UDC 581.14

INVERSE PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL CONDITIONS

Abdiraimova Shoxista Toji qizi,
Teacher at Secondary School № 13,
Karshi District, Kashkadarya Region,
Republic of Uzbekistan
abdiraimovashoxista1@gmail.com

Abstract. *This paper investigates the classical solvability of an inverse boundary value problem for a time-fractional parabolic equation involving the Caputo derivative. The temperature distribution and the unknown time-dependent lowest-order coefficient are simultaneously determined using nonlocal integral conditions and periodic boundary conditions. By applying the separation of variables method via a biorthogonal system of eigenfunctions and sharp estimates of generalized Mittag-Leffler functions, the existence and uniqueness of the classical solution in Banach spaces are established.*

Key words: *time-fractional parabolic equation, Caputo derivative, inverse boundary value problem, nonlocal conditions, biorthogonal system, Riesz basis, Mittag-Leffler function, Banach contraction mapping principle.*

INTEGRAL SHARTLI KASR TARTIBLI TENGLAMA UCHUN TESKARI MASALA

Annotatsiya. *Ushbu maqolada Kaputo hosilali vaqt bo'yicha kasr tartibli parabolik tenglama uchun teskari chegaraviy masalaning klassik yechilishi o'rganiladi. Nofokal integral va davriy chegaraviy shartlar yordamida harorat taqsimoti hamda vaqtga bog'liq bo'lmag'in noma'lum quyi koeffitsiyent bir vaqtning o'zida aniqlanadi. O'zgaruvchilarni ajratish usuli va xususiy funksiyalarning biorthogonal sistemasi yordamida Banax fazosida klassik yechimning mavjudligi va yagonaligi isbotlangan.*

Kalit so'zlar:*. kasr tartibli parabolik tenglama, Kaputo hosilasi, teskari chegaraviy masala, nofokal shartlar, biorthogonal sistema, Riss bazisi, Mittag-Leffler funksiyasi, qisqarib boruvchi akslantirishlar prinsipi.*

ОБРАТНАЯ ЗАДАЧА ДЛЯ УРАВНЕНИЯ ДРОБНОГО ПОРЯДКА С ИНТЕГРАЛЬНЫМИ УСЛОВИЯМИ

Аннотация. *В данной статье исследуется классическая разрешимость обратной краевой задачи для параболического уравнения дробного по времени порядка с производной Капуто. Одновременно определяются распределение температуры и неизвестный коэффициент при младшем члене, зависящий от времени, с использованием нелокальных интегральных и периодических краевых условий. С помощью метода разделения переменных через биортогональную систему собственных функций доказаны существование и единственность классического решения в пространстве Банаха.*

Ключевые слова*: параболическое уравнение дробного порядка, производная Капуто, обратная краевая задача, нелокальные условия, биортогональная система, базис Рисса, функция Миттаг-Леффлера, принцип сжимающих отображений.*

Introduction. Let $0 < T < +\infty$ be a fixed moment of time and D_T be a rectangular domain defined by the inequalities $0 < x < 1$ and $0 < t \le T$. We consider the problem of finding a pair of functions $\{u(x,t), a(t)\}$ satisfying the following time-fractional parabolic equation:

$$ {}^{C}D_t^{\alpha} u(x,t) = u_{xx}(x,t) + a(t)u(x,t) + f(x,t), \quad 0 < \alpha < 1, \quad . \tag{1}$$

The nonlocal initial condition has the following form:

$$ u(x,0) + \delta u(x,T) + \int_0^T p(t)u(x,t)dt = \varphi(x), \quad 0 \le x \le 1. \tag{2}$$

The boundary conditions are periodic:

$$ u(0,t) = \beta u(1,t), \quad 0 \le t \le T. \tag{3}$$

The nonlocal integral condition is:

$$\int_0^1 u(x,t)dx = 0, \quad 0 \le t \le T, \tag{4}$$

and the overdetermination condition for determining the inverse problem is as follows:

$$u(x_0,t) = h(t), \quad x_0 \in (0,1). \tag{5}$$

Here $T > 0, \beta, \delta \ge 0$ are given numbers, and $p(t)$, $\varphi(x)$, $h(t)$, $f(x,t)$ are given positive functions. ${}^C D_t^\alpha u(x,t)$ is the Caputo fractional derivative of $u(x,t)$ with respect to time, defined as follows:

$${}^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} \frac{d\tau}{(t-\tau)^\alpha}.$$

The symbol $\Gamma(\cdot)$ in the last equation denotes the Euler gamma function.

Definition 1 *A pair of functions $\{u(x,t), a(t)\}$ is called a solution to problem (1)–(5) if:*

1. the function $u(x,t)$ and its derivatives ${}^C D_t^\alpha u(x,t)$ and $u_{xx}(x,t)$ are continuous in the domain D_T
2. the time-dependent function $a(t)$ is continuous on the interval $[0,T]$;
3. the pair of functions $\{u(x,t), a(t)\}$ satisfies equation (1) and conditions (2)–(5).

Then the following theorem holds.

Theorem 1 *Assume that $\delta \ge 0$, $p(t) \in C[0,T]$, $f(x,t) \in C(\overline{D}_T)$, $\varphi(x) \in C[0,1]$, and ${}^C D_t^\alpha h(t) \in C[0,T]$. Let the following equality hold for $0 \le t \le T$:*

$$\int_0^1 f(x,t)dx = 0,$$

and let $h(t) \ne 0$ for $0 \le t \le T$. Furthermore, let the following compatibility conditions be satisfied:

$$\int_0^1 \varphi(x)dx = 0, \tag{6}$$

$$h(0) + \delta h(T) + \int_0^T p(t)h(t)dt = \varphi(x_0). \tag{7}$$

Then the problem of finding a classical solution to problem (1)–(5) is equivalent to the problem of determining functions $u(x,t)$ and $a(t)$ satisfying equation (1), conditions (2)–(3), and the following relations (the case $\beta \ne \pm 1$ is studied):

$$u_x(0,t) = u_x(1,t), \quad 0 \le t \le T, \tag{8}$$

$${}^C D_t^\alpha h(t) = u_{xx}(x_0,t) + a(t)h(t) + f(x_0,t), \quad 0 \le t \le T. \tag{9}$$

Proof. Let the pair of functions $\{u(x,t), a(t)\}$ be a classical solution of problem (1)–(5). Integrating equation (1) with respect to the variable x from 0 to 1, we obtain:

$${}^C D_t^\alpha \int_0^1 u(x,t)dx = u_x(1,t) - u_x(0,t) + a(t)\int_0^1 u(x,t)dx + \int_0^1 f(x,t)dx. \tag{10}$$

Taking into account equality (6), we obtain equality (8). Evaluating equation (1) at the point $x = x_0$, we obtain:

$${}^C D_t^\alpha u(x_0,t) = u_{xx}(x_0,t) + a(t)u(x_0,t) + f(x_0,t), \quad 0 \le t \le T. \tag{11}$$

Since ${}^C D_t^\alpha h(t) \in C[0,T]$, using condition (5) we obtain the relation:

$${}^C D_t^\alpha u(x_0,t) = {}^C D_t^\alpha h(t). \tag{12}$$

As a result, using relations (5) and (12), we obtain (9). Now assume that the pair of functions $\{u(x,t), a(t)\}$ is a solution to problem (1)–(3), (8), (9). Then, taking into account conditions (3) and (8), from relation (10) we have:

$${}^C D_t^\alpha \int_0^1 u(x,t)dx = a(t)\int_0^1 u(x,t)dx, \quad 0 \le t \le T. \tag{13}$$

From condition (2), it is easy to see that:

$$\int_0^1 u(x,0)dx + \delta\int_0^1 u(x,T)dx + \int_0^1\int_0^T p(t)u(x,t)dtdx = \int_0^1 \varphi(x)dx = 0. \quad (14)$$

Since problem (13) and (14) has only the trivial solution, it follows that:

$$\int_0^1 u(x,t)dx = 0, \quad 0 \le t \le T,$$

which means condition (4) is satisfied.

Next, from relations (9) and (11) we find:

$${}^C D_t^{\alpha}(u(x_0,t) - h(t)) = a(t)(u(x_0,t) - h(t)), \quad 0 \le t \le T. \quad (15)$$

According to condition (2) and compatibility condition (7), we obtain:

$$u(x_0,0) - h(0) = \varphi(x_0) - h(0) = 0. \quad (16)$$

From relation (15), taking into account equality (16), it becomes evident that condition (5) is also satisfied.

Auxiliary Spectral Problem

Now, to investigate problems (1)–(3) and (8), we present some known facts. Consider the following spectral problem:

$$X''(x) + \lambda X(x) = 0, \quad 0 \le x \le 1, \quad (17)$$

$$X(0) = \beta X(1), \quad X'(0) = X'(1), \quad \beta \ne \pm 1. \quad (18)$$

Problem (17)–(18) is not self-adjoint, and the adjoint problem to this problem is as follows:

$$Y''(x) + \lambda Y(x) = 0, \quad 0 \le x \le 1,$$

$$(19) \quad Y(0) = Y(1), \quad Y'(1) = \beta Y'(0). \quad (20)$$

According to the literature [1], we denote the system of eigenfunctions and associated functions of problem (19)–(20) in the following form:

$$X_0(x) = ax + b, \quad X_{2k-1}(x) = (ax+b)\cos\lambda_k x, \quad X_{2k}(x) = \sin\lambda_k x, \quad (21)$$

$$Y_0(x) = 2, \quad Y_{2k-1}(x) = 4\cos\lambda_k x, \quad Y_{2k}(x) = 4(1-b-ax)\sin\lambda_k x. \quad (22)$$

Here

$$\lambda_k = 2k\pi \quad (k = 1,2,\ldots), \quad a = \frac{1-\beta}{1+\beta} \ne 0, \quad b = \frac{\beta}{1+\beta}. \quad (23)$$

It is not difficult to verify the fulfillment of the biorthogonality conditions: $(X_i, Y_j) = \int_0^1 X_i(x)Y_j(x)dx = \delta_{ij}$.

Theorem 1.1. *The system of functions (21)–(22) forms a Riesz basis in the space* $L_2(0,1)$, *and for any function* $\psi(x) \in L_2(0,1)$, *the following estimates hold:*

$$r \parallel \psi(x) \parallel^2_{L_2(0,1)} \le \sum_{k=0}^{\infty}\psi_k^2 \le R \parallel \psi(x) \parallel^2_{L_2(0,1)}. \quad (24)$$

where

$$\psi_k = (\psi(x), Y_k(x)) = \int_0^1 \psi(x)Y_k(x)dx \quad (k = 0,1,\ldots),$$

When the appropriate conditions are satisfied, we establish the validity of the following estimates:

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i}\psi_{2k-1})^2\right)^{\frac{1}{2}} \le 2\sqrt{2} \parallel \psi^{(2i)}(x) \parallel_{L_2(0,1)}, \quad (25)$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i}\psi_{2k})^2\right)^{\frac{1}{2}} \le 2\sqrt{2} \parallel \psi^{(2i)}(x)(1-b-ax) - 2ai\psi^{(2i-1)}(x) \parallel_{L_2(0,1)}. \quad (26)$$

Furthermore, under certain conditions:

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i+1}\psi_{2k-1})^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\parallel \psi^{(2i+1)}(x)\parallel_{L_2(0,1)}, \tag{27}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i+1}\psi_{2k})^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\parallel \psi^{(2i+1)}(x)(1-b-ax)-a(2i+1)\psi^{(2i)}(x)\parallel_{L_2(0,1)}. \tag{28}$$

Existence and Uniqueness of the Solution to the Inverse Problem

Since relation (21) forms a Riesz basis in the space $L_2(0,1)$ and relations (21)–(22) form a biorthogonal system of functions, we seek the function $u(x,t)$ of problem (1)–(3), (8)–(9) in the following form:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x), \tag{29}$$

where

$$u_k(t)=\int_0^1 u(x,t)Y_k(x)dx \quad (k=0,1,2,\ldots). \tag{30}$$

Then, applying the method of separation of variables, from relations (1) and (2) we obtain the following:

$$^{C}D_t^{\alpha}u_0(t)=F_0(t,u,a),$$

(31) $$^{C}D_t^{\alpha}u_{2k-1}(t)+\lambda_k^2u_{2k-1}(t)=F_{2k-1}(t,u,a) \quad (k=1,2,\ldots;0\le t\le T), \tag{32}$$

$$^{C}D_t^{\alpha}u_{2k}(t)+\lambda_k^2u_{2k}(t)=F_{2k}(t,u,a)-2a\lambda_k u_{2k-1}(t), \tag{33}$$

$$u_k(0)+\delta u_k(T)+\int_0^T p(t)u_k(t)dt=\varphi_k. \tag{34}$$

Solving problem (31)–(34), we arrive at the following results:

$$u_0(t)=u_0(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}F_0(\tau,u,a)d\tau, \tag{35}$$

$$u_{2k-1}(t)=u_{2k-1}(0)E_{\alpha}(-\lambda_k^2t^{\alpha})+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^{\alpha})F_{2k-1}(\tau,u,a)d\tau, \tag{36}$$

$$u_{2k}(t)=u_{2k}(0)E_{\alpha}(-\lambda_k^2t^{\alpha})+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^{\alpha})[F_{2k}(\tau,u,a)-2a\lambda_k u_{2k-1}(\tau)]d\tau. \tag{37}$$

$$u_{2k}(t)=u_{2k}(0)E_{\alpha}(-\lambda_k^2t^{\alpha})+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^{\alpha})[F_{2k}(\tau,u,a)-2a\lambda_k u_{2k-1}(\tau)]d\tau. \tag{38}$$

Here, $E_{\alpha}(z)\equiv E_{\alpha,1}(z)$, and $E_{\alpha,\beta}(z)$ is the generalized two-parameter Mittag-Leffler function defined by the series

$$E_{\alpha,\beta}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+\beta)}, \quad (\alpha>0,\beta\in\mathbb{R},z\in\mathbb{C}).$$

Using the nonlocal integral condition (34), we determine the unknown initial value coefficients. Using the nonlocal integral condition (34), we determine the unknown initial value coefficients. Substituting solutions (35), (36), and (37) into condition (34), we find respectively:

$$u_0(t)=\frac{1}{1+\delta}\left(\varphi_0-\frac{\delta}{\Gamma(\alpha)}\int_0^T(T-\tau)^{\alpha-1}F_0(\tau,u,a)d\tau-\int_0^T p(t)u_0(t)dt\right)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}F_0(\tau,u,a)d\tau, \tag{39}$$

$$u_{2k-1}(t)=\frac{E_{\alpha}(-\lambda_k^2t^{\alpha})}{1+\delta E_{\alpha}(-\lambda_k^2T^{\alpha})}\left(\varphi_{2k-1}-\delta\int_0^T(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(T-\tau)^{\alpha})F_{2k-1}(\tau,u,a)d\tau\right.$$
$$\left.-\int_0^T p(t)u_{2k-1}(t)dt\right)+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^{\alpha})F_{2k-1}(\tau,u,a)d\tau, \tag{40}$$

$$u_{2k}(t)=\frac{E_\alpha(-\lambda_k^2t^\alpha)}{1+\delta E_\alpha(-\lambda_k^2T^\alpha)}\left[\varphi_{2k}-\int_0^T p(t)u_{2k}(t)dt-\delta\int_0^T(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(T-\tau)^\alpha)F_{2k}(\tau,u,a)d\tau\right]$$
$$+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^\alpha)F_{2k}(\tau,u,a)d\tau+q_k(t,T), \quad (41)$$

where

$$q_k(t,T)=-2a\lambda_k\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^\alpha)u_{2k-1}(\tau)d\tau$$
$$+2a\lambda_k\frac{\delta E_\alpha(-\lambda_k^2t^\alpha)}{1+\delta E_\alpha(-\lambda_k^2T^\alpha)}\int_0^T(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(T-\tau)^\alpha)u_{2k-1}(\tau)d\tau. \quad (42)$$

Substituting the found Fourier coefficients into series (29), we obtain the form of the general solution of the inverse problem:

$$u(x,t)=u_0(t)(ax+b)+\sum_{k=1}^{\infty}[u_{2k-1}(t)(ax+b)\cos\lambda_kx+u_{2k}(t)\sin\lambda_kx]. \quad (43)$$

Now, using (9), we find the unknown coefficient $a(t)$:

$$a(t)=\frac{{}^CD_t^\alpha h(t)-u_{xx}(x_0,t)-f(x_0,t)}{h(t)}, \quad (44)$$

$$a(t)=\frac{1}{h(t)}\left\{{}^CD_t^\alpha h(t)+\sum_{k=1}^{\infty}\lambda_k^2u_{2k-1}(t)X_{2k-1}(x_0)+\sum_{k=1}^{\infty}\lambda_k^2u_{2k}(t)X_{2k}(x_0)-f(x_0,t)\right\}. \quad (45)$$

Using relations (39) and (40), we bring (44) to the following form:

$$a(t)=\frac{1}{h(t)}\ {}^CD_t^\alpha h(t)-f(x_0,t)$$
$$+\sum_{k=1}^{\infty}\lambda_k^2X_{2k-1}(x_0)\left[\frac{E_\alpha(-\lambda_k^2t^\alpha)}{1+\delta E_\alpha(-\lambda_k^2T^\alpha)}\left(\varphi_{2k-1}-\delta\int_0^T(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(T-\tau)^\alpha)F_{2k-1}(\tau)d\tau\right)\right.$$
$$\left.+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^\alpha)F_{2k-1}(\tau,u,a)d\tau\right]$$
$$+\sum_{k=1}^{\infty}\lambda_k^2X_{2k}(x_0)\left[\frac{E_\alpha(-\lambda_k^2t^\alpha)}{1+\delta E_\alpha(-\lambda_k^2T^\alpha)}\left(\varphi_{2k}-\int_0^T p(t)u_{2k}(t)dt-\delta\int_0^T(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(T-\tau)^\alpha)F_{2k}(\tau,u,a)d\tau\right)\right.$$
$$\left.\left.+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^\alpha)F_{2k}(\tau,u,a)d\tau+q_k(t,T)\right]\right\}. \quad (46)$$

Lemma 1 *If the set* $\{u(x,t),a(t)\}$ *is a classical solution of problem (1)–(3), (8), and (9), then the functions*

$$u_0(t)=\int_0^1u(x,t)Y_0(x)dx,$$
$$u_{2k-1}(t)=\int_0^1u(x,t)Y_{2k-1}(x)dx \quad (k=1,2,\ldots),$$
$$u_{2k}(t)=\int_0^1u(x,t)Y_{2k}(x)dx \quad (k=1,2,\ldots)$$

satisfy the system (42), (43), and (44) on the interval $[0,T]$.

Corollary 1 *According to Lemma 1, proving the uniqueness of the solution to problem (1)–(3), (8), (9) reduces to proving the uniqueness of the solution to system (39), (40), (41).*

To investigate problem (1)–(3), (8), (9), we consider the following function spaces: Let $B_{2,T}^3$ denote set of all functions considered in the domain D_T and having the following form:

$$u(x,t)=u_0(t)(ax+b)+\sum_{k=1}^{\infty}u_{2k-1}(t)(ax+b)\cos\lambda_kx+\sum_{k=1}^{\infty}u_{2k}(t)\sin\lambda_kx, \quad (47)$$

where the functions $u_0(t)$, $u_{2k-1}(t)$ $(k=1,2,\ldots)$, and $u_{2k}(t)$ $(k=1,2,\ldots)$ involved in the sum are continuous on the interval $[0,T]$, and

$$\| u_0(t) \|_{C[0,T]} + \left(\sum_{k=1}^{\infty} (\lambda_k^3 \| u_{2k-1}(t) \|_{C[0,T]})^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} (\lambda_k^3 \| u_{2k}(t) \|_{C[0,T]})^2 \right)^{\frac{1}{2}} < +\infty. \quad (48)$$

The operations of addition and scalar multiplication in the space $B_{2,T}^3$ are defined as usual, and the norm is defined by the following formula:

$$\| u(x,t) \|_{B_{2,T}^3} = \| u_0(t) \|_{C[0,T]} + \left(\sum_{k=1}^{\infty} (\lambda_k^3 \| u_{2k-1}(t) \|_{C[0,T]})^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} (\lambda_k^3 \| u_{2k}(t) \|_{C[0,T]})^2 \right)^{\frac{1}{2}}. \quad (49)$$

Furthermore, let E_T^3 denote the space consisting of the topological product $B_{2,T}^3 \times C[0,T]$. The norm of its element $z(x,t) = \{u(x,t), a(t)\}$ is defined as follows:

$$\| z(x,t) \|_{E_T^3} = \| u(x,t) \|_{B_{2,T}^3} + \| a(t) \|_{C[0,T]}. \quad (50)$$

It is known that $B_{2,T}^3$ and E_T^3 are Banach spaces.

In the space E_T^3, we consider the following operator:

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\}, \quad (51)$$

where

$$\Phi_1(u,a) = \tilde{u}(x,t) = \sum_{k=0}^{\infty} \tilde{u}_k(t) X_k(x), \quad (52)$$

$$\Phi_2(u,a) = \tilde{a}(t), \quad (53)$$

and the functions $\tilde{u}_k(t)$ $(k=0,1,2,\ldots)$ as well as the function $\tilde{a}(t)$ are defined by the right-hand sides of(38), (39), (40), and (45), respectively.

Assume that the given data of problem (1)–(3) and (5) satisfy the following conditions:

- $\varphi(x) \in C^2[0,1]$, $\varphi''(x) \in L_2(0,1)$, $\varphi(0) = \beta\varphi(1)$, $\varphi'(0) = \varphi'(1)$, $\varphi''(0) = \beta\varphi''(1)$ $(\beta \neq \pm 1)$.
- $f(x,t), f_x(x,t), f_{xx}(x,t) \in C(D_T)$, $f_{xxx}(x,t) \in L_2(D_T)$, $f(0,t) = \beta f(1,t)$, $f_x(0,t) = f_x(1,t)$, $f_{xx}(0,t) = \beta f_{xx}(1,t)$ $(\beta \neq \pm 1)$ $(0 \le t \le T)$.
- $h(t), {}^C D_t^{\alpha} h(t) \in C[0,T]$, $h(t) \neq 0$ $(0 \le t \le T)$.

From these assumptions, according to the properties of the Mittag-Leffler functions established in [2], the following estimates hold:

$$| E_{\alpha}(-\lambda_k^2 t^{\alpha}) | \le \frac{M_1}{1+\lambda_k^2 t^{\alpha}}, \quad M_1 = \text{const} \ge 0,$$

$$| E_{\alpha,\alpha}(-\lambda_k^2 (t-\tau)^{\alpha}) | \le \frac{M_2}{1+\lambda_k^2 (t-\tau)^{\alpha}}, \quad M_2 = \text{const} \ge 0.$$

Consequently, we obtain

$$\| \tilde{u}_0(t) \|_{C[0,T]} \le (1+\delta)^{-1} (| \varphi_0 | + T \| p(t) \|_{C[0,T]} \| u_0(t) \|_{C[0,T]})$$

$$+ (1 + \delta(1+\delta)^{-1}) \left[\frac{T^{\alpha-0.5}}{\Gamma(\alpha)\sqrt{2\alpha-1}} \left(\int_0^T | f_0(\tau) |^2 \, d\tau \right)^{\frac{1}{2}} + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \| a(t) \|_{C[0,T]} \| u_0 \|_{C[0,T]} \right], \quad (54)$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel \tilde{u}_{2k-1}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}} \le 2\left[M_1\left(\sum_{k=1}^{\infty}(\lambda_k^3 \mid \varphi_{2k-1} \mid)^2\right)^{\frac{1}{2}} + M_1 T \parallel p(t) \parallel_{C[0,T]} \left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel u_{2k-1}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}}\right]$$

$$+M_2(1+\delta M_1)\frac{T^{\alpha-0.5}}{\sqrt{2\alpha-1}}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3 \mid f_{2k-1}(\tau) \mid)^2 d\tau\right)^{\frac{1}{2}} + M_2(1+\delta M_1)\frac{T^{\alpha}}{\alpha} \parallel a(t) \parallel_{C[0,T]} \left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel u_{2k-1}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{55}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel q_k(t,T) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}} \le \tilde{M}\frac{T^{\alpha}}{\alpha}(1+\delta M_1)\times\left[2\sqrt{2}M_1 \parallel \varphi'''(x) \parallel_{L_2} + 2\sqrt{2}M_2(1+\delta M_1)\frac{T^{\alpha-0.5}}{\sqrt{2\alpha-1}} \parallel f_{xxx}(x,t) \parallel_{L_2}\right.$$

$$\left.+\left(M_1 T \parallel p(t) \parallel_{C[0,T]} + M_2(1+\delta M_1)\frac{T^{\alpha}}{\alpha} \parallel a(t) \parallel_{C[0,T]}\right)\left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel u_{2k-1}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}}\right], \tag{56}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel \tilde{u}_{2k}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{5}\left[M_1\left(\sum_{k=1}^{\infty}(\lambda_k^3 \mid \varphi_{2k} \mid)^2\right)^{\frac{1}{2}} + M_1 T \parallel p(t) \parallel_{C[0,T]} \left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel u_{2k}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}}\right]$$

$$+\sqrt{5}M_2(1+\delta M_1)\left[\frac{T^{\alpha-0.5}}{\sqrt{2\alpha-1}}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3 \mid f_{2k}(\tau) \mid)^2 d\tau\right)^{\frac{1}{2}} + \frac{T^{\alpha}}{\alpha} \parallel a(t) \parallel_{C[0,T]} \left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel u_{2k}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}}\right]$$

$$+\sqrt{5}\left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel q_k(t,T) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}}. \tag{57}$$

Taking into account inequalities (54), (55), (56), and (57), and based on (25)–(28), we obtain the following estimates:

$$\parallel \tilde{u}_0(t) \parallel_{C[0,T]} \le A_1(T) + B_1(T) \parallel a(t) \parallel_{C[0,T]} + C_1(T) \parallel u(x,t) \parallel_{B_{2,T}^3}, \tag{58}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel \tilde{u}_{2k-1}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}} \le A_2(T) + B_2(T) \parallel a(t) \parallel_{C[0,T]} \parallel u(x,t) \parallel_{B_{2,T}^3} + C_2(T) \parallel u(x,t) \parallel_{B_{2,T}^3}, \tag{59}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3 \parallel \tilde{u}_{2k}(t) \parallel_{C[0,T]})^2\right)^{\frac{1}{2}} \le A_3(T) + B_3(T) \parallel a(t) \parallel_{C[0,T]} \parallel u(x,t) \parallel_{B_{2,T}^3} + C_3(T) \parallel u(x,t) \parallel_{B_{2,T}^3}, \tag{60}$$

where

Now, using (58)–(60), we compute the following:

$$\parallel \tilde{u}(x,t) \parallel_{B_{2,T}^3} \le A_4(T) + B_4(T) \parallel a(t) \parallel_{C[0,T]} \parallel u(x,t) \parallel_{B_{2,T}^3} + C_4(T) \parallel u(x,t) \parallel_{B_{2,T}^3}, \tag{61}$$

where:

$A_4(T) = A_1(T) + A_2(T) + A_3(T)$, $B_4(T) = B_1(T) + B_2(T) + B_3(T)$,

$C_4(T) = C_1(T) + C_2(T) + C_3(T)$.

From (45), we estimate: $\parallel \tilde{a}(t) \parallel_{C[0,T]}$

Using (57), taking into account (25)–(28), and from relations (63), the following expression follows:

$$\parallel \tilde{a}(t) \parallel_{C[0,T]} \le A_5(T) + B_5(T) \parallel a(t) \parallel_{C[0,T]} \parallel u(x,t) \parallel_{B_{2,T}^3} + C_5(T) \parallel u(x,t) \parallel_{B_{2,T}^3}, \tag{63}$$

From inequalities (61) and (63), we deduce the following:

$$\parallel \tilde{u}(x,t) \parallel_{B_{2,T}^3} + \parallel \tilde{a}(t) \parallel_{C[0,T]} \le A(T) + B(T) \parallel \tilde{a}(t) \parallel_{C[0,T]} \parallel \tilde{u}(x,t) \parallel_{B_{2,T}^3} + C(T) \parallel \tilde{u}(x,t) \parallel_{B_{2,T}^3},$$

where

$A(T) = A_4(T) + A_5(T)$, $B(T) = B_4(T) + B_5(T)$, $C(T) = C_4(T) + C_5(T)$.

Theorem 2 *If conditions (1)–(3) and the inequality*

$$(A(T)+2)(B(T)A(T)+2)+C(T)\le 1 \tag{66}$$

hold, then problem (1)–(3) and (8)–(9) has a unique solution in the space E_T^3 under the condition

$$K=\{z\in E_T^3 : \| z \|_{E_T^3}\le R\le A(T)+2\}.$$

Remark 1 *Inequalities (65) are satisfied for sufficiently small values of* T .

Proof. In the space E_T^3, we consider the following equation:

$$z=\Phi(z), \tag{67}$$

where $z=(u,a)$, and the components $\Phi_i(u,a)$ $(i=1,2)$ of the operator $\Phi(u,a)$ are defined by the right-hand sides of equations (51) and (52).

Consider the operator $\Phi(u,a)$ in the ball $K=K_R$ of the space E_T^3. Using similar relations, for any $z_1,z_2\in K_R$ we obtain the validity of the following inequalities:

$$\| \Phi z \|_{E_T^3}\le A(T)+B(T)\| a(t) \|_{C[0,T]}\| u(x,t) \|_{B_{2,T}^3} +C(T)\| u(x,t) \|_{B_{2,T}^3}\le A(T)+B(T)(A(T)+2)^2 +C(T)(A(T)+$$

(68)

$$\| \Phi z_1-\Phi z_2 \|_{E_T^3}\le B(T)R(\| a_1(t)-a_2(t) \|_{C[0,T]} +\| u_1(x,t)-u_2(x,t) \|_{B_{2,T}^3})+C(T)\| u_1(x,t)-u_2(x,t) \|_{B_{2,T}^3} .$$

(69)

Then, according to (66), from (67) and (68) it is clearly seen that the operator satisfies the conditions of the contraction mapping principle on the set $K=K_R$. Therefore, the operator has a unique fixed point $z=(u,a)$ in the ball $K=K_R$, which is the solution to equation (66), meaning it is the unique solution of the system in the ball $K=K_R$.

After that, the function $u(x,t)$, as an element of the space $B_{2,T}^3$, is continuous in the domain D_T and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$. Furthermore, it can be verified that equation (1) and conditions (2), (3), (8), and (9) are satisfied in the usual sense. As a result, $(u(x,t),a(t))$ is a solution of problem (1)–(3), (8), and (9), and according to Corollary 1, it is unique in the ball $K=K_R$.

The following statement follows directly from Theorems 1 and 2.

Theorem 3 *Assume that all conditions of Theorem 2 and the compatibility conditions*

$$\int_0^1\varphi(x)dx=0,\quad h(0)+\delta h(T)+\int_0^T p(t)h(t)dt=\varphi(x_0)$$

are satisfied. If the conditions

$$\int_0^1 f(x,t)dx=0,\quad 0\le t\le T$$

hold, then problem (1)–(5) has a unique classical solution in the ball $K=K_R$.

REFERENCES:

1. E. I. Azizbayov, The nonlocal inverse problem of the identification of the lowest coefficient and the right-hand side in a second-order parabolic equation with integral conditions, Boundary Value Problems, 2019 (2019), 2019:11.

2. E. I. Azizbayov and Y. T. Mehraliyev, Determination of source terms in a time-fractional parabolic equation via nonlocal measurements, Journal of Contemporary Applied Mathematics, 16(1) (2026).

3. A. N. Tikhonov, On stability of inverse problems, Dokl. Akad. Nauk SSSR, 39(5) (1943), 195–198.

4. M. M. Lavrent'ev, V. G. Romanov, and S. T. Shishatskii, Ill-Posed Problems of Mathematical Physics and Analysis, Nauka, Moscow, 1980.

5. V. M. Isakov, Inverse parabolic problems with the final overdetermination, Commun. Pure Appl. Math..44(2) (1991), 185–209.

6. J. R. Cannon, Y. P. Lin, and S. M. Wang, Determination of a control parameter in a parabolic partial-differential equation, J. Aust. Math. Soc. Ser. B. Appl. Math., 33(2) (1991), 149–163.

7. A. M. Denisov, Elements of the Theory of Inverse Problems, Inverse and Ill-Posed Problems Series, VSP, Utrecht, 1999.

8. V. L. Kamynin, The inverse problem of the simultaneous determination of the right-hand side and the lowest coefficient in parabolic equation with many space variables, Math. Notes, 97(3-4) (2015), 349–361.

9. M. I. Ivanchov, Inverse Problem for Equations of Parabolic Type, Mathematical Studies Monograph Series, vol. 10, VNTL Publishers, Lviv, 2003.

10. A. I. Kozhanov, Parabolic equations with unknown time-dependent coefficients, Comput. Math. Math. Phys., 57(6) (2017), 961–972.

11. I. A. Vasin and V. L. Kamynin, On the asymptotic behavior of solutions of inverse problems for parabolic equations, Sib. Math. J., 38(4) (1997), 647–662.

12. S. I. Kabanikhin, Inverse and Ill-Posed Problems: Theory and Applications, De Gruyter, Berlin, 2012.

13. N. B. Kerimov and M. I. Ismailov, Direct and inverse problems for the heat equation with a dynamic-type boundary condition, IMA J. Appl. Math., 80(5) (2015), 1519–1533.

14.E. I. Azizbayov and Y. T. Mehraliyev, Solvability of nonlocal inverse boundary-value problem for a second-order parabolic equation with integral conditions, Electron. J. Differ. Equ., 2017(125) (2017), 1–17.

15. N. I. Ionkin, Solution of a boundary-value problem in heat conduction with a nonclassical boundary condition, Differ. Equ., 13 (1977), 204–211.

16. T. B. Kasumov and V. S. Mirzoyev, On one generalization of Ionkin's example, In: The Abstracts of Scientific Conference Devoted to 100th Anniversary of the Academician A. I. Huseynov, Baku, 2007, pp. 87–88.

17. K. L. Khudaverdiyev and E. I. Azizbekov, Investigation of classical solution of a one-dimensional not self-adjoint mixed problem for a class of semi-linear pseudohyperbolic equations of fourth order, News Baku State Univ., Ser. Phys-Math. Sci., 2 (2002), 114–121.