left(\partial^{2n-1}/\partial x^{2n-1}\right)\left[x^\alpha\left(\partial^{2n}/\partial x^{2n}\right)G(x,\eta)\right]$ is continuous for $x\neq\eta$. Accounting for this continuity, we obtain:

$$\left[G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)\Bigg|_{x=s-0}-G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\right)\Bigg|_{x=s+0}\right]+$$

$$+\left[G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\Bigg|_{x=\xi+0}-G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\left(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\right)\Bigg|_{x=\xi-0}\right]=0.$$

Hence, by virtue of equality (11), we arrive at $-G(s,\xi)+G(\xi,s)=0$, which completes the proof.

Reduction to an integral equation. Following the technique set out in [20, §3, pp. 40-42], one shows that problem {(8), (9)} is equivalent to the integral equation

$$v(x)=\lambda\int_0^1 G(x,s)v(s)ds. \tag{27}$$

The proof follows the same line of reasoning as that of Theorem 2 in (21 §3, pp. 40-42). Although the boundary conditions considered there differ from (9), the argument remains valid in the present case, since it uses only Properties 1)-4) of the Green function established above. Repeating the proof of Theorem 2 with the Green function constructed here, one verifies that the function

$$v(x)=\int_0^1 G(x,s)\,f(s)ds$$

satisfies $M[v]=f$ together with the boundary conditions (9). Therefore, setting $f(x)=\lambda v(x)$, we arrive at the homogeneous Fredholm integral equation (27).

This equivalence is established because $G(x,s)$ is the Green's function of the differential operator M under the specified boundary conditions, which allows the differential operator to be represented as an inverse integral operator. Since the kernel $G(x,s)$ is real, symmetric (as shown above) and continuous, by the theory of symmetric integral equations the corresponding integral operator possesses a countable set of characteristic numbers $|\lambda_1|\leqslant|\lambda_2|\leqslant\ldots\leqslant|\lambda_k|\leqslant\ldots$, (see [20], §26, pp. 140-145), and, by the Hilbert-Schmidt theorem (see [20], §27, Theorem 1, p. 150), the corresponding system of eigenfunctions is complete in $L_2(0,1)$. Recall that, as shown above, every eigenvalue satisfies $\lambda>0$; hence the eigenvalues can be arranged as

$$0<\lambda_1\leqslant\lambda_2\leqslant\lambda_3\leqslant\ldots\leqslant\lambda_k<\ldots,\quad \lambda_k\to+\infty,$$

and the corresponding system of eigenfunctions $v_1(x),v_2(x),v_3(x),\ldots,v_k(x)\ldots$ forms an orthonormal system in space $L_2(0,1)$ [20].

In addition, it is not difficult to verify that the system of functions $s^{\alpha/2}v_k^{(n)}(s)/\sqrt{\lambda}_k$ also $k=1,2,\ldots$ forms an orthonormal system in $L_2(0,1)$.

The following lemma demonstrates which type of functions can be expanded into series by the system of eigenfunctions of the problem {(8), (9)}.

Auxiliary lemmas. In all that follows, g_k stands for the Fourier coefficient of the function $g(x)$,

$$g_k = \int_0^1 g(x)v_k(x)dx, \qquad k \in \mathbb{N}. \tag{28}$$

Lemma 2.1. Let $g(x)$ comply with conditions (9) and let $Mg(x) \in C(0,1) \cap L_2(0,1)$. Then $g(x)$ can be expanded on $[0,1]$ into an absolutely and uniformly convergent series over the eigenfunctions of problem {(8), (9)}

Proof. By integrating by parts and drawing on the properties of $G(x,s)$ along with the hypotheses placed on g, one confirms the identity

$$\int_0^1 G(x,s)Mg(s)ds = \int_0^1 G(x,s)\left[s^\alpha g^{(2n)}(s)\right]^{(2n)} ds = g(x).$$

As $Mg(x) \in L_2(0,1)$, the identity establishes that g can be represented through the kernel $G(x,s)$. Since, moreover, $G(x,s)$ is continuous on $\bar{\Omega}$, the claim is a consequence of Theorem 2 of [20, p. 153]. Lemma 2.1 has been proved.

Lemma 2.2. The series

$$\sum_{k=1}^{+\infty} \frac{\left[v_k^{(j)}(x)\right]^2}{\lambda_k}, \qquad \sum_{k=1}^{+\infty} \frac{\left(\left[x^\alpha v_k^{(2n)}(x)\right]^{(j)}\right)^2}{\lambda_k^2}, \qquad j = \overline{0, 2n-1}, \tag{29}$$

converge uniformly on $[0,1]$.

Proof. Using (8) and the properties of the Green's function, from (27) with $v(x) \equiv v_k(x)$ one derives

$$v_k^{(j)}(x) = \lambda_k \int_0^1 \frac{\partial^j}{\partial x^j} G(x,s)v_k(s)ds = \int_0^1 \left[s^\alpha v_k^{(2n)}(s)\right]^{(2n)} \frac{\partial^j}{\partial x^j} G(x,s)ds.$$

After $2n$ integrations by parts and an appeal to (9),

$$v_k^{(j)}(x) = \int_0^1 s^\alpha v_k^{(2n)}(s) \frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s)ds,$$

whence, as $\lambda_k > 0$,

$$\frac{v_k^{(j)}(x)}{\sqrt{\lambda_k}} = \int_0^1 \left(s^{\alpha/2} \frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s) \right) \left(\frac{s^{\alpha/2} v_k^{(2n)}(s)}{\sqrt{\lambda_k}} \right) ds. \tag{30}$$

Thus $v_k^{(j)}(x)/\sqrt{\lambda_k}$ is the Fourier coefficient of $s^{\alpha/2}\partial^{2n+j}G(x,s)/(\partial x^j \partial s^{2n})$ in the orthonormal system $\{s^{\alpha/2}v_k^{(2n)}(s)/\sqrt{\lambda_k}\}$, and Bessel's inequality [see: [22], pp. 23-24)] gives

$$\sum_{k=1}^{+\infty} \frac{\left[v_k^{(j)}(x)\right]^2}{\lambda_k} \le \int_0^1 s^\alpha \left[\frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s) \right]^2 ds, \quad j = \overline{0, 2n-1}. \tag{31}$$

The integral on the right-hand side of (31) can be rewritten as

$$\int_0^1 s^\alpha \left[\frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s) \right]^2 ds = \int_0^1 s^{-\alpha} \left[\frac{\partial^j}{\partial x^j} \left(s^\alpha \frac{\partial^{2n}}{\partial s^{2n}} G(x,s) \right) \right]^2 ds, \quad j = \overline{0, 2n-1}.$$

Since

$$s^\alpha \frac{\partial^{2n} G(x,s)}{\partial s^{2n}}, \quad \frac{\partial^j G(x,s)}{\partial x^j} \in C(\bar{\Omega}), \quad j = \overline{0, 2n-1},$$

the function in the square bracket is continuous on $\bar{\Omega}$. Then, due to $0 < \alpha < 1$, the integral on the right-hand side, and therefore the integral in (31), is uniformly bounded at $j = \overline{0, 2n-1}$, which implies that the first series in (29) converges uniformly.

The convergence of the remaining series can be proved similarly. Lemma 2.2 has been proved.

Lemma 2.3. If $g^{(j)}(x) \in C[0,1]$, $j = \overline{0, 2n-1}$, $x^{\alpha/2} g^{(2n)}(x) \in C(0,1) \cap L_2(0,1)$, and g satisfies the corresponding boundary conditions in $g^{(j)}(0) = 0, \quad g^{(n+j)}(0) + g^{(n+j)}(1) = 0, \quad j = \overline{0, n-1}$, then

$$\sum_{k=1}^{+\infty} \lambda_k g_k^2 \le \int_0^1 x^\alpha \left[g^{(2n)}(x) \right]^2 dx \tag{32}$$

in particular, the series on the left converges.}

Proof. By (8),

$$\lambda_k^{1/2} g_k = \lambda_k^{-1/2} \int_0^1 g(x)[x^\alpha v_k^{(2n)}(x)]^{(2n)}\,dx.$$

Carrying out $2n$ integrations by parts and using the properties of g and v_k,

$$\lambda_k^{1/2} g_k = \int_0^1 \left\{x^{\alpha/2} g^{(2n)}(x)\right\}\left\{\lambda_k^{-1/2} x^{\alpha/2} v_k^{(2n)}(x)\right\}dx.$$

Thus $\lambda_k^{1/2} g_k$ is the Fourier coefficient of $x^{\alpha/2} g^{(2n)}(x)$ with respect to the orthonormal family $\{x^{\alpha/2} v_k^{(2n)}(x)/\sqrt{\lambda_k}\}$, and (32) issues from Bessel's inequality. Lemma 2.3 has been proved.

Lemma 2.4. If $g(x)$ obeys (9) and $Mg(x) \in C(0,1) \cap L_2(0,1)$, then

$$\sum_{k=1}^{+\infty} \lambda_k^2 g_k^2 \le \int_0^1 \left[Mg(x)\right]^2 dx; \tag{33}$$

in particular, the series on the left converges.

Proof. From (8), $\lambda_k g_k = \int_0^1 g(x)[x^\alpha v_k^{(2n)}(x)]^{(2n)}\,dx$. Performing $4n$ integrations by parts and accounting for the properties of g gives $\lambda_k g_k = \int\limits_0^1 \left[Mg\left(x\right)\right] v_k\left(x\right) dx$. Accordingly, $\lambda_k g_k$ is the Fourier coefficient of $Mg(x)$ in the system $\{v_k\}$, and (33) results from Bessel's inequality. Lemma 2.4 has been proved.

Lemma 2.5. If $g(x)$ satisfies (9) and, in addition, $[Mg(x)]^{(j)} \in C[0,1]$, $j = \overline{0,2n-1}$, $x^{\alpha/2}[Mg(x)]^{(2n)} \in C(0,1) \cap L_2(0,1)$, $\left[Mg\left(x\right)\right]^{(j)} = 0, \left[Mg\left(x\right)\right]^{(n+j)}\Big|_{x=0} + \left[Mg\left(x\right)\right]^{(n+j)}\Big|_{x=1} = 0,\ \overline{j=0,n-1},$ then

$$\sum_{k=1}^{+\infty} \lambda_k^3 g_k^2 \le \int_0^1 x^\alpha \left\{[Mg(x)]^{(2n)}\right\}^2 dx. \tag{34}$$

Proof. The function Mg satisfies the assumptions of Lemma 2.3, and, by what was shown in Lemma 2.4, the quantities $\lambda_k g_k$ are the Fourier coefficients of Mg in $\{v_k\}$. Invoking Lemma 2.3 for Mg (its Fourier coefficients being $\lambda_k g_k$) produces $\sum\limits_{k=1}^{+\infty} \lambda_k (\lambda_k g_k)^2 \le \int_0^1 x^\alpha \{[Mg]^{(2n)}\}^2 dx$, which is precisely (34). Lemma 2.5 has been proved.

Existence, uniqueness and stability of the solution

We look for the solution in the form

$$u(x,t) = \sum_{k=1}^{+\infty} u_k(t) v_k(x), \tag{35}$$

in which $v_k(x)$ denote the eigenfunctions of {(8), (9)} and the functions $u_k(t)$ remain to be found. Inserting (35) into (5) and (6) and exploiting the orthonormality of $\{v_k\}$, we obtain, for every $k \in \mathbb{N}$, the degenerate Cauchy problem

$$t^a (1-t)^b u_{k'}(t) + (c+\lambda_k) u_k(t) = f_k(t), \qquad t \in (0,1), \tag{36}$$

$$u_k(0) = \varphi_k, \tag{37}$$

with $\varphi_k = \int_0^1 \varphi(x) v_k(x)\,dx$ and $f_k(t) = \int_0^1 f(x,t) v_k(x)\,dx$. Relation (36) is a linear ordinary differential equation of the first order that degenerates at the endpoints $t=0$ and $t=1$, because $0 \le a < 1$ and $0 \le b < 1$, the weight $1/[t^a(1-t)^b]$ is integrable over $(0,1)$. Dividing (36) through by $t^a(1-t)^b$ and first treating the homogeneous equation, we evaluate

$$\int_0^t \frac{dz}{z^a (1-z)^b} = \frac{t^{1-a}}{1-a}\, {}_2F_1\left(1-a, b; 2-a; t\right),$$

with ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ the Gauss hypergeometric function. Setting $\Phi(t)=\dfrac{t^{1-a}}{1-a}{}_2F_1(1-a,b;2-a;t)$, the homogeneous equation has the solution $u_k(t)=A_k e^{-(c+\lambda_k)\Phi(t)}$, and (37) forces $A_k=\varphi_k$. Varying the constant in the inhomogeneous equation, we find the unique solution of (36)-(37) to be

$$u_k(t)=\varphi_k e^{-(c+\lambda_k)\Phi(t)}+\int_0^t \frac{f_k(z)}{z^a(1-z)^b}e^{-(c+\lambda_k)[\Phi(t)-\Phi(z)]}dz. \tag{38}$$

As $c+\lambda_k>0$ and, for $t>z$, $\Phi(t)-\Phi(z)=\int_z^t s^{-a}(1-s)^{-b}ds\ge 0$, both $e^{-(c+\lambda_k)[\Phi(t)-\Phi(z)]}\le 1$ and $e^{-(c+\lambda_k)\Phi(t)}\le 1$ hold, consequently

$$|u_k(t)|\le|\varphi_k|+\int_0^t\frac{|f_k(z)|}{z^a(1-z)^b}dz,\qquad 0\le t\le 1. \tag{39}$$

Denoting $\rho(t)=1/[t^a(1-t)^b]$, $B_0=(\int_0^1\rho(z)dz)^{1/2}<\infty$ and $\|f_k\|^2_{L_{2,\rho}(0,1)}=\int_0^1\rho(z)f_k^2(z)dz$, appealing to the Cauchy-Schwarz inequality one obtains

$$|u_k(t)|\le|\varphi_k|+B_0\|f_k\|_{L_{2,\rho}(0,1)},\qquad 0\le t\le 1. \tag{40}$$

Theorem 3.1. Let $\varphi(x)$ satisfy the hypotheses of Lemma 2.5. Let $f(x,t)$ be continuous on $\bar{\Omega}$ and, for each fixed $t\in[0,1]$, satisfy with respect to x the hypotheses of Lemma 2.5 uniformly in t, with $x^{\alpha/2}[Mf]^{(2n)}\in L_{2,\rho}(\Omega)$, assume moreover that $f(x,\cdot)$ is Hölder continuous in t uniformly in x, so that $f_k(t)$ is continuous and $f_k(z)/[z^a(1-z)^b]$ is integrable on $(0,1)$, as required in (38). Then the series (35), whose coefficients are given by (38), determines a solution of the problem.

Proof. It has to be shown that (35) and the series obtained from it by formal term-by-term differentiation,

$$\partial^j u/\partial x^j=\sum_{k=1}^{+\infty}u_k(t)v_k^{(j)}(x),\quad j=\overline{1,2n-1},\quad\text{and}\quad(\partial^j/\partial x^j)(x^\alpha\partial^{2n}u/\partial x^{2n})=\sum_{k=1}^{+\infty}u_k(t)(x^\alpha v_k^{(2n)}(x))^{(j)},$$

$j=\overline{0,2n-1}$, converge uniformly on $\bar{\Omega}$, and that the series $(\partial^{2n}/\partial x^{2n})(x^\alpha\partial^{2n}u/\partial x^{2n})=\sum_{k=1}^{+\infty}u_k(t)(x^\alpha v_k^{(2n)}(x))^{(2n)}$ and $t^a(1-t)^b u_t=\sum_{k=1}^{+\infty}t^a(1-t)^b u_{k'}(t)v_k(x)$. By (40), for $(x,t)\in\bar{\Omega}$, the Cauchy—Schwarz inequality gives

$$|u(x,t)|\le\sqrt{\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}}\left(\sqrt{\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2}+B_0\sqrt{\int_0^1\rho(z)\sum_{k=1}^{+\infty}\lambda_k f_k^2(z)dz}\right). \tag{41}$$

By Lemmas 2.2 and 2.3 the right-hand series converge, and so (35) is uniformly convergent on $\bar{\Omega}$. As for the series $\sum_{k=1}^{+\infty}\lambda_k u_k(t)v_k(x)$ - which, by (8), stands for $(\partial^{2n}/\partial x^{2n})(x^\alpha\partial^{2n}u/\partial x^{2n})$ - it is enough, in light of (40), to secure the convergence of $\sum_{k=1}^{+\infty}\lambda_k\varphi_k v_k(x)$ and $\sum_{k=1}^{+\infty}\lambda_k\|f_k\|_{L_{2,\rho}(0,1)}v_k(x)$, the Cauchy-Schwarz inequality majorizes them by

$$\sqrt{\sum_{k=1}^{+\infty}\lambda_k^3\varphi_k^2}\sqrt{\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}},\qquad\sqrt{\int_0^1\rho(z)\sum_{k=1}^{+\infty}\lambda_k^3 f_k^2(z)dz}\sqrt{\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}},$$

both of which converge by virtue of Lemmas 2.2, 2.4 and 2.5. The series therefore converges uniformly on Ω_0, uniform convergence of the final series is a consequence of (5). The series left over are treated in an analogous fashion. The proof of Theorem 3.1 is complete.

Theorem 3.2. The problem cannot possess more than one solution.

Proof. Assume the problem has two solutions $u_1(x,t)$ and $u_2(x,t)$, and put $u=u_1-u_2$. Then u obeys (5) with $f\equiv 0$, the initial datum (6) with $\varphi\equiv 0$, and (7). Choose $T_0\in(0,1)$ and write $\Omega_{T_0}=\{(x,t):0<x<1,0<t<T_0\}$. Multiply (5) (with $f\equiv 0$) by u and integrate over Ω_{T_0}, using

$uu_t = \frac{1}{2}(u^2)_t$ in the first term and integrating by parts $2n$ times in the spatial one - where the boundary form disappears thanks to the combined local-and-paired relations (7), just as in Section 2 - we arrive at

$$\frac{1}{2}\int_0^1 T_0^a(1-T_0)^b u^2(x,T_0)dx + c\int_0^{T_0}\int_0^1 u^2\,dxdt + \int_0^{T_0}\int_0^1 x^\alpha\left[\frac{\partial^{2n}u}{\partial x^{2n}}\right]^2 dxdt = \frac{1}{2}\int_0^{T_0}\int_0^1 \frac{\partial}{\partial t}\left[t^a(1-t)^b\right]u^2(x,t)dxdt.$$

As $\partial_t[t^a(1-t)^b] = at^{a-1}(1-t)^b - bt^a(1-t)^{b-1}$, one can find $M_0 = M_0(a,b) > 0$ with $|\partial_t[t^a(1-t)^b]| \le M_0\rho_1(t)$, where we set $\rho_1(t) = t^{a-1} + (1-t)^{b-1}$, since $a,b<1$, the integrals $\int_0^1 t^{a-1}dt$ and $\int_0^1 (1-t)^{b-1}dt$ are finite, so $\rho_1 \in L_1(0,1)$ and the right-hand member does not exceed $K\int_0^{T_0}\int_0^1 u^2\,dxdt$ for a suitable $K>0$. We stress that ρ_1 is the Gronwall coefficient appearing here and differs from the weight $\rho(t) = t^{-a}(1-t)^{-b}$ used earlier. Because the two integral terms on the left are nonnegative, $\frac{1}{2}T_0^a(1-T_0)^b\int_0^1 u^2(x,T_0)dx \le K\Psi(T_0)$, $\Psi(T_0) = \int_0^{T_0}\int_0^1 u^2\,dxdt$, and, as $\Psi'(T_0) = \int_0^1 u^2(x,T_0)dx$, this becomes $\Psi'(T_0) \le 2K\rho(T_0)\Psi(T_0)$, $\Psi(0)=0$, the factor $2K\rho(T_0)$ being integrable over $(0,1)$. Gronwall's inequality now forces $\Psi(T_0)\equiv 0$, so that $u\equiv 0$ on $\bar{\Omega}$ that is, $u_1 \equiv u_2$. The proof of Theorem 3.2 is complete.

Theorem 3.3. Let $\varphi(x)$ and $f(x,t)$ satisfy the hypotheses of Theorem 3.1. Then the solution obeys the estimates

$$\| u(x,t)\|^2_{L_2(0,1)} \le K_0\left[\| \varphi(x)\|^2_{L_2(0,1)} + \| f(x,t)\|^2_{L_{2,\rho}(\Omega)}\right], \tag{42}$$

$$\| u(x,t)\|_{C(\bar{\Omega})} \le K_1\left[\| \varphi^{(2n)}(x)\|_{L_{2,r}(0,1)} + \| f(x,t)\|_{L_{2,\rho}(\Omega)}\right], \tag{43}$$

Where $\| \varphi(x)\|_{L_{2,r}(0,1)} = [\int_0^1 x^\alpha \varphi^2(x)dx]^{1/2}$, $r(x) = x^\alpha$, and K_0, K_1 are positive constants. These entail the continuous dependence of the solution on the data φ and f .}

Proof. By the orthonormality of $\{v_k\}$ and (40), in conjunction with $(p+q)^2 \le 2(p^2+q^2)$,

$$\| u(x,t)\|^2_{L_2(0,1)} = \sum_{k=1}^{+\infty} u_k^2(t) \le 2\sum_{k=1}^{+\infty}\left[\varphi_k^2 + B_0^2\| f_k\|^2_{L_{2,\rho}(0,1)}\right].$$

Bessel's inequality, combined with the identity $\| f\|^2_{L_{2,\rho}(\Omega)} = \sum_{k=1}^{+\infty}\| f_k\|^2_{L_{2,\rho}(0,1)}$ (which holds because ρ depends on t alone while $\{v_k\}$ is orthonormal in x), delivers (42) with $K_0 = 2\max(1, B_0^2)$. As for (43), drawing on Lemmas 2.2, 2.3 and on (41) we get

$$\| u(x,t)\|_{C(\bar{\Omega})} \le K_1\left\{\sqrt{\int_0^1 x^\alpha\left[\varphi^{(2n)}(x)\right]^2 dx} + \sqrt{\int_0^1 \rho(z)\sum_{k=1}^{+\infty} f_k^2(z)dz}\right\},$$

with $K_1 = \sup_{x\in[0,1]}(\sum_{k=1}^{+\infty} v_k^2(x)/\lambda_k)^{1/2}$, which is (43). The proof of Theorem 3.3 is complete.

Conclusion. Over a rectangular region we have set up and studied an initial-boundary value problem carrying a blend of local and paired (linked) boundary conditions, for a high even-order partial differential equation that loses its type in space at $x=0$ and in time at $t=0$ and $t=1$, and that additionally involves a subordinate term. Uniqueness of the solution was secured through an energy-integral technique tailored to the temporal degeneracy and rounded off with an inequality of Gronwall type. The solution itself was constructed as a Fourier expansion in the eigenfunctions of the attendant spectral problem, whose Green's function was produced in explicit form - displayed in closed form for $n=1$ and, for $n=2$, through the explicit determination of all its coefficients – shown to be symmetric, and converted into a Fredholm integral equation of the second kind with a continuous and symmetric kernel that generates a positive-definite operator. We isolated sufficient conditions on the data guaranteeing solvability and derived stability bounds in the spaces $L_2(0,1)$ and $C(\bar{\Omega})$.

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