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EXACT SOLUTIONS FOR THE NONISOSPECTRAL LOADED KORTEWEG-DE VRIES EQUATION VIA INVERSE SCATTERING TRANSFORM

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Abstract. *In this paper, soliton-like solutions for the Korteweg–de Vries (KdV) equation with variable coefficients are investigated using the Inverse Scattering Transform (IST) method. By applying appropriate transformations, the original equation is reduced to an integrable form. Subsequently, new general soliton solutions of the resulting equation are derived via the IST method, and explicit analytical solutions are obtained in closed form. The proposed approach can be effectively extended to analyze other classes of integrable nonlinear systems.*

Keywords: *variable-coefficient KdV equation, nonisospectral KdV equation, inverse scattering transform, Jost solutions, Lax pair, Cauchy problem.*

TESKARI SOCHILISH TRANSFORMATSIYASI YORDAMIDA NOIZOSPEKTRAL YUKLANGAN KORTEWEG-DE VRIES TENGLAMASI UCHUN ANIQ YECHIMLAR

Annotatsiya. *Ushbu ishda sochilish nazariyasining teskari masalalari usulidan foydalanib, o‘zgaruvchan koeffitsiyentli Korteveg–de Friz (KdF) tenglamasining solitonsimon yechimlari o‘rganiladi. Munosib almashtirishlar yordamida dastlabki tenglama integrallanuvchi ko‘rinishga keltiriladi. So‘ngra sochilish nazariyasining teskari masalalari usuli asosida ushbu tenglamaning yangi umumiy soliton yechimlari olinadi hamda ularning ochiq analitik ko‘rinishdagi aniq yechimlari hosil qilinadi. Maqolada taqdim etilgan yondashuv boshqa turdagi integrallanuvchi nochiziqli tenglamalarni tadqiq etishda ham keng qo‘llanilishi mumkin.*

Kalit so‘zlar: *o‘zgaruvchan koeffitsiyentli KdF tenglamasi, nonisospektral KdF tenglamasi, sochilish nazariyasining teskari masalalari, Yost yechimlari, Laks juftligi, Koshi masalasi.*

ТОЧНЫЕ РЕШЕНИЯ НЕИЗОСПЕКТРАЛЬНОГО НАГРУЖЕННОГО УРАВНЕНИЯ КОРТЕВЕГА-ДЕ ФРИЗА МЕТОДОМ ОБРАТНОЙ ЗАДАЧИ РАССЕЯНИЯ

Аннотация. *В данной работе методом обратной задачи рассеяния исследуются солитоноподобные решения уравнения Кортевега-де Фриза (КдФ) с переменными коэффициентами. С помощью соответствующих преобразований исходное уравнение сводится к интегрируемому виду. На основе метода обратной задачи рассеяния выводятся новые общие солитонные решения полученного уравнения, и строятся их точные аналитические решения в явном виде. Предложенный подход может быть успешно применён к исследованию других классов интегрируемых нелинейных систем.*

Ключевые слова: *уравнение КдФ с переменными коэффициентами, неизоспектральное уравнение КдФ, метод обратной задачи рассеяния, решения Йоста, пара Лакса, задача Коши.*

Introduction. Nonlinear evolution equations play an important role in the mathematical modeling of a wide range of physical phenomena. Among them, integrable nonlinear evolution equations are of particular

interest because their analytical properties allow exact solutions to be constructed and their dynamical behavior to be studied in detail. The KdV equation is one of the fundamental integrable nonlinear evolution equations and has been extensively studied in connection with the theory of solitons and nonlinear wave propagation.

The well-known IST [1] has been widely used to solve nonlinear evolution equations [2–4]. One of the advantages of the IST is that it can be applied to a broad set of evolution equations associated with a given spectral problem. Evolution equations associated with a given spectral problem can generally be classified into isospectral and non-isospectral types, depending on whether the spectral parameter remains constant or evolves with time. Isospectral equations often describe solitary waves in lossless and homogeneous media. Non-isospectral equations, which arise from spectral problems with a time-dependent spectral parameter η [5–6], can also be solved using the IST, and their solutions demonstrate the existence of solitary waves in certain types of nonuniform media [7–10].

In this paper, using the IST method, we solve the initial-value problem for the following non-isospectral loaded KdV equation

$$u_t - \gamma(t)u(x_0,t)(u_{xxx} - 6uu_x) + 4u_x - (xu_x + 2u) = 0, \tag{1}$$

where $\gamma(t)$ is a given continuous function, and $x_0 \in \mathbb{R}$. Equation (1) is considered under the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{2}$$

where the function $u_0(x)$ satisfies the following criteria:

$$\int_{-\infty}^{\infty} (1+|x|)(|u_0(x)| + |u_{0xx}(x)|)dx < \infty. \tag{3}$$

The solution $u(x,t)$ is assumed to exhibit sufficient smoothness and sufficiently rapidly decreasing as $x \to \pm\infty$, quantified by the condition:

$$\int_{-\infty}^{\infty} (1+|x|)\left(|u(x,t)| + \sum_{k=1}^{3}\left|\frac{\partial^k u(x,t)}{\partial x^k}\right|\right)dx < \infty. \tag{4}$$

The aim of this work is to derive explicit representations for the solution $u(x,t)$ of the problem (1)-(4) by means of the IST method.

Preliminaries

In this section, we provide some basic information on the scattering theory for the Sturm–Liouville problem on the whole real axis [4]. We temporarily omit the dependence of the function $u(x,t)$ on the variable t.

Consider the equation:

$$-\psi_{xx} + u\psi = \lambda\psi, \quad x \in \mathbb{R}, \tag{5}$$

where the potential $u(x)$ satisfies the condition

$$\int_{-\infty}^{\infty} (1+|x|)(|u(x)| + |u''(x)|)dx < \infty. \tag{6}$$

Under the assumption (6), equation (5) possesses the Jost solutions with the following asymptotics:

$$\psi_1(x,k) = e^{ikx} + o(1), \quad x \to +\infty,$$
$$\varphi_1(x,k) = e^{-ikx} + o(1), \quad x \to -\infty,$$

The solution $\psi_1(x,k)$ admits the representation

$$\psi_1(x,k) = e^{ikx} + \int_x^{\infty} K_+(x,z)e^{ikz}\,dz, \tag{7}$$

where the kernel $K_+(x,z)$ is a real-valued function related to the potential $u(x)$ by the relations:

$$u(x) = -2\frac{d}{dx}K_+(x,x).$$

For real k, the pairs $\{\varphi_1(x,k), \varphi_1(x,-k)\}$ and $\{\psi_1(x,k), \psi_1(x,-k)\}$ are pairs of linearly independent solutions for equation (5). Therefore,

$$\varphi_1(x,k) = a(k)\psi_1(x,-k) + b(k)\psi_1(x,k). \tag{8}$$

One can readily see that

$$a(k) = -\frac{1}{2ik} W\{\psi_1(x,k), \varphi_1(x,k)\}.$$

The function $a(k)$ admits an analytic continuation into the upper half-plane and has a finite number of zeroes $k = ik_n$, $k_n > 0$. Meanwhile,

$$\varphi_1(x, ik_n) = b_n \psi_1(x, ik_n), \qquad n = 1, 2, ..., N.$$

The scattering data for the spectral problem (5) are defined by

$$\left\{ R(k) = \frac{b(k)}{a(k)},\ k \in \mathbb{R},\ k_n,\ c_n,\ n = 1, 2, ..., N \right\},$$

where $c_n^2 = -\frac{b_n}{\dot{a}(ik_n)}$, $\dot{a}(ik_n) = \left.\frac{da(k)}{dk}\right|_{k=ik_n}$ and each c_n is so-called normalization constant of eigenfunction $\psi_1(x, ik_n)$.

The kernel $K_+(x,y)$ in representation (7) is a solution to the Gelfand–Levitan–Marchenko integral equation:

$$K_+(x,y) + F_+(x+y) + \int_x^\infty K_+(x,z) F_+(z+y) dz = 0, \quad (y > x),$$

where

$$F_+(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k) e^{ikx} dk + \sum_{j=1}^{N} c_j^2 e^{-k_j x}. \tag{9}$$

The time evolution of scattering data

In this section, we derive the time evolution of the scattering data associated with the non-isospectral loaded KdV equation (1). To this end, consider the following Lax pair:

$$-\psi_{xx} + u\psi = \lambda\psi,$$
$$\psi_t = A\psi + B\psi_x, \tag{10}$$

where $A = -\frac{1}{2} B_x$, $B = x - 2\gamma(t) u(x_0, t)(u + 2\lambda) - 4$, $\lambda_t = 2\lambda$. The compatibility condition $\psi_{xxt} = \psi_{txx}$ of the Lax pair (10) yields the non-isospectral loaded KdV equation (1).

Lemma 1. Suppose that $\psi(x,k,t)$ is a solution of (5), A and B satisfy the compatibility condition $\psi_{xxt} = \psi_{txx}$, i.e.,

$$2A_x + B_{xx} = 0,$$
$$u_t = A_{xx} + 2(u - \lambda) B_x + u_x B + \lambda_t.$$

Then

$$P(x,k,t) = \psi_t(x,k,t) - A\psi(x,k,t) - B\psi_x(x,k,t) \tag{11}$$

is also a solution of the spectral problem (5).

Proof. We show that $P(x,k,t)$ is a solution of the spectral problem (5). Direct calculation of $-P_{xx} + uP - \lambda P$ yields:

$$-P_{xx} + uP - \lambda P = -(\psi_t - A\psi - B\psi_x)_{xx} + (u - \lambda)(\psi_t - A\psi - B\psi_x).$$

By using the relation $\psi_{xx} = (u - \lambda)\psi$, the second-order spatial derivatives are evaluated as follows:

$$(\psi_t)_{xx} = (\psi_{xx})_t = u_t\psi + (u - \lambda)\psi_t - \lambda_t\psi,$$
$$(A\psi)_{xx} = A_{xx}\psi + 2A_x\psi_x + A(u - \lambda)\psi,$$
$$(B\psi_x)_{xx} = B_{xx}\psi_x + 2B_x(u - \lambda)\psi + B(u_x\psi + (u - \lambda)\psi_x).$$

Substituting these derivatives into $-P_{xx} + uP - \lambda P$ and collecting terms with respect to ψ_t, ψ_x and ψ, we obtain

$$-P_{xx} + uP - \lambda P = (-(u - \lambda) + (u - \lambda))\psi_t + (2A_x + B_{xx})\psi_x + (-u_t + \lambda_t + A_{xx} + 2(u - \lambda)B_x + u_x B)\psi.$$

Applying the compatibility conditions $2A_x + B_{xx} = 0$ and $u_t = A_{xx} + 2(u - \lambda)B_x + u_x B + \lambda_t$, all the coefficients vanish identically. Therefore,

$$-P_{xx}+uP=\lambda P.$$

The lemma 1 is proved.

Theorem 1. If the function $u(x,t)$ is a solution to the problem (1)-(4), then the scattering data of the equation (5) with the potential $u(x,t)$ vary with t as follows:

$$\frac{dR}{dt}=-8ik\left(\gamma(t)u(x_0,t)k^2+1\right)R,\quad (\operatorname{Im}k=0), \tag{12}$$

$$\frac{dc_n}{dt}=\left(\frac{1}{2}+4k_n-\gamma(t)u(x_0,t)4k_n^3\right)c_n, \tag{13}$$

$$\frac{dk_n}{dt}=k_n. \tag{14}$$

Proof. It is clear that the systam (5) and (10) has a solution

$$f(x,k,t)=s_1(t)\psi_1(x,-k,t)+s_2(t)\psi_1(x,k,t), \tag{15}$$

where $s_1^2(t)+s_2^2(t)\neq 0$. Substituting (15) into (10), and as $x\to\infty$,

$$\frac{ds_1(t)}{dt}e^{-ikx}-ixs_1(t)\frac{dk}{dt}e^{-ikx}+\frac{ds_1(t)}{dt}e^{ikx}+ixs_2(t)\frac{dk}{dt}e^{ikx}=-\frac{1}{2}\left(s_1(t)e^{-ikx}+s_2(t)e^{ikx}\right)-$$
$$-ixk\,s_1(t)\,e^{-ikx}+ixk\,s_2(t)e^{ikx}+\left(-4\gamma(t)u(x_0,t)k^2-4\right)\left(ik\,s_1(t)e^{-ikx}+ik\,s_2(t)e^{ikx}\right), \tag{16}$$

Comparing the coefficients of the terms xe^{-ikx} on both sides of (16), we have

$$\frac{dk(t)}{dt}=k(t). \tag{17}$$

Consequently, using (17), Eq. (16) becomes

$$\frac{ds_1(t)}{dt}e^{-ikx}+\frac{ds_2(t)}{dt}e^{ikx}=\left(-\frac{1}{2}+4ik^3\gamma(t)u(x_0,t)+4ik\right)s_1(t)e^{-ikx}+$$
$$+\left(-\frac{1}{2}-4ik^3\gamma(t)u(x_0,t)-4ik\right)s_2(t)e^{ikx}. \tag{18}$$

Equating the coefficients of $\exp(-ikx)$ and $\exp(ikx)$ on both sides of (18), we obtain

$$\frac{ds_1(t)}{dt}=s_1(t)\left(-\frac{1}{2}+4ik^3\gamma(t)u(x_0,t)+4ik\right),$$
$$\frac{ds_2(t)}{dt}=s_2(t)\left(-\frac{1}{2}-4ik^3\gamma(t)u(x_0,t)-4ik\right).$$

Solving for $s_1(t)$ and $s_2(t)$, we deduce

$$s_1(t)=s_1(0)\exp\left\{-\frac{t}{2}\right\}\exp\left\{\int_0^t\left(4ik^3\gamma(\tau)u(x_0,\tau)+4ik\right)d\tau\right\}, \tag{19}$$

$$s_2(t)=s_2(0)\exp\left\{-\frac{t}{2}\right\}\exp\left\{-\int_0^t\left(4ik^3\gamma(\tau)u(x_0,\tau)+4ik\right)d\tau\right\}. \tag{20}$$

In particular, taking

$$f(x,k,t)=\varphi_1(x,k,t)\exp\left\{-\frac{t}{2}\right\}\exp\left\{\int_0^t\left(4ik^3\gamma(\tau)u(x_0,\tau)+4ik\right)d\tau\right\}.$$

From (8), we have

$$f(x,k,t)=a(k,t)\psi_1(x,-k,t)W_0(t)+b(k,t)\psi_1(x,k,t)W_0(t),$$

where

$$W_0(t)=\exp\left\{-\frac{t}{2}\right\}W(t),\ W(t)=\exp\left\{\int_0^t\left(4ik^3\gamma(\tau)u(x_0,\tau)+4ik\right)d\tau\right\}.$$

In this case, $s_1(t)$ and $s_2(t)$ given by (19) and (20) become $s_1(t)=a(k,t)W_0(t)$, $s_2(t)=b(k,t)W_0(t)$ and $s_1(0)=a(k,0)$, $s_2(0)=b(k,0)$. Hence we obtain

$$a(k,t)=a(k,0),$$
$$b(k,t)=b(k,0)W^{-2}(t).$$

Now, using the definition of the function $R(k,t)$, we get (12).

Further, we calculate the dependence of $c_n(t)$ and $k_n(t)$, $n=1,2,...,N$, on time t.

Consider the following three solutions of (5) with $k=ik_n$, $(k_n>0)$,

$$P(x,ik_n,t)=\psi_t(x,ik_n,t)-A\psi(x,ik_n,t)-B\psi_x(x,ik_n,t),$$

$\psi(x,ik_n,t)$ and $\bar{\psi}(x,ik_n,t)$, where $\psi(x,ik_n,t)=c_n(t)\psi_1(x,ik_n,t)$ is a normalized Jost function, $c_n=c_n(t)$ is the related normalization constant, and $\bar{\psi}(x,ik_n,t)$ is another Jost function linearly independent of $\psi(x,ik_n,t)$.

There exist two constants α, β such that

$$\psi_t(x,ik_n,t)-A\psi(x,ik_n,t)-B\psi_x(x,ik_n,t)=\alpha\psi(x,ik_n,t)+\beta\bar{\psi}(x,ik_n,t),$$

Noting the fact of $\psi(x,ik_n,t)\to 0$ as $x\to+\infty$, it is easy to see $\beta=0$, i.e.,

$$\psi_t(x,ik_n,t)-A\psi(x,ik_n,t)-B\psi_x(x,ik_n,t)=\alpha\psi(x,ik_n,t), \tag{21}$$

To derive the value of α, multiplying (21) by $2\psi(x,ik_n,t)$, and noting that $A=-\frac{1}{2}B_x$, yields

$$2\psi(x,ik_n)\psi_t(x,ik_n,t)+B_x\psi^2(x,ik_n,t)-2B\psi(x,ik_n,t)\psi_x(x,ik_n,t)=2\alpha\psi^2(x,ik_n,t).$$

Further we have

$$\frac{d}{dt}\int_{-\infty}^{\infty}\psi^2(x,ik_n)dx+\int_{-\infty}^{\infty}\left[B_x\psi^2(x,ik_n,t)-B\left(\psi^2(x,ik_n,t)\right)_x\right]dx=2\alpha\int_{-\infty}^{\infty}\psi^2(x,ik_n,t)dx.$$

Since $\int_{-\infty}^{\infty}\psi^2(x,ik_n,t)dx=1$ and $B\psi^2(x,ik_n,t)\to 0$ as $x\to\pm\infty$, we obtain

$$\alpha=-\int_{-\infty}^{\infty}B(\psi^2(x,ik_n,t))_x\,dx=1.$$

Thus (21) becomes

$$\psi_t(x,ik_n,t)-A\psi(x,ik_n,t)-B\psi_x(x,ik_n,t)=\psi(x,ik_n,t). \tag{22}$$

Setting $x\to+\infty$ in (22) immediately yields

$$\frac{dc_n}{dt}-x\frac{dk_n}{dt}c_n+\frac{1}{2}c_n+c_n k_n\left(x+\gamma(t)u(x_0,t)4k_n^2-4\right)=c_n,$$

which further means

$$\frac{dk_n}{dt}=k_n,\qquad \frac{dc_n}{dt}=c_n\left(\frac{1}{2}+4k_n-\gamma(t)u(x_0,t)4k_n^3\right).$$

The theorem is proved.

An Illustrative example

Let us consider the following Cauchy problem:

$$u_t-\gamma(t)u(0,t)(u_{xxx}-6uu_x)+4u_x-(xu_x+2u)=0, \tag{23}$$

$$u_0(x)=-\frac{1}{2}\operatorname{sech}^2\frac{x}{2}. \tag{24}$$

Here

$$\gamma(t)=-\frac{16e^{-5t}}{\operatorname{sech}^2\left(4t-2\left(e^t-1\right)\right)}.$$

Solving the direct problem for the equation

$$-\psi_{xx}+u_0\psi=\lambda\psi\ ,$$

we find the scattering data for $t=0$:

$$c_1(0)=1,\ k_1(0)=\frac{1}{2},\ R(k,0)=0.$$

By theorem 1, we have

$$R(k,t)=0,\ k_1(t)=\frac{e^t}{2},\ c_1(t)=\exp\left\{\frac{t}{2}+2\left(e^t-1\right)-\frac{1}{2}\mu(t)\right\},$$

where

$$\mu(t)=\int_0^t e^{3\tau}\gamma(\tau)u(0,\tau)d\tau.$$

Substituting these data into Eq. (9), we obtain the following:

$$F_+(x,t)=\exp\left\{-\frac{e^t}{2}x+t+4(e^t-1)-\mu(t)\right\}.$$

Now, solving the Gelfand–Levitan–Marchenko integral equation, we obtain:

$$K_+(x,x,t)=-\frac{e^t}{2}\left[1-\tanh\left(\frac{e^t x-4(e^t-1)+\mu(t)}{2}\right)\right].$$

According to the formula $u(x,t)=-2\frac{d}{dx}K_+(x,x,t)$, it follows that the solution to the Cauchy problem (23)–(24) takes the following form:

$$u(x,t)=-\frac{1}{2}e^{2t}\operatorname{sech}^2\left(\frac{xe^t}{2}-2\left(e^t-1\right)+4t\right).$$

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