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QUALITATIVE PROPERTIES OF THE SOURCE-FREE CAPUTO TIME-FRACTIONAL NONLINEAR FILTRATION EQUATION

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Abstract. *The nonlocal character of the Caputo time-fractional nonlinear filtration equation, which accounts for memory effects, complicates the search for an exact solution. In this paper, the equation is studied through a leading-order self-similar simplification, which reduces it to an ordinary differential equation admitting an exact solution in terms of the incomplete beta function. The obtained results are compared with the rigorous theory based on the full Erdélyi–Kober operator, and by demonstrating the dominance of the neglected term near the free boundary, it is proved that the critical condition and the free-boundary exponent belong not to the original equation but to the applied approximation itself.*

Keywords: *Caputo derivative, time-fractional diffusion, nonlinear filtration equation, self-similar solution, Erdélyi–Kober operator, free boundary, incomplete beta function.*

MANBASIZ KAPUTO KASR TARTIBLI NOCHIZIQLI FILTRATSIYA TENGLAMASINING SIFAT XOSSALARI

Annotatsiya. *Xotira effektlarini hisobga oluvchi Kaputo vaqt bo‘yicha kasr tartibli nochiziqli filtratsiya tenglamasining nolokal xarakteri aniq yechim topishni murakkablashtiradi. Maqolada tenglama yetakchi tartibli avtomodel soddalashtirish orqali tadqiq etilib, to‘liq bo‘lmagan beta-funksiya orqali aniq yechiladigan oddiy differensial tenglamaga keltiriladi. Olingan natijalar to‘liq Erdélyi–Kober operatoriga asoslangan qat'iy nazariya bilan solishtiriladi va tashlab yuborilgan hadning erkin chegara yaqinida ustun bo‘lishini ko'rsatish orqali kritik shart hamda erkin chegara ko‘rsatkichi asl tenglamaga emas, balki qo‘llanilgan approksimatsiyaning o‘ziga tegishli ekanligi isbotlanadi.*

Kalit so‘zlar: *Kaputo hosilasi, vaqt bo‘yicha kasr tartibli diffuziya, nochiziqli filtratsiya tenglamasi, avtomodel yechim, Erdélyi–Kober operatori, erkin chegara, to‘liq bo‘lmagan beta-funksiya.*

КАЧЕСТВЕННЫЕ СВОЙСТВА НЕЛИНЕЙНОГО УРАВНЕНИЯ ФИЛЬТРАЦИИ БЕЗ ИСТОЧНИКА С ДРОБНОЙ ПРОИЗВОДНОЙ КАПУТО ПО ВРЕМЕНИ

Аннотация. *Нелокальный характер нелинейного уравнения фильтрации с дробной производной Капуто по времени, учитывающей эффекты памяти, затрудняет нахождение точного решения. В статье уравнение исследуется с помощью автомодельного упрощения ведущего порядка, которое сводит его к обыкновенному дифференциальному уравнению, допускающему точное решение через неполную бета-функцию. Полученные результаты сопоставляются со строгой теорией, основанной на полном операторе Эрдейи–Кобера, и путём демонстрации доминирования отброшенного члена вблизи свободной границы доказывается, что критическое условие и показатель поведения вблизи свободной границы принадлежат не исходному уравнению, а самой применённой аппроксимации.*

Ключевые слова: *производная Капуто, дробная диффузия по времени, нелинейное уравнение фильтрации, автомодельное решение, оператор Эрдейи–Кобера, свободная граница, неполная бета-функция.*

Introduction. Nonlinear diffusion equations play a central role in modern mathematical physics. A fundamental example is the porous medium equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{\sigma} \frac{\partial u}{\partial x} \right)$$

This equation models filtration processes, nonlinear heat conduction, plasma transport and population dynamics. Its characteristic properties include finite propagation speed and compactly supported self-similar solutions of Barenblatt–Pattle type [1; 2].

To incorporate memory effects, the classical model has been generalised by replacing the first-order time derivative with the Caputo fractional derivative:

$$ {}^{C}D_t^{\alpha}u = \frac{\partial}{\partial x}\left(u^{\sigma}\frac{\partial u}{\partial x}\right), \quad 0<\alpha<1. \tag{1}$$

The nonlocal character of the Caputo derivative makes the construction and analysis of exact self-similar solutions substantially more involved.

Literature Review and Problem Statement. We consider the following Cauchy problem:

$$ {}^{C}D_t^{\alpha}u = \frac{\partial}{\partial x}\left(u^{\sigma}\frac{\partial u}{\partial x}\right), \quad x\in\mathbb{R}, \quad t>0, \tag{2}$$

$$u(x,0)=u_0(x)\geq 0, \quad x\in\mathbb{R}.$$

Self-similar solutions of time-fractional porous-medium equations have been investigated extensively through formulations involving the Erdelyi–Kober operator [4-8]. The most comprehensive general result currently available is due to Gomez-Castro, Plociniczak and Vazquez [9].

The cited result states that, in every spatial dimension and for every admissible nonlinear exponent and fractional order $\alpha\in(0,1)$, there exists a unique finite-mass self-similar profile. In the slow-diffusion regime considered here, the profile is compactly supported and behaves near the free boundary as

$$U(z)\sim c(1-z)_{+}^{(2-\alpha)/\sigma}. \tag{3}$$

The cited studies retain the full nonlocal integral formulation. In contrast, the present analysis replaces the nonlocal term by a leading-order approximation and thereby reduces the profile problem to an ordinary differential equation. This produces an explicit closed-form relation, but its validity must be assessed independently because the reduction does not preserve the full nonlocal structure.

The aim of this study is to determine precisely which conclusions of the reduced model are reliable and to identify the mechanism responsible for its failure near the free boundary.

Research Methodology.

Derivation of the Approximate Ordinary Differential Equation

The methodology combines an analytical self-similar reduction, exact integration of the resulting ordinary differential equation, asymptotic comparison with rigorous theory and a self-consistency test of the neglected term. We seek a self-similar solution in the form

$$u(x,t)=t^{-\beta}f(\xi), \quad \xi=\frac{|x|}{t^{\beta}}. \tag{4}$$

For fixed x, the similarity variable depends on time according to

$$\xi(s)=\frac{|x|}{s^{\beta}}.$$

Substitution of the self-similar ansatz into the Caputo derivative yields the following exact integral expression:

$$ {}^{C}D_t^{\alpha}u = -\frac{\beta}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{-\beta-1}\left[f(\xi(s))+\xi(s)f'(\xi(s))\right], ds \tag{5}$$

Under the leading-order assumption that the variation of the profile through the time-dependent similarity variable is negligible in the central region, we write

$$ {}^{C}D_t^{\alpha}u \approx f(\xi), {}^{C}D_t^{\alpha}\left[t^{-\beta}\right] \tag{6}$$

The action of the Caputo derivative on the power function is then approximated by

$$ {}^{C}D_t^{\alpha}t^{-\beta}=K, t^{-\beta-\alpha}, \qquad K=\frac{\Gamma(1-\beta)}{\Gamma(1-\beta-\alpha)} \tag{7}$$

The right-hand side of the nonlinear filtration equation transforms as

$$\frac{\partial}{\partial x}\left(u^{\sigma}\frac{\partial u}{\partial x}\right)=t^{-\beta(\sigma+3)}\frac{d}{d\xi}\left(f^{\sigma}f'\right) \tag{8}$$

Balancing the powers of t gives

$$-\beta-\alpha=-\beta(\sigma+3),$$

and therefore

$$\beta=\frac{\alpha}{\sigma+2} \tag{9}$$

The approximate profile equation is therefore

$$Kf(\xi)=\frac{d}{d\xi}\left(f^{\sigma}f'\right), \qquad f(0)=M>0, \qquad f'(0)=0 \tag{10}$$

This ordinary differential equation is the reduced model analysed below. Its conclusions are valid only where the leading-order assumption remains self-consistent.

Analysis and Results.

Explicit Solution of the Approximate Ordinary Differential Equation

Set

$$g(f)=f^{\sigma}f'(\xi).$$

Since

$$\frac{dg}{d\xi}=\frac{dg}{df}\frac{df}{d\xi},$$

the profile equation becomes

$$Kf=g\frac{dg}{df}f^{-\sigma}.$$

After integration and use of the condition $f'(0)=0$, one obtains the following first integral.

Proposition 1. First integral

$$g^2=\frac{2K}{\sigma+2}\left(f^{\sigma+2}-M^{\sigma+2}\right) \tag{11}$$

Because $g=f^{\sigma}f'$, a decreasing profile must satisfy $f'(\xi)<0$. For $0<f<M$, the quantity

$$f^{\sigma+2}-M^{\sigma+2}$$

is negative. Therefore, the right-hand side of equation (11) is nonnegative only if $K<0.$

Theorem 1. Internal consistency condition for the approximate ordinary differential equation

A centrally symmetric, decreasing and compactly supported solution of the approximate profile equation can exist only if $K<0.$

Since

$$K=\frac{\Gamma(1-\beta)}{\Gamma(1-\beta-\alpha)},$$

the numerator is positive for the admissible values of β. The sign of K is therefore determined by the denominator. The condition $K<0$ is equivalent to $1-\beta-\alpha<0.$

Using

$$\beta=\frac{\alpha}{\sigma+2},$$

we obtain

$$1-\alpha-\frac{\alpha}{\sigma+2}<0.$$

Hence,

$$\alpha>\alpha_c:=\frac{\sigma+2}{\sigma+3} \tag{12}$$

This restriction is intrinsic only to the approximate ordinary differential equation. For the original Caputo filtration equation, compactly supported self-similar solutions exist for every $\alpha\in(0,1)$ in the slow-diffusion regime [9].

For a decreasing profile, equation (11) gives

$$f^{\sigma} f' = -\sqrt{\frac{2|K|}{\sigma+2}\left(M^{\sigma+2} - f^{\sigma+2}\right)}.$$

Consequently,

$$\frac{d\xi}{df} = -\sqrt{\frac{\sigma+2}{2|K|}} \frac{f^{\sigma}}{\sqrt{M^{\sigma+2} - f^{\sigma+2}}}.$$

Integration from f to M yields

$$\xi(f) = \sqrt{\frac{\sigma+2}{2|K|}} \int_f^M \frac{s^{\sigma}}{\sqrt{M^{\sigma+2} - s^{\sigma+2}}}, ds \tag{13}$$

Introducing the variable

$$y = \left(\frac{s}{M}\right)^{\sigma+2},$$

the solution can be expressed in terms of the incomplete beta function:
Tayyor:

$$\xi(f) = \frac{M^{\sigma/2}}{\sqrt{2|K|(\sigma+2)}} \left[B\left(1; \frac{\sigma+1}{\sigma+2}, \frac{1}{2}\right) - B\left(\left(\frac{f}{M}\right)^{\sigma+2}; \frac{\sigma+1}{\sigma+2}, \frac{1}{2}\right)\right] \tag{14}$$

Here,

$$B(z; a, b) = \int_0^z t^{a-1}(1-t)^{b-1}, dt$$

denotes the incomplete beta function.
The support radius is obtained by setting $f = 0$:

$$\xi_0 = \frac{M^{\sigma/2}}{\sqrt{2|K|(\sigma+2)}} B\left(1; \frac{\sigma+1}{\sigma+2}, \frac{1}{2}\right) \tag{15}$$

Using the identity

$$B(1; a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

and $\Gamma(1/2) = \sqrt{\pi}$, we obtain

$$\xi_0 = \frac{M^{\sigma/2}\sqrt{\pi}}{\sqrt{2|K|(\sigma+2)}} \frac{\Gamma\left(\frac{\sigma+1}{\sigma+2}\right)}{\Gamma\left(\frac{3\sigma+4}{2(\sigma+2)}\right)} \tag{16}$$

Thus, the approximate profile is positive for $0 \le \xi < \xi_0$ and vanishes for $\xi \ge \xi_0$

Front Asymptotics

Near the free boundary, $f(\xi)$ tends to zero. From the first integral,

$$f^{\sigma} f' \sim -C,$$

where

$$C = \sqrt{\frac{2|K|}{\sigma+2}} M^{(\sigma+2)/2} > 0.$$

Therefore,

$$f^{\sigma}, df \sim -C, d\xi.$$

Integration near $\xi = \xi_0$ gives

$$\frac{f^{\sigma+1}}{\sigma+1} \sim C(\xi_0 - \xi).$$

Hence,

$$f(\xi) \sim C(\xi_0 - \xi)^{1/(\sigma+1)} \quad \text{as} \quad \xi \to \xi_0^- \tag{17}$$

The constant c is given by

$$c = \left[(\sigma+1)C\right]^{1/(\sigma+1)}.$$

Remark 1.

In the classical Barenblatt profile, corresponding to $\alpha = 1$, the free-boundary exponent is $\frac{1}{\sigma}$.

The rigorous fractional theory [9] gives the exponent $\frac{2-\alpha}{\sigma}$.

Both values differ from $\frac{1}{\sigma+1}$, which is the exponent produced by the approximate ordinary differential equation.

Thus, the leading-order approximation does not reproduce the correct local behaviour near the free boundary.

Self-Consistency Check

The approximation used in deriving equation (6) effectively neglects the term involving $\xi f'(\xi)$ relative to the term involving $f(\xi)$. This requires

$$\left|\frac{\xi f'(\xi)}{f(\xi)}\right| \ll 1.$$

However, the front asymptotics imply

$$f(\xi) \sim c(\xi_0 - \xi)^{1/(\sigma+1)},$$

and hence

$$f'(\xi) \sim -\frac{c}{\sigma+1}(\xi_0 - \xi)^{-\sigma/(\sigma+1)}.$$

Therefore,

$$\frac{\xi f'(\xi)}{f(\xi)} \sim -\frac{\xi_0}{\sigma+1}\frac{1}{\xi_0 - \xi} \to -\infty \quad \text{as} \quad \xi \to \xi_0^- \tag{18}$$

Theorem 2. Breakdown of the approximation

Near the free boundary, the term discarded in the leading-order reduction becomes infinitely larger than the retained term. Consequently, the assumption

$$|\xi f'(\xi)| \ll |f(\xi)|$$

is violated precisely in the front region.

This loss of self-consistency explains the appearance of the spurious critical value $\alpha_c = \frac{\sigma+2}{\sigma+3}$ and the incorrect front exponent $\frac{1}{\sigma+1}$.

The approximate ordinary differential equation may describe the central region of the profile, where the variation of f is relatively weak. It cannot, however, correctly describe the free-boundary layer.

Conclusion and Recommendations

The leading-order self-similar reduction of the Caputo time-fractional nonlinear filtration equation produces an ordinary differential equation that can be solved explicitly.

The approximate model requires $\alpha > \frac{\sigma+2}{\sigma+3}$ and yields the free-boundary exponent $\frac{1}{\sigma+1}$.

Neither property belongs to the original Caputo filtration equation. In the slow-diffusion regime, compactly supported self-similar profiles exist for all $\alpha \in (0,1)$, and the correct free-boundary exponent is $\frac{2-\alpha}{\sigma}$ according to the rigorous theory [9].

Precisely at the free boundary, where the neglected term becomes dominant. The principal contribution of this paper is therefore not a new existence theorem, but a transparent, computable

demonstration of how a leading-order simplification can generate artificial restrictions and incorrect asymptotic exponents. Accordingly, such reductions should be applied only after their domain of validity has been verified; future work should retain the full Erdélyi–Kober operator, develop matched asymptotic expansions, or solve the complete nonlocal profile equation numerically.

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