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$L_2^{(1,0)}(0,1)$ FAZOSIDA OPTIMAL INTERPOLYATSION FORMULANING EKSTREMAL FUNKSIYASI

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Annotatsiya. *$L_2^{(1,0)}(0,1)$ fazosida optimal interpolyatsion formulaning ekstremal funksiyasini qurish va uning xatolik funksionalini baholash masalalari o‘rganilgan. Tadqiqot jarayonida xatolik funksionalining normasi aniq analitik ko‘rinishda topilgan bo‘lib, bu olingan natijalarning nazariy ahamiyatini yanada oshiradi. Shuningdek, $L_2^{(1,0)}(0,1)$ fazosida optimal interpolyatsion formulaning ekstremal funksiyasi aniqlangan. Ushbu yondashuvlar zamonaviy matematika, xususan, funksional analiz hamda hisoblash matematikasi sohalarida qator masalalarni hal etishga xizmat qiladi. Olingan natijalar nazariy hisob-kitoblarni soddalashtirish va ularning aniqligini oshirish imkonini beradi.*

Kalit so‘zlar: *$L_2^{(1,0)}(0,1)$ fazosi, optimal interpolyatsion formula, ekstremal funksiya, xatolik funksionali, xatolik funksionalining normasi.*

THE EXTREMAL FUNCTION OF THE OPTIMAL INTERPOLATION FORMULA IN THE $L_2^{(1,0)}(0,1)$ SPACE

Abstract. *The problems of constructing the extremal function of the optimal interpolation formula in the $L_2^{(1,0)}(0,1)$ space and evaluating its error functional are studied. Throughout the research, the norm of the error functional is derived in an explicit analytical form, which significantly enhances the theoretical significance of the obtained results. Furthermore, the extremal function of the optimal interpolation formula in the $L_2^{(1,0)}(0,1)$ space is determined. These approaches serve to solve a number of problems in modern mathematics, particularly in the fields of functional analysis and computational mathematics. The obtained results provide a means to simplify theoretical calculations and improve their accuracy.*

Keywords: *$L_2^{(1,0)}(0,1)$ space, optimal interpolation formula, extremal function, error functional, norm of the error functional.*

ЭКСТРЕМАЛЬНАЯ ФУНКЦИЯ ОПТИМАЛЬНОЙ ИНТЕРПОЛЯЦИОННОЙ ФОРМУЛЫ В ПРОСТРАНСТВЕ $L_2^{(1,0)}(0,1)$

Аннотация. *Изучены вопросы построения экстремальной функции оптимальной интерполяционной формулы в пространстве $L_2^{(1,0)}(0,1)$ и оценки её функционала погрешности. В ходе исследования норма функционала погрешности найдена в явном аналитическом виде, что существенно повышает теоретическую значимость полученных результатов. Кроме того, определена экстремальная функция оптимальной интерполяционной формулы в пространстве $L_2^{(1,0)}(0,1)$. Представленные подходы служат для решения ряда задач современной математики, в частности в области функционального анализа и вычислительной математики. Полученные результаты позволяют упростить теоретические расчеты и повысить их точность.*

Ключевые слова: *пространство $L_2^{(1,0)}(0,1)$, оптимальная интерполяционная формула, экстремальная функция, функционал погрешности, норма функционала погрешности.*

Kirish va masalaning qo‘yilishi. Splayn funksiyalar va ularning tatbiqlari zamonaviy hisoblash matematikasi, interpolyatsiya va yaqinlashtirish nazariyasida muhim o‘rin tutadi. Splaynlar nazariyasining poydevori va ularning asosiy xossalari J.H. Ahlberg, E.N. Nilson, J.L. Walsh [1], J.C. Holladay [11] hamda C. de Boor [9] kabi olimlarning dastlabki ilmiy ishlarida batafsil yoritilgan. Keyinchalik bu nazariya ko‘p

o‘lchovli va variatsion splaynlar yo‘nalishida R. Arcangeli va boshqalar [2], M. Attea [3], A.Y. Bezhayev va V.A. Vasilenko [8],

L.L. Schumaker [12], shuningdek S.B. Stechkin va Y.N. Subbotinlar [19] tomonidan kengaytirildi va jadal rivojlantirildi.

Splaynlarni optimal kubatur va interpolyatsion formulalarini qurishda qo‘llash asosan S.L. Sobolevning [16], [17], [18] fundamental ishlariga borib taqaladi. Uning ko‘p o‘lchovli funksiyalarni interpolyatsiya qilish va kubatur formulalari nazariyasi bugungi kundagi tadqiqotlar uchun kuchli matematik apparat bo‘lib xizmat qilmoqda.

So‘nggi yillarda turli xil Gilbert va Sobolev fazolarida optimal interpolyatsiya formulalarini qurish, ularning xatoliklarini baholash va koeffitsiyentlarini hisoblash algoritmlarini yaratish bo‘yicha qator muhim amaliy natijalarga erishildi. Xususan, X.M. Shadimetov va A.R. Hayotovlar tomonidan $\frac{d^{2m}}{dx^{2m}}-\frac{d^{2m-2}}{dx^{2m-2}}$ differensial operatorining diskret analoglari qurildi va ularning xossalari chuqur o‘rganildi [13], [14]. Ushbu olingan natijalar asosida $W_2^{(m,m-1)}(0,1)$ fazosida yarim normani minimallashtiruvchi interpolyatsion splaynlarni qurish [15], shuningdek, maxsus fazolarda optimal interpolyatsiya formulalari koeffitsiyentlarini hisoblash algoritmlari [10] ishlab chiqildi.

Shu bilan bir qatorda, S.S. Babayev, A.R. Hayotov va boshqa hammualliflar tomonidan $K_2(P_2)$, $L_2^{(m)}(0,1)$ hamda $W_2^{(m,m-1)}$ fazolarida optimal interpolyatsion formulalar qurish masalalari hal etildi [5], [6], [7], [20]. Ushbu izlanishlar asosida olingan matematik modellar hisoblash matematikasining texnik va amaliy masalalarini yuqori aniqlikda yechish imkoniyatini taqdim etadi. Ushbu maqolada yuqorida keltirilgan fundamental va zamonaviy tadqiqotlarga tayangan holda bajarilgan.

Biz quyidagi ko‘rinishdagi interpolyatsion formulani qaraymiz

$$\varphi(x)\cong P_\varphi(x)=\sum_{\beta=0}^{N}C_\beta(x)\cdot\varphi(x_\beta) \tag{1}$$

yig‘indi va fundamental splayn orasidagi quyidagi ayirmaga

$$(\ell(x,z),\varphi)=\varphi(x)-\sum_{\beta=0}^{N}C_\beta(x)\cdot\varphi(x_\beta) \tag{2}$$

(1)-interpolyatsion formulaning xatoligi deyiladi.

Yuqoridagi (2)-xatolikga mos xatolik funksionali quyidagi

$$\ell(x,z)=\delta(x-z)-\sum_{\beta=0}^{N}C_\beta(z)\delta(x-x_\beta) \tag{3}$$

ko‘rinishda bo‘ladi.

Bu yerda $C_\beta(z)$-koeffitsiyentlar x_β-tugun nuqtalar, φ funksiyalarimiz esa $L_2^{(1,0)}(0,1)$ fazosiga tegishli bo‘lib, ushbu fazo quyidagicha ta‘riflanadi:

$$L_2^{(1,0)}(0,1)=\left\{\varphi:[0,1]\to R\,\middle|\,\varphi-abs.\,uzl.,\ \varphi'(x)\in L_2(0,1)\right\}.$$

Ushbu fazoga tegishli ixtiyoriy ikkita φ va ψ funksiyalarning skalyar ko‘paytmasi quyidagicha aniqlanadi.

$$\langle\varphi,\psi\rangle_{L_2^{(1,0)}}=\int_0^1\left(\varphi'(x)\psi'(x)+\varphi(x)\psi(x)\right)dx \tag{4}$$

$L_2^{(1,0)}(0,1)$ fazosi Gilbert fazosi bo‘lgani uchun norma skalyar ko‘paytma yordamida kiritiladi

$$\|\varphi\|=\left\{\int_0^1\left(\left(\varphi'(x)\right)^2+\left(\varphi(x)\right)^2\right)dx\right\}^{1/2}. \tag{5}$$

Funksionalning normasini ta’rifiga ko‘ra ushbu o‘rinli

$$\|\ell\|_{L_2^{(1,0)*}}=\sup_{\varphi,\|\varphi\|\neq0}\frac{|(\ell,\varphi)|}{\|\varphi\|_{L_2^{(1,0)}}}.$$

Koshi-Shvars tengsizligiga ko‘ra quyidagi munosabat o‘rinli

$$|(\ell,\varphi)|\le\|\ell\|\cdot\|\varphi\|.$$

Bundan ko‘rinib turibdiki xatolikning absolyut qiymati yuqoridan $L_2^{(1,0)}(0,1)$ fazoga tegishli φ funskiyaning normasi va $L_2^{(1,0)*}(0,1)$ fazoga tegishli $\ell(x,z)$ funksionalning normasi yordamida yuqoridan baholanadi. Shuning uchun ham xatolikning absolyut qiymatini yuqoridan bahosini topish uchun xatolik funksionalini normasini ko‘rinishini aniqlashimiz zarur.

1-masala: Yuqoridagi (1)-ko‘rinishdagi interpolyatsion formulaning xatolik funksionalining normasini ko‘rinishini topish.

Xatolik funksionalining ko‘rinishidan ma‘lumki xatolik funksionali x_β tugun nuqtalar va $C_\beta(z)$ koeffitsiyentlarga bog‘liq. Sard ma’nosida interpolyatsion formula qurish uchun x_β tugun nuqtalarni fiksrlab $C_\beta(z)$ koeffitsiyentlar bo‘yicha minimum qiymatini topamiz.

2-masala: Xatolik funksionali normasiga eng kichik qiymat beruvchi $C_\beta(z)$ koeffitsiyentlarini topish.

Endi biz keyingi paragrafda 1-masalani yechish bilan shug‘ullanamiz.

Ektremal funksiya va xatolik funksionalining normasi

Yuqoridagi 1-masalalarni yechish uchun bizga ekstremal funksiya tushunchasi kerak, ya’ni

$$\left|(\ell,\psi_\ell)\right| = \|\ell\|\cdot\|\psi_\ell\|$$

ψ_ℓ funksiyaga ekstremal funksiya deyiladi.

$L_2^{(1,0)}(0,1)$ fazosi Gilbert fazosi bo‘lgani uchun chiziqli uzluksiz funksionalning umumiy ko‘rinishi haqidagi Riss teoremasidan foydalanib quyidagi munosabatni yozib olamiz:

Qo‘shma fazoda aniqlangan har qanday $\ell \in L_2^{(1,0)*}(0,1)$ funksioanl uchun $L_2^{(1,0)}(0,1)$ fazoda $\exists!\psi_\ell(x)$ funksiya topilib ushbu munosabatlar o‘rinli bo‘ladi

$$(\ell,\varphi) = \langle \psi_\ell, \varphi\rangle \tag{6}$$

$$\|\ell\| = \|\psi_\ell\| \tag{7}$$

(6)-munosabatdan foydalanib skalyar ko‘paytmani bo‘laklab integrallaymiz

$$(\ell,\varphi) = \int_0^1 \left(\varphi'(x)\psi_\ell'(x) + \varphi(x)\psi_\ell(x)\right)dx = \varphi(x)\psi_\ell'(x)\Big|_0^1 + \int_0^1 \varphi(x)\left(\psi_\ell(x) - \psi_\ell''(x)\right)dx. \tag{8}$$

Biz ψ_ℓ funksiyalarni $\psi_\ell'(x)\Big|_0^1 = 0$ shartni qanoatlantiradigan qilib topib olamiz.

$$-\ell(x,z) = \psi_\ell''(x) - \psi_\ell(x) \tag{9}$$

$$\psi_\ell'(x)\Big|_0^1 = 0 \tag{10}$$

(9)-differensial tenglamani umumiy yechimi bu: ushbu (9)-tenglamaga mos bir jinsli differensial yenglamaning umumiy yechimi bilan bir jinsli bo‘lmagan holatidagi bitta xususiy yechimi yig‘indisi orqali ifodalash mumkin. Ushbu bir jinsli differensial tenglamaning

$$y_1''(x) - y_1(x) = 0$$

Yechimini quyidagicha qidiramiz e^{kx}. Differensial tenglamaga keltirib qo‘ysak ushbu ko‘rinishdagi tenglamaga kelamiz

$$k^2 - 1 = 0,$$

Bundan esa $k = \pm 1$ ekanligi kelib chiqadi. Demak bir jinsli differensial tenglamaning umumiy yechimi $y_1(x) = d_1e^x + d_2e^{-x}$ ko‘rinishda bo‘ladi. Endi bitta xususiy yechimini topish bilan shug‘ullanamiz buning uchun esa bir jinsli bo`lmagan differensial tenglamaning fundamental yechimini topishimiz kerak.

Fundamental yechim tushunchasi: Faraz qilaylik ushbu

$$L\left[u(x)\right] = f(x) \tag{11}$$

ko‘rinishda differensial tenglama berilgan bo‘lsin ushbu tenglamaning fundamental yechimi deb,

$$L\left[g(x)\right] = \delta(x)$$

tenglamaning yechimiga aytiladi. (11)-differensal tenglamaning xususiy yechimi esa $u(x) = f(x) * g(x)$ ko`rinishda bo‘ladi. Bu yerda $g(x)$ bu L operatorning fundamental yechimi, $(*)$ – o‘rama amali ushbu amal uzluksiz funksiyalar uchun quyidagicha hisoblanadi

$$f(x)*g(x)=\int_{-\infty}^{+\infty} f(y)g(x-y)dy.$$

(9)-differensial tenglamaning xususiy yechimi quyidagi ko‘rinishda bo‘ladi

$$y_2(x)=-\ell(x,z)*G_1(x) \tag{12}$$

Bu yerda $G_1(x)$ funksiya $\left(\frac{d^2}{dx^2}-1\right)$ differensial operatorning fundamental yechimi va uning ko‘rinishi quyidagicha:

$$G_1(x)=\frac{sign(x)}{2}\cdot\sinh(x).$$

Bundan foydalanib (9)-differensial tenglamaning umumiy yechimi quyidagicha

$$\psi_\ell(x)=-\ell(x,z)*G_1(x)+d_1e^x+d_2e^{-x}$$

(10)-chegaraviy shartlarni hisobga olsak u holda $d_1=d_2=0$ bo‘ladi. Bundan esa quyidagi teorema o‘rinli.

1-teorema: Ushbu $L_2^{(1,0)}(0,1)$ fazosida (1)-interpolyatsion formulaning ekstremal funksiyasi quyidagi

$$\psi_\ell(x)=-\ell(x,z)*G_1(x) \tag{11}$$

ko‘rinishida bo‘ladi.

O‘rama amalining ta’rifiga asosan quyidagini yozamiz

$$\psi_\ell(x)=-\ell(x,z)*G_1(x)=-\int_{-\infty}^{+\infty}\left(\delta(y-z)-\sum_{\beta=0}^{N}C_\beta(z)\delta(y-x_\beta)\right)G_1(x-y)dy=$$

$$=-G_1(x-z)+\sum_{\beta=0}^{N}C_\beta(z)G_1(x-x_\beta)$$

Normani kvadratini ko‘rinishini topish uchun ekstremal funksiya va Riss teoremasidan foydalanamiz.

$$\|\ell\|^2=\|\ell\|\cdot\|\ell\|=\|\ell\|\cdot\|\psi_\ell\|=(\ell,\psi_\ell)=\int_{-\infty}^{+\infty}\ell(x,z)\psi_\ell(x)dx$$

$$\int_{-\infty}^{+\infty}\ell(x,z)\psi_\ell(x)dx=\int_{-\infty}^{+\infty}\left(\delta(x-z)-\sum_{\gamma=0}^{N}C_\gamma(z)\delta(x-x_\gamma)\right)\psi_\ell(x)dx=$$

$$=\sum_{\beta=0}^{N}C_\beta(z)G_1(z-x_\beta)+\sum_{\gamma=0}^{N}C_\gamma(z)G_1(z-x_\gamma)-\sum_{\gamma=0}^{N}\sum_{\beta=0}^{N}C_\beta(z)C_\gamma(z)G_1(x_\beta-x_\gamma)$$

$G_1(x)$ funksiyamiz juft funksiya bo‘lgani uchun $G_1(-x)=G_1(x)$ munosabat o‘rinli. Bundan esa quyidagi teorema kelib chiqadi.

2-teorema: Ushbu (1)-ko‘rinishdagi interpolyatsion formulaning (3)-ko‘rinishdagi xatolik funksionalini normasining kvadrati quyidagi ko‘rinishda

$$\|\ell\|^2=2\sum_{\beta=0}^{N}C_\beta(z)G_1(z-x_\beta)-\sum_{\gamma=0}^{N}\sum_{\beta=0}^{N}C_\beta(z)C_\gamma(z)G_1(x_\beta-x_\gamma) \tag{12}$$

bo‘ladi.

1-masala to‘liq yechildi.

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