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CONVERGENCE ARIES OF A NEW TRIVARIATE MITTAG-LEFFLER-TYPE FUNCTION

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Abstract. *This article studies the main analytical properties of a new three-variable Mittag-Leffler type function. The convergence domains of the corresponding triple power series are determined. The convergence analysis is carried out based on Horn's convergence theory and the asymptotic properties of the gamma function for large arguments. Twenty-seven different combinations of parameters are considered, and for each case necessary and sufficient conditions for the absolute convergence of the series are established, along with some particular cases of ML. The obtained results enrich the theory of fractional differential equations and provide an effective analytical basis for investigating problems involving the Prabhakar fractional derivative.*

Key words: *Mittag-Leffler function, trivariate function, fractional calculus, Prabhakar derivative, triple power series, region of convergence, gamma function, special functions.*

О СВОЙСТВАХ СХОДИМОСТИ НОВОЙ ТРЁХПЕРЕМЕННОЙ ФУНКЦИИ ТИПА МИТТАГА-ЛЕФФЛЕРА

Аннотация. *В данной статье исследуются основные аналитические свойства новой функции типа Миттаг-Леффлера от трёх переменных. Определены области сходимости соответствующего тройного степенного ряда. Анализ сходимости проведён на основе теории сходимости Хорна, а также асимптотических свойств гамма-функции при больших значениях аргументов. Рассмотрены двадцать семь различных комбинаций параметров, и для каждого случая установлены необходимые и достаточные условия абсолютной сходимости ряда, а также приведены некоторые частные случаи ML. Полученные результаты дополняют теорию дифференциальных уравнений дробного порядка и представляют эффективную аналитическую основу для исследования задач, включающих дробную производную Прабхакара.*

Ключевые слова: *функция Миттага-Леффлера, трёхпеременная функция, дробное исчисление, производная Прабхакара, тройной степенной ряд, область сходимости, гамма-функция, специальные функции.*

YANGI UCH O‘ZGARUVCHILI MITTAG -LEFFLER TURIDAGI FUNKSIYANING YAQINLASHISH XOSSALARI

Annotatsiya. *Ushbu maqolada yangi uch o‘zgaruvchili Mittag-Leffler turidagi funksiyaning asosiy analitik xossalari o‘rganilgan. Mos uch karrali darajali qatorning yaqinlashish sohalari aniqlangan. Yaqinlashish tahlili Hornning yaqinlashish nazariyasi hamda gamma-funksiyaning katta argumentlardagi asimptotik xossalaridan foydalanish asosida amalga oshirilgan. Parametrlarning yigirma yettita turli kombinatsiyasi ko‘rib chiqilgan va har bir holat uchun qatorning mutlaq yaqinlashishi uchun zarur va yetarli shartlar aniqlangan, hamda MLning ba’zi xususiy hollari keltirilgan. Olingan natijalar kasr tartibli differensial tenglamalar nazariyasini boyitadi hamda Prabhakar kasr hosilasini o‘z ichiga oluvchi masalalarni tadqiq etishda samarali analitik asosni taqdim etadi.*

Kalit so‘zlar: *Mittag-Leffler funksiyasi, uch o‘zgaruvchili funksiya, kasr tartibli hisob, Prabhakar hosilasi, uch karrali darajali qator, yaqinlashish sohasi, gamma-funksiya, maxsus funksiyalar.*

Introduction. It should be noted that, numerous studies have investigated special functions of three variables. For instance, the bivariate and $\bar{F}_E^{(3)}(x,y,z)$ trivariate Mittag-Leffler (ML)-type functions were introduced in [1], where their properties were examined and their application to a generalized telegraph equation involving the regularized Prabhakar derivative was demonstrated. In [2], confluent three-variable hypergeometric functions and their fundamental properties were explored. Furthermore, several additional three-variable ML-type functions were proposed in [3] and [4]: $\bar{F}_A^{(3)}(x,y,z)$, $\bar{F}_B^{(3)}(x,y,z)$, $\bar{F}_C^{(3)}(x,y,z)$,

$\bar{F}_D^{(3)}(x,y,z)$ and $E^{(A)}_{(\alpha_1,\beta_1,\gamma_1,\delta_1),(\alpha_2,\beta_2,\gamma_2,\delta_2),(\alpha_3,\beta_3,\gamma_3,\delta_3)}(x,y,z)$, $E^{(B,C)}_{(\alpha_1,\beta_1,\gamma_1,\delta_1),(\alpha_2,\beta_2,\gamma_2,\delta_2),(\alpha_3,\beta_3,\gamma_3,\delta_3)}(x,y,z)$ $E^{(P,Q,R)}_{(\alpha_1,\beta_1,\gamma_1,\delta_1),(\alpha_2,\beta_2,\gamma_2,\delta_2),(\alpha_3,\beta_3,\gamma_3,\delta_3)}(x,y,z)$,

the latter being applied to coupled systems of linear constant-coefficient fractional differential equations with Caputo derivatives. The three-variable ML-type function $E^{(\eta)}_{\alpha,\beta,\gamma,\delta}(x,y,z)$ was introduced and studied in [5]. Additionally, learn the $E_{13}\left(\begin{matrix}\alpha_1,\beta_1,\delta_1;\alpha_2,\delta_2;\\ \alpha_3,\beta_2,\delta_3;\alpha_4,\delta_4;\alpha_5,\delta_5;\beta_3,\delta_6;\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)$ functions with three parameters that were covered in [6]. In [7], new trivariate ${}_3E_{13}\left(\begin{matrix}\alpha_1,\beta_1,\delta_1;\beta_2,\gamma_1,\delta_2;\\ \alpha_2,\beta_3,\gamma_2,\delta_3;\beta_4,\delta_4;\alpha_3,\delta_5;\beta_5,\delta_6;\gamma_3,\delta_7;\end{matrix}\middle|\begin{matrix}x,y\\ z\end{matrix}\right)$ and quadrivariate

$$ {}_4E_1\left(\begin{matrix}\alpha_1,\beta_1,\gamma_1,\delta_1;\gamma_2,\theta_1,\delta_2;\\ \alpha_2,\beta_2,\gamma_3,\theta_2,\delta_3;\gamma_4,\delta_4;\alpha_3,\delta_5;\beta_3,\delta_6;\gamma_5,\delta_7;\theta_3,\delta_8;\end{matrix}\middle|\begin{matrix}x;y\\ z;t\end{matrix}\right) $$

Mittag-Leffler-type functions were introduced, and their fundamental analytical properties were investigated. In particular, their regions of convergence, integral representations, and explicit estimates were studied. Several notable special cases were also obtained. Furthermore, the applicability of the proposed functions to the study of a partial differential equation was demonstrated. In [8], a new trivariate ${}_3E_1\left(\begin{matrix}\alpha_1,\delta_1,\mu_1,\sigma_1;\alpha_2,\delta_2,\sigma_2;\\ \alpha_3,\delta_3,\mu_2,\sigma_3;\alpha_4,\delta_4,\sigma_4;\alpha_5,\sigma_5;\delta_5,\sigma_6;\mu_3,\delta_7;\end{matrix}\middle|x;y;z\right)$ Mittag-Leffler-type function was introduced and a Cauchy problem was solved using this function. In [9], an integral representation of this function was obtained, while in [10], an estimate for the function was established. However, its region of convergence has not been investigated. In this paper, we establish the region of convergence of this new trivariate Mittag–Leffler-type function and present several corresponding cases and particular cases.

New trivariate Mittag-Leffler-type function

Let us, recall a trivariate ML-type function investigated in [8].

Definition 2.1.

Let the assume that, $x,\ y,\ z,\ \alpha_l,\ \delta_l,\ \mu_i,\sigma_n \in \mathbb{R};\ \min\{\alpha_l,\delta_l,\mu_i,\sigma_n\}>0,\ \left(l=\overline{1,5},\ i=\overline{1,3},\ n=\overline{1,7}\right)$ .Then the new trivariate ML-type function was defined as follows:

$$ {}_3E_1\left(\begin{matrix}\alpha_1,\delta_1,\mu_1,\sigma_1;\alpha_2,\delta_2,\sigma_2;\\ \alpha_3,\delta_3,\mu_2,\sigma_3;\alpha_4,\delta_4,\sigma_4;\alpha_5,\sigma_5;\delta_5,\sigma_6;\mu_3,\sigma_7;\end{matrix}\middle|x;y;z\right)= $$

$$ =\sum_{m,j,k=0}^{+\infty}\frac{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2)}{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4)}\times $$

$$ \times\frac{x^m y^j z^k}{\Gamma(\alpha_5 m+\sigma_5)\Gamma(\delta_5 j+\sigma_6)\Gamma(\mu_3 k+\sigma_7)}. \qquad (1) $$

Now, we are going to determine convergence areas of the function (1) by using the Horn method which used in [11]. The triple series in (1) converges absolutely for $|x|<\rho_1$, $|y|<\rho_2$ and $|z|<\rho_3$, if $\Delta_1=0$, $\Delta_2=0$, $\Delta_3=0$ as described in

$$ \rho_1=\min_{\eta,\nu,\xi>0}(E_1),\ \rho_2=\min_{\eta,\nu,\xi>0}(E_2),\ \rho_3=\min_{\eta,\nu,\xi>0}(E_3),\ \eta,\nu,\xi>0 $$

where $\Delta_1=\alpha_3+\alpha_4+\alpha_5-\alpha_1-\alpha_2$, $\Delta_2=\delta_3+\delta_4+\delta_5-\delta_1-\delta_2$, $\Delta_3=\mu_2+\mu_3-\mu_1$.

Then, we obtain

$$ E_1=\eta^{\alpha_5}\frac{(\alpha_3\eta+\delta_3\nu+\mu_2\xi)^{\alpha_3}(\alpha_4\eta+\delta_4\nu)^{\alpha_4}(\alpha_5)^{\alpha_5}}{(\alpha_1\eta+\delta_1\nu+\mu_1\xi)^{\alpha_1}(\alpha_2\eta+\delta_2\nu)^{\alpha_2}}, $$

$$E_2 = \nu^{\delta_5}\frac{\left(\alpha_3\eta+\delta_3\nu+\mu_2\xi\right)^{\delta_3}\left(\alpha_4\eta+\delta_4\nu\right)^{\delta_4}\left(\delta_5\right)^{\delta_5}}{\left(\alpha_1\eta+\delta_1\nu+\mu_1\xi\right)^{\delta_1}\left(\alpha_2\eta+\delta_2\nu\right)^{\delta_2}},\ E_3 = \xi^{\mu_3}\frac{\left(\alpha_3\eta+\delta_3\nu+\mu_2\xi\right)^{\mu_2}\left(\mu_3\right)^{\mu_3}}{\left(\alpha_1\eta+\delta_1\nu+\mu_1\xi\right)^{\mu_1}},$$

respectively.

Theorem 2.1. Let $\Delta_1, \Delta_2, \Delta_3$ be the quantities defined above. Then the series associated with the trivariate Mittag-Leffler-type function converges absolutely according to the following cases:

Case 1. Let $\Delta_1 > 0$, $\Delta_2 > 0, \Delta_3 > 0$. Making use of equations (3)-(6), we obtain

$$\phi_1(\eta,\nu,\xi) = \lim_{t\to\infty} f_1(\eta t,\nu t,\xi t) = 0,\ \phi_2(\eta,\nu,\xi) = \lim_{t\to\infty} f_2(\eta t,\nu t,\xi t) = 0,$$
$$\phi_3(\eta,\nu,\xi) = \lim_{t\to\infty} f_3(\eta t,\nu t,\xi t) = 0.$$

Thus $\rho_1,\ \rho_2, \rho_3 \to \infty$, and the series converges absolutely for any x, y, z.

Case 2. Let $\Delta_1 = 0,\ \Delta_2 = 0,\ \Delta_3 = 0$. It follows from equations (3), (4), (5), and (6) that

$$\phi_1(\eta,\nu,\xi) = \frac{1}{E_1},\ \phi_2(\eta,\nu,\xi) = \frac{1}{E_2},\ \phi_3(\eta,\nu,\xi) = \frac{1}{E_3},$$

which yields the parametric form of curve C shown in the (ρ_1,ρ_2,ρ_3) - plane as $\rho_1 = G_1, \rho_2 = G_2$ and $\rho_3 = G_3$. Hence, absolute convergence of the series is ensured for $|x| < \rho_1,\ |y| < \rho_2$ and $|z| < \rho_3$, with $\rho_1 = \min_{(\eta,\nu,\xi)}(E_1),\ \rho_2 = \min_{(\eta,\nu,\xi)}(E_2)$ and $\rho_3 = \min_{(\eta,\nu,\xi)}(E_3)$.

Case 3. Let $\Delta_1 < 0,\ \Delta_2 < 0$ and $\Delta_3 < 0$. The series diverges for $\rho_1 = \rho_2 = \rho_3 = 0$. Convergence of the series occurs exclusively at the point $x = y = z = 0$.

Case 4. Assume that $\Delta_1 = 0,\ \Delta_2 > 0$ and $\Delta_3 > 0$. The series converges uniquely at the point $|x| < \rho_1,\ |y| < \infty$ and $|z| < \infty$.

Case 5. Suppose that $\Delta_1 > 0,\ \Delta_2 = 0$ and $\Delta_3 = 0$. The series is convergent solely at the point $|x| < \infty,\ |y| < \rho_2$ and $|z| < \rho_3$.

Case 6. Let $\Delta_1 > 0,\ \Delta_2 > 0$ and $\Delta_3 = 0$. Thus, the series converges the absolutely in the domain where $|x| < \infty,\ |y| < \infty$ and $|z| < \rho_3$.

Case 7. Let $\Delta_1 > 0,\ \Delta_2 > 0$ and $\Delta_3 < 0$. Thus, the series converges the absolutely in the domain where $|x| < \infty,\ |y| < \infty$ and $z = 0$.

Case 8. Let $\Delta_1 > 0,\ \Delta_2 < 0$ and $\Delta_3 > 0$. Thus, the series converges the absolutely in the domain where $|x| < \infty,\ y = 0$ and $|z| < \infty$.

Case 9. Let $\Delta_1 > 0,\ \Delta_2 < 0$ and $\Delta_3 < 0$. Thus, the series converges absolutely for $|x| < \infty,\ y = 0$, $z = 0$.

Case 10. Let $\Delta_1 > 0,\ \Delta_2 < 0$ and $\Delta_3 = 0$. Thus, the series converges absolutely for $|x| < \infty,\ y = 0$, $|z| < \rho_3$.

Case 11. Let $\Delta_1 > 0,\ \Delta_2 = 0$ and $\Delta_3 > 0$. Thus, the series converges absolutely for $|x| < \infty,\ |y| < \rho_2,\ |z| < \infty$.

Case 12. Let $\Delta_1 > 0,\ \Delta_2 = 0$ and $\Delta_3 < 0$. Thus, the series converges absolutely for $|x| < \infty,\ |y| < \rho_2,\ z = 0$.

Case 13. Let $\Delta_1 < 0,\ \Delta_2 > 0$ and $\Delta_3 > 0$. Thus, the series converges absolutely for $x = 0,\ |y| < \infty$, $|z| < \infty$.

Case 14. Let $\Delta_1 < 0,\ \Delta_2 > 0$ and $\Delta_3 < 0$. Thus, the series converges absolutely for $x = 0,\ |y| < \infty$, $z = 0$.

Case 15. Let $\Delta_1 < 0,\ \Delta_2 > 0$ and $\Delta_3 = 0$. Thus, the series converges absolutely for $x = 0,\ |y| < \infty$, $|z| < \rho_3$.

Case 16. Let $\Delta_1 < 0,\ \Delta_2 < 0$ and $\Delta_3 > 0$. Thus, the series converges absolutely for $x = 0, y = 0$, $|z| < \infty$.

Case 17. Let $\Delta_1 < 0,\ \Delta_2 < 0$ and $\Delta_3 = 0$. Thus, the series converges absolutely for $x = 0, y = 0$, $|z| < \rho_3$.

Case 18. Let $\Delta_1 < 0,\ \Delta_2 = 0$ and $\Delta_3 > 0$. Thus, the series converges absolutely for $x = 0,\ |y| < \rho_2$, $|z| < \infty$.

Case 19. Let $\Delta_1 < 0,\ \Delta_2 = 0$ and $\Delta_3 < 0$. Thus, the series converges absolutely for $x = 0,\ |y| < \rho_2$, $z = 0$.

Case 20. Let $\Delta_1 < 0,\ \Delta_2 = 0$ and $\Delta_3 = 0$. Thus, the series converges absolutely for $x = 0,\ |y| < \rho_2$, $z = 0$.

Case 21. Let $\Delta_1 = 0,\ \Delta_2 > 0$ and $\Delta_3 < 0$. Thus, the series converges absolutely for $|x| < \rho_1,\ |y| < \infty,\ z = 0$.

Case 22. Let $\Delta_1 = 0,\ \Delta_2 > 0$ and $\Delta_3 = 0$. Thus, the series converges absolutely for $|x| < \rho_1,\ |y| < \infty$, $|z| < \rho_3$.

Case 23. Let $\Delta_1 = 0,\ \Delta_2 < 0$ and $\Delta_3 > 0$. Thus, the series converges absolutely for $|x| < \rho_1,\ y = 0$, $|z| < \infty$.

Case 24. Let $\Delta_1 = 0,\ \Delta_2 < 0$ and $\Delta_3 < 0$. Thus, the series converges absolutely for $|x| < \rho_1,\ y = 0$, $z = 0$.

Case 25. Let $\Delta_1 = 0,\ \Delta_2 < 0$ and $\Delta_3 = 0$. Thus, the series converges absolutely for $|x| < \rho_1,\ y = 0$, $|z| < \rho_3$.

Case 26. Let $\Delta_1 = 0,\ \Delta_2 = 0$ and $\Delta_3 > 0$. Thus, the series converges absolutely for $|x| < \rho_1,\ |y| < \rho_2,\ |z| < \infty$.

Case 27. Let $\Delta_1 = 0,\ \Delta_2 = 0$ and $\Delta_3 < 0$. Thus, the series converges absolutely for $|x| < \rho_1,\ |y| < \rho_2,\ z = 0$.

It should be noted that the results for the remaining cases can be derived from these twenty seven cases.

Proof. Following Horn's approach [11], we determine the convergence region of the trivariate Mittag-Leffler-type function ${}_3E_1(.)$.To simplify the notation, let the following expression be denoted by $A(m, j, k)$:

$$A(m,j,k) = \frac{\Gamma(\alpha_1 m + \delta_1 j + \mu_1 k + \sigma_1)\Gamma(\alpha_2 m + \delta_2 j + \sigma_2)}{\Gamma(\alpha_3 m + \delta_3 j + \mu_2 k + \sigma_3)\Gamma(\alpha_4 m + \delta_4 j + \sigma_4)\Gamma(\alpha_5 m + \sigma_5)\Gamma(\delta_5 j + \sigma_6)\Gamma(\mu_3 k + \sigma_7)}, \quad (2)$$

We introduce the following notations:

$$f_1(m,j,k) = \frac{A(m+1,j,k)}{A(m,j,k)},\quad f_2(m,j,k) = \frac{A(m,j+1,k)}{A(m,j,k)},\quad f_3(m,j,k) = \frac{A(m,j,k+1)}{A(m,j,k)}. \quad (3)$$

Using the notations, we obtain $f_1(m,j,k)$, $f_2(m,j,k)$ and $f_3(m,j,k)$:

$$f_1(m,j,k) = \frac{\Gamma(\alpha_1 m + \delta_1 j + \mu_1 k + \sigma_1 + \alpha_1)\Gamma(\alpha_2 m + \delta_2 j + \sigma_2 + \alpha_2)}{\Gamma(\alpha_3 m + \delta_3 j + \mu_2 k + \sigma_3 + \alpha_3)\Gamma(\alpha_4 m + \delta_4 j + \sigma_4 + \alpha_4)} \times$$

$$\times \frac{1}{\Gamma(\alpha_5 m + \sigma_5 + \alpha_5)\Gamma(\delta_5 j + \sigma_6)\Gamma(\mu_3 k + \sigma_7)} \times$$

$$\times\frac{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4)}{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2)}\times\frac{\Gamma(\alpha_5 m+\sigma_5)\Gamma(\delta_5 j+\sigma_6)\Gamma(\mu_3 k+\sigma_7)}{1}=$$

$$=\frac{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1+\alpha_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2+\alpha_2)}{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2)}\times$$

$$\times\frac{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4)\Gamma(\alpha_5 m+\sigma_5)}{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3+\alpha_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4+\alpha_4)\Gamma(\alpha_5 m+\sigma_5+\alpha_5)},$$

similarly, we define

$$f_2(m,j,k)=\frac{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1+\delta_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2+\delta_2)}{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3+\delta_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4+\delta_4)}\times$$

$$\times\frac{1}{\Gamma(\alpha_5 m+\sigma_5)\Gamma(\delta_5 j+\sigma_6+\delta_5)\Gamma(\mu_3 k+\sigma_7)}\times$$

$$\times\frac{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4)}{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2)}\times\frac{\Gamma(\alpha_5 m+\sigma_5)\Gamma(\delta_5 j+\sigma_6)\Gamma(\mu_3 k+\sigma_7)}{1}=$$

$$=\frac{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1+\delta_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2+\delta_2)}{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2)}\times$$

$$\times\frac{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4)\Gamma(\delta_5 j+\sigma_6)}{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3+\delta_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4+\delta_4)\Gamma(\delta_5 j+\sigma_6+\delta_5)},$$

in the same manner

$$f_3(m,j,k)=\frac{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1+\mu_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2)}{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3+\mu_2)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4)}\times$$

$$\times\frac{1}{\Gamma(\alpha_5 m+\sigma_5)\Gamma(\delta_5 j+\sigma_6)\Gamma(\mu_3 k+\sigma_7+\mu_3)}\times$$

$$\times\frac{\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3)\Gamma(\alpha_4 m+\delta_4 j+\sigma_4)}{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1)\Gamma(\alpha_2 m+\delta_2 j+\sigma_2)}\times\frac{\Gamma(\alpha_5 m+\sigma_5)\Gamma(\delta_5 j+\sigma_6)\Gamma(\mu_3 k+\sigma_7)}{1}=$$

$$=\frac{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1+\mu_1)\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3)\Gamma(\mu_3 k+\sigma_7)}{\Gamma(\alpha_1 m+\delta_1 j+\mu_1 k+\sigma_1)\Gamma(\alpha_3 m+\delta_3 j+\mu_2 k+\sigma_3+\mu_2)\Gamma(\mu_3 k+\sigma_7+\mu_3)}.$$

Then, we have

$$f_1(\eta t,\nu t,\xi t)=\frac{\Gamma(\alpha_1\eta t+\delta_1\nu t+\mu_1\xi t+\sigma_1+\alpha_1)\Gamma(\alpha_2\eta t+\delta_2\nu t+\sigma_2+\alpha_2)}{\Gamma(\alpha_1\eta t+\delta_1\nu t+\mu_1\xi t+\sigma_1)\Gamma(\alpha_2\eta t+\delta_2\nu t+\sigma_2)}\times$$

$$\times\frac{\Gamma(\alpha_3\eta t+\delta_3\nu t+\mu_2\xi t+\sigma_3)\Gamma(\alpha_4\eta t+\delta_4\nu t+\sigma_4)\Gamma(\alpha_5\eta t+\sigma_5)}{\Gamma(\alpha_3\eta t+\delta_3\nu t+\mu_2\xi t+\sigma_3+\alpha_3)\Gamma(\alpha_4\eta t+\delta_4\nu t+\sigma_4+\alpha_4)\Gamma(\alpha_5\eta t+\sigma_5+\alpha_5)},$$

$$f_2(\eta t,\nu t,\xi t)=\frac{\Gamma(\alpha_1\eta t+\delta_1\nu t+\mu_1\xi t+\sigma_1+\delta_1)\Gamma(\alpha_2\eta t+\delta_2\nu t+\sigma_2+\delta_2)}{\Gamma(\alpha_1\eta t+\delta_1\nu t+\mu_1\xi t+\sigma_1)\Gamma(\alpha_2\eta t+\delta_2\nu t+\sigma_2)}\times$$

$$\times\frac{\Gamma(\alpha_3\eta t+\delta_3\nu t+\mu_2\xi t+\sigma_3)\Gamma(\alpha_4\eta t+\delta_4\nu t+\sigma_4)\Gamma(\delta_5\nu t+\sigma_6)}{\Gamma(\alpha_3\eta t+\delta_3\nu t+\mu_2\xi t+\sigma_3+\delta_3)\Gamma(\alpha_4\eta t+\delta_4\nu t+\sigma_4+\delta_4)\Gamma(\delta_5\nu t+\sigma_6+\delta_5)},$$

$$f_3(\eta t,\nu t,\xi t)=\frac{\Gamma(\alpha_1\eta t+\delta_1\nu t+\mu_1\xi t+\sigma_1+\mu_1)\Gamma(\alpha_3\eta t+\delta_3\nu t+\mu_2\xi t+\sigma_3)\Gamma(\mu_3\xi t+\sigma_7)}{\Gamma(\alpha_1\eta t+\delta_1\nu t+\mu_1\xi t+\sigma_1)\Gamma(\alpha_3\eta t+\delta_3\nu t+\mu_2\xi t+\sigma_3+\mu_2)\Gamma(\mu_3\xi t+\sigma_7+\mu_3)}.$$

Using the asymptotic properties of the Gamma function for large arguments given in [4],

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}\sim z^{\alpha-\beta}\left[1+\frac{(\alpha-\beta)(\alpha+\beta-1)}{2z}+O\left(z^{-2}\right)\right],\ \ |\arg(z)|\le\pi,$$

we obtain

$$f_1(\eta t,\nu t,\xi t)\to t^{\alpha_1+\alpha_2-\alpha_3-\alpha_4-\alpha_5}\,\frac{(\alpha_1\eta+\delta_1\nu+\mu_1\xi)^{\alpha_1}(\alpha_2\eta+\delta_2\nu)^{\alpha_2}}{(\alpha_3\eta+\delta_3\nu+\mu_2\xi)^{\alpha_3}(\alpha_4\eta+\delta_4\nu)^{\alpha_4}(\alpha_5\eta)^{\alpha_5}},$$

$$f_1(\eta t,\nu t,\xi t)\sim\frac{1}{E_1}t^{-\Delta_1},\ \Delta_1=\alpha_3+\alpha_4+\alpha_5-\alpha_1-\alpha_2, \tag{4}$$

$$E_1=\eta^{\alpha_5}\frac{(\alpha_3\eta+\delta_3\nu+\mu_2\xi)^{\alpha_3}(\alpha_4\eta+\delta_4\nu)^{\alpha_4}(\alpha_5)^{\alpha_5}}{(\alpha_1\eta+\delta_1\nu+\mu_1\xi)^{\alpha_1}(\alpha_2\eta+\delta_2\nu)^{\alpha_2}},\ G_1=\frac{(\alpha_3)^{\alpha_3}(\alpha_4)^{\alpha_4}(\alpha_5)^{\alpha_5}}{(\alpha_1)^{\alpha_1}(\alpha_2)^{\alpha_2}},$$

likewise we define

$$f_2(\eta t,\nu t,\xi t)\to t^{\delta_1+\delta_2-\delta_3-\delta_4-\delta_5}\,\frac{(\alpha_1\eta+\delta_1\nu+\mu_1\xi)^{\delta_1}(\alpha_2\eta+\delta_2\nu)^{\delta_2}}{(\alpha_3\eta+\delta_3\nu+\mu_2\xi)^{\delta_3}(\alpha_4\eta+\delta_4\nu)^{\alpha_4}(\delta_5\nu)^{\delta_5}},$$

$$f_2(\eta t,\nu t,\xi t)\sim\frac{1}{E_2}t^{-\Delta_2},\quad \Delta_2=\delta_3+\delta_4+\delta_5-\delta_1-\delta_2, \tag{5}$$

$$E_2=\nu^{\delta_5}\frac{(\alpha_3\eta+\delta_3\nu+\mu_2\xi)^{\delta_3}(\alpha_4\eta+\delta_4\nu)^{\alpha_4}(\delta_5)^{\delta_5}}{(\alpha_1\eta+\delta_1\nu+\mu_1\xi)^{\delta_1}(\alpha_2\eta+\delta_2\nu)^{\delta_2}},\ G_2=\frac{(\delta_3\nu)^{\delta_3}(\delta_4)^{\alpha_4}(\delta_5)^{\delta_5}}{(\delta_1)^{\delta_1}(\delta_2)^{\delta_2}},$$

analogously, we define

$$f_3(\eta t,\nu t,\xi t)\to t^{\mu_1-\mu_2-\mu_3}\,\frac{(\alpha_1\eta+\delta_1\nu+\mu_1\xi)^{\mu_1}}{(\alpha_3\eta+\delta_3\nu+\mu_2\xi)^{\mu_2}(\mu_3\xi)^{\mu_3}},\ f_3(\eta t,\nu t,\xi t)\sim\frac{1}{E_3}t^{-\Delta_3},$$

$$\Delta_3=\mu_2+\mu_3-\mu_1, \tag{6}$$

$$E_3=\xi^{\mu_3}\frac{(\alpha_3\eta+\delta_3\nu+\mu_2\xi)^{\mu_2}(\mu_3)^{\mu_3}}{(\alpha_1\eta+\delta_1\nu+\mu_1\xi)^{\mu_1}},\quad G_3=\frac{(\mu_2)^{\mu_2}(\mu_3)^{\mu_3}}{(\mu_1)^{\mu_1}}.$$

Theorem 2.1 is proved.

Now, we present some particular cases of the trivariate ML-type function.

2.1. Particular cases

For $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\delta_1,\delta_2,\delta_3,\delta_4,\mu_1,\mu_2\to 0$, $\sigma_1=\sigma_2=\sigma_3=\sigma_4=1$,

$${}_3E_1\left(\begin{matrix}0,0,0,1;0,0,1;\\0,0,0,1;0,0,1;\alpha_5,\sigma_5;\delta_5,\sigma_6;\mu_3,\sigma_7;\end{matrix}\middle|\,x;y;z\right)=E_{\alpha_5,\sigma_5}(x)E_{\delta_5,\sigma_6}(y)E_{\mu_3,\sigma_7}(z).$$

For $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\delta_1,\delta_2,\delta_3,\mu_1,\mu_2\to 0$, $\sigma_1=\sigma_2=\sigma_3=\sigma_4=1$,

$${}_3E_1\left(\begin{matrix}0,0,0,1;0,0,1;\\0,0,0,1;0,\delta_4,1;\alpha_5,\sigma_5;\delta_5,\sigma_6;\mu_3,\sigma_7;\end{matrix}\middle|\,x;y;z\right)=E_{\alpha_5,\sigma_5}(x)e^{1,\sigma_6}_{\delta_4,-\delta_5}(y)E_{\mu_3,\sigma_7}(z).$$

For $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\delta_5\to 0$, $\sigma_1=\sigma_2=\sigma_3=\sigma_6=\delta_4=1$,

$${}_3E_1\left(\begin{matrix}0,\delta_1,\mu_1,1;0,\delta_2,1;\\0,\delta_3,\mu_2,1;0,1,\sigma_4;\alpha_5,\sigma_5;0,1;\mu_3,\sigma_7;\end{matrix}\middle|\,x;y;z\right)=E_{\alpha_5,\sigma_5}(x)E_2\left(\begin{matrix}\delta_1,\mu_1,1;\delta_2,1;\\\delta_3,\mu_2,1;1,\sigma_4;\mu_3,\sigma_7;\end{matrix}\middle|\,\begin{matrix}y\\z\end{matrix}\right).$$

For $\alpha_2,\alpha_4,\delta_1,\delta_2,\delta_3,\delta_4\to 0$, $\sigma_2=\sigma_4=1$,

$${}_3E_1\left(\begin{matrix}\alpha_1,0,\mu_1,\sigma_1;0,0,1;\\\alpha_3,0,\mu_2,\sigma_3;0,0,1;\alpha_5,\sigma_5;\delta_5,\sigma_6;\mu_3,\sigma_7;\end{matrix}\middle|\,x;y;z\right)=E_{\delta_5,\sigma_6}(y)E_8\left(\begin{matrix}\alpha_1,\mu_1,\sigma_1;\\\alpha_3,\mu_2,\sigma_3;\alpha_5,\sigma_5;\mu_3,\sigma_7;\end{matrix}\middle|\,\begin{matrix}x\\z\end{matrix}\right)$$

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