UDC 517.98

DYNAMICS OF A CUBIC STOCHASTIC OPERATOR

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Abstract. *It is well-known that Lotka-Volterra operators play important role in studying predator-prey models. In the present paper, we consider discrete-time dynamical systems generated by cubic operators which we are called separable cubic stochastic operators. That is, we consider cubic stochastic operators defined as a product of three linear operators defined on a simplex. It is shown that for a separable cubic stochastic operator defined on a two-dimensional simplex, the vertices of the simplex are fixed points. It is also proven that the orbit of a separable cubic stochastic operator converges for any starting point taken from the simplex.*

Keywords: *cubic stochastic operator, separable cubic stochastic operator, orbit, simplex.*

KUBIK STOXASTIK OPERATORNING DINAMIKASI

Annotatsiya*. Ma'lumki, Lotka–Volterra operatorlari "yirtqich–o'lja" turidagi modellarni o'rganishda muhim ahamiyat kasb etadi. Ushbu maqolada biz separabel kubik stoxastik operatorlar deb ataluvchi kubik operatorlar tomonidan hosil qilingan diskret vaqtli dinamik tizimlarni ko'rib chiqamiz. Ya'ni, simpleksda aniqlangan uchta chiziqli operatorlarning ko'paytmasi sifatida aniqlangan kubik stoxastik operatorlarni ko'rib chiqamiz. Ikki o'lchamli simpleksda aniqlangan separabel kubik stoxastik operator uchun simpleksning uchlari qo'zg'almas nuqtalar ekanligi ko'rsatilgan. Shuningdek, separabel kubik stoxastik operatorning orbitasi simpleksdan olingan har qanday boshlang'ich nuqta uchun yaqinlashuvchi ekanligi isbotlangan.*

Kalit so'zlar: *kubik stoxastik operator, separabel kubik stoxastik operator, orbita, simpleks.*

ДИНАМИКА КУБИЧЕСКОГО СТОХАСТИЧЕСКОГО ОПЕРАТОРА

Аннотация. *Хорошо известно, что операторы Лотки–Вольтерры играют важную роль в исследовании моделей типа «хищник–жертва». В настоящей работе рассматриваются дискретные динамические системы, порождённые кубическими операторами, которые мы называем сепарабельными кубическими стохастическими операторами. То есть, рассматриваются кубические стохастические операторы, заданные как произведение трёх линейных операторов, заданных на симплексе. Показано, что для сепарабельного кубического стохастического оператора, заданного на двумерном симплексе, вершины симплекса являются неподвижными точками.*

Ключевые слова: *кубический стохастический оператор, сепарабельный кубический стохастический оператор, орбита, симплекс.*

Introduction. The evolution of biological populations can be studied by dynamical systems of nonlinear operators. One of the simplest nonlinear case is quadratic (QSO) one. This operator models the evolution of a population and was originally introduced by Bernstein in [1]. For more than 80 years, the theory of QSOs has been developed and many papers were published (see e.g. [4]-[8], [16]-[17]). In recent years it has again become of interest in connection with its numerous applications in many branches of mathematics, biology and physics.

Let $E=\{1,2,...,m\}$ be a finite set and the set of all probability distribution on E

$$S^{m-1}=\left\{x=(x_1,\dots x_m)\in\mathbb{R}^m : x_i\geq 0, \sum_{i=1}^{m} x_i=1\right\} \tag{1}$$

be the $(m-1)$-dimensional simplex. A QSO is a mapping defined as $V:S^{m-1}\to S^{m-1}$ of the simplex into itself, of the form $V(\mathbf{x})=\mathbf{x}'\in S^{m-1}$, where

$$x_k'=\sum_{i,j=1}^{m} P_{ij,k}x_i x_j, \quad k\in E, \tag{2}$$

and the coefficients $P_{ij,k}$ satisfy

$$P_{ij,k} = P_{ji,k} \geq 0, \quad \sum_{k=1}^{m} P_{ij,k} = 1 \qquad \text{for all} \quad i, j \in E. \tag{3}$$

The trajectory (orbit) $\left\{\mathbf{x}^{(n)}\right\}_{n\geq 0}$, of V for an initial value $\mathbf{x}^{(0)} \in S^{m-1}$ is defined by

$$\mathbf{x}^{(n+1)} = V\left(\mathbf{x}^{(n)}\right) = V^{(n+1)}\left(\mathbf{x}^{(0)}\right), \qquad n = 0,1,2,\ldots .$$

One of the main problems in mathematical biology is to study the asymptotic behavior of the trajectories. This problem was solved completely for the Volterra QSO.

The operator V is called Volterra QSO, if $P_{ij,k} = 0$ for any $k \notin \{i, j\}$, $i, j, k \in E$. For the Volterra QSO the general formula was given in [4],

$$x_k' = x_k\left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \tag{4}$$

where $a_{ki} = 2P_{ik,k} - 1$ for $i \neq k$ and $a_{kk} = 0$. Moreover, $a_{ki} = -a_{ik}$ and $|a_{ki}| \leq 1$ for all $i, k \in E$.

In [4], the theory of Volterra QSO was developed using theory of the Lyapunov functions and tournaments. But non-Volterra QSOs were not completely studied. Because, there is no general theory that can be applied for study of non-Volterra operators.

In [16] Separable Quadratic Stochastic Operators (SQSOs) where introduced. Volterra QSO (4) has a form as SQSO, but in [17] it was proved that it coincides with a SQSO if and only if it is a linear operator.

In recent years, Cubic Stochastic Operators (CSOs) have begun to be studied, which different from quadratic operators [8-10].

In this paper we consider another class of cubic operators which we call separable cubic stochastic operators [11-13].

In Section 2, we recall the definition of CSOs and definitions and known results. In Section 3 for a SCSO defined on the two-dimensional simplex, we prove that it has three fixed points and we find the boundary ∂S^2 is an invariant set.

Preliminaries and known results

Separable quadratic stochastic operator. Let us recall some necessary definition and notations. In [18] Separable Quadratic Stochastic Operators were introduced as follows: The QSO (2), (3) with additional condition

$$P_{ij,k} = a_{ik} b_{jk} \qquad \text{for all} \qquad i, j, k \in E \tag{5}$$

where $a_{ik}, b_{jk} \in \mathbb{R}$ entries of matrices $A = (a_{ik})$ and $B = (b_{jk})$ such that the conditions (3) are satisfied for the coefficients (5).

Then the QSO V corresponding to the coefficients (5) has the form.

$$x_k' = \left(V(\mathbf{x})\right)_k = \left(A(\mathbf{x})\right)_k \left(B(\mathbf{x})\right)_k, \tag{6}$$

where $\left(A(\mathbf{x})\right)_k = \sum_{i=1}^{m} a_{ik} x_i$, $\left(B(\mathbf{x})\right)_k = \sum_{j=1}^{m} b_{jk} x_j$.

Definition 1. [16] *The QSO (6) is called separable quadratic stochastic operator (SQSO).*

Cubic stochastic operator. The CSO is a mapping $W : S^{m-1} \to S^{m-1}$ of the form

$$x_l' = \sum_{i,j,k=1}^{m} P_{ijk,l} x_i x_j x_k, \quad l \in E, \tag{7}$$

where $P_{ijk,l}$ are coefficients of heredity such that

$$P_{ijk,l} \geq 0, \quad \sum_{l=1}^{m} P_{ijk,l} = 1, \quad \forall i, j, k \in E. \tag{8}$$

and we suppose that the coefficients $P_{ijk,l}$ do not change for any permutation of i, j, k.

For a given $\mathbf{x}^{(0)} \in S^{m-1}$, the trajectory $\left\{\mathbf{x}^{(n)}\right\}_{n\geq 0}$ of initial point $\mathbf{x}^{(0)}$ under action of CSO (8) is defined by $\mathbf{x}^{(n+1)} = W\left(\mathbf{x}^{(n)}\right)$, where $n = 0,1,2,\ldots$ with $\mathbf{x} = \mathbf{x}^{(0)}$. Denote by $\omega\left(\mathbf{x}^{(0)}\right)$ the set of limit points of the trajectory $\left\{\mathbf{x}^{(n)}\right\}_{n=0}^{\infty}$. Since $\left\{\mathbf{x}^{(n)}\right\}_{n=0}^{\infty} \subset S^{m-1}$ and S^{m-1} is a compact set, it is follows that $\omega\left(\mathbf{x}^{(0)}\right) \neq \varnothing$. If $\omega\left(\mathbf{x}^{(0)}\right)$ consists of a single point, they the trajectory converges and $\omega\left(\mathbf{x}^{(0)}\right)$ is a fixed point of the operator W. A

point $\mathbf{x}\in S^{m-1}$ is called a fixed of the W if $W(\mathbf{x})=\mathbf{x}$. Denote by $\mathrm{Fix}(W)$ the set of all fixed points of the operator W, i.e.

$$\mathrm{Fix}(W)=\{\mathbf{x}\in S^{m-1}: W(\mathbf{x})=\mathbf{x}\}.$$

Let $DW(\mathbf{x}^*)=(\partial W_i/\partial x_j)(\mathbf{x}^*)$ be a Jacobian of W at the point $\mathbf{x}^*$.

Definition 2 ([3]): *A fixed point $\mathbf{x}^*$ is called hyperbolic if its Jacobian $DW(\mathbf{x}^*)$ has no eigenvalues on the unit circle in $\mathbb{C}$.*

Definition 3 ([3]): *A hyperbolic fixed point $\mathbf{x}^*$ is called*:

(i) attracting if all the eigenvalues of the Jacobian $DW(\mathbf{x}^*)$ are in the unit disk;

(ii) repelling if all the eigenvalues of the Jacobian $DW(\mathbf{x}^*)$ are outside the closed unit disk;

(iii) a saddle otherwise;

Main result

In this section we consider CSO (7), (8) with additional condition

$$P_{ijk,l}=a_{il}b_{jl}c_{kl},\quad \text{for all}\quad i,j,k,l\in E, \tag{9}$$

where $a_{il},b_{jl},c_{kl}\in\mathbb{R}$ entries of matrices $A=(a_{il})$, $B=(b_{jl})$ and $C=(c_{kl})$ such that the conditions (8) are satisfied for the coefficients (9).

Then the CSO W corresponding to the coefficients (9) has the form

$$x'_l=(W(\mathbf{x}))_l=(A(\mathbf{x}))_l(B(\mathbf{x}))_l(C(\mathbf{x}))_l \tag{10}$$

where $(A(\mathbf{x}))_l=\sum_{i=1}^{m}a_{il}x_i$, $(B(\mathbf{x}))_l=\sum_{j=1}^{m}b_{jl}x_j$ and $(C(\mathbf{x}))_l=\sum_{k=1}^{m}c_{kl}x_k$.

Definition 4. [11] *The CSO* (10) *is called separable cubic stochastic operator (SCSO).*

Let us consider the following matrices:

$$A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ B=\begin{pmatrix}1&0&2\\2&1&1\\1&1&1\end{pmatrix},\ C=\begin{pmatrix}1&1&1\\1&1&3/2\\0&1/2&1\end{pmatrix}. \tag{11}$$

Then corresponding SCSO $W:S^2\to S^2$ is:

$$W:\begin{cases}x'_1=x_1(x_1+2x_2+x_3)(x_1+x_2),\\ x'_2=x_2(x_2+x_3)(x_1+x_2+0{,}5x_3),\\ x'_3=x_3(2x_1+x_2+x_3)(x_1+1{,}5x_2+x_3).\end{cases} \tag{12}$$

Using the equation $x_1+x_2+x_3=1$ we rewrite the operator (12) as follows

$$W:\begin{cases}x'_1=x_1(1+x_2)(1-x_3),\\ x'_2=x_2(1-x_1)(1-0{,}5x_3),\\ x'_3=x_3(1+x_1)(1+0{,}5x_2).\end{cases} \tag{13}$$

Let a face of the simplex S^2 be the set $\Gamma_\alpha=\{\mathbf{x}\in S^2: x_i=0,\ i\notin\alpha\subset\{1,2,3\}\}$.

Let the set $\operatorname{int}S^2=\{\mathbf{x}\in S^2: x_1x_2x_3>0\}$ and let the set $\partial S^2=S^2\setminus\operatorname{int}S^2$ be the interior and the boundary of the simplex S^2, respectively. Let $\mathbf{e}_1=(1,0,0)$, $\mathbf{e}_2=(0,1,0)$, $\mathbf{e}_3=(0,0,1)$ be the vertexes of the two-dimensional simplex.

Lemma 1. *For the SCSO W* (13), *the following assertions true*:

(i) *The face $\Gamma_{\{1,2\}}$, $\Gamma_{\{1,3\}}$, $\Gamma_{\{2,3\}}$ of the simplex S^2 are invariant sets;*

(ii) $\mathrm{Fix}(W)=\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$;

(iii) The fixed point $\mathbf{e}_1$ is a saddle point, $\mathbf{e}_2$ is a repelling fixed point and the fixed point $\mathbf{e}_3$ is a attracting point.

Proof: (i) Obviously.

(ii) To find the fixed points we consider the equation $W(\mathbf{x})=\mathbf{x}$, that is the following system of equations

$$\begin{cases} x_1 = x_1(1+x_2)(1-x_3), \\ x_2 = x_2(1-x_1)(1-0,5x_3), \\ x_3 = x_3(1+x_1)(1+0,5x_2). \end{cases} \tag{14}$$

(a) Let $x_1=0$. Then from the third equation of (14) it follows

$$x_3 = x_3 + 0,5x_2x_3 \;\Rightarrow\; x_3=0 \quad \text{and} \quad 1=1+0,5x_2.$$

It is clear that if $x_3=0$ then we have $x_2=1$. If $x_3>0$ then from the last equation one has

$$x_2=0 \;\Rightarrow\; x_3=1.$$

We take solution $x_2=1$, then it follows that $x_3=0$. Consequently, if $x_1=0$, we obtain the fixed points $\mathbf{e}_2$, $\mathbf{e}_3$

Similarly, in the case $x_2=0$ and $x_3=0$, we have the fixed points $\mathbf{e}_1$, $\mathbf{e}_3$ and $\mathbf{e}_1$, $\mathbf{e}_2$, respectively.

(b) Suppose that $x_1x_2x_3\neq 0$. Then from the system (14), one has

$$\begin{cases} 1 = (1+x_2)(1-x_3), \\ 1 = (1-x_1)(1-0,5x_3), \\ 1 = (1+x_1)(1+0,5x_2). \end{cases} \tag{15}$$

The second equation of (15) we get

$$1=(1-x_1)(1-0,5x_3) \;\Rightarrow\; x_1 = 0,5x_3(x_1-1).$$

Last equation it follows that $x_1=0$ and $x_3=0$, this contradicts to $x_1x_2x_3\neq 0$.

Consequently, we have that $\mathrm{Fix}(W)\cap \mathrm{int}\,S^2=\varnothing$.

(iii) To find the type of fixed point of the SCSO (13), we rewrite it in the form

$$\begin{cases} x_1' = x_1\left(x_1+2x_2+(1-x_1-x_2)\right)(x_1+x_2), \\ x_2' = x_2\left(x_2+(1-x_1-x_2)\right)(x_1+x_2+0,5(1-x_1-x_2)), \end{cases} \tag{16}$$

where $(x_1,x_2)\in\{(x,y): x,y\geq 0,\ 0\leq x+y\leq 1\}$ and x_1,x_2 are the first two coordinates of a point lying in the simplex S^2.

The Jacobian of the operator (16) at a fixed point $\mathbf{e}_1$ has the following eigenvalues $\lambda_1=2$, $\lambda_2=0$, at the $\mathbf{e}_2$ has the following eigenvalues $\lambda_1=1,5$, $\lambda_2=2$ and at the $\mathbf{e}_3$ has the following eigenvalues $\lambda_1=0,5$, $\lambda_2=0$. Therefore, it follow that the fixed point $\mathbf{e}_1$ is a saddle point, $\mathbf{e}_2$ is a repelling fixed point and the fixed point $\mathbf{e}_3$ is a attracting point.

The theorem is proved. □

The set of limit points

Let $\mathbf{x}^{(0)}\in S^2$ be the initial point. Then the trajectory of $\mathbf{x}^{(0)}$ is denote by $\{\mathbf{x}^{(n)}\}_{n\geq 0}$ and is defined as $W(\mathbf{x}^{(n)})=\mathbf{x}^{(n+1)}$ where $n=0,1,2,\ldots$.

$$W:\begin{cases} x_1' = x_1(1+x_2)(1-x_3), \\ x_2' = x_2(1-x_1)(1-0,5x_3), \\ x_3' = x_3(1+x_1)(1+0,5x_2). \end{cases}$$

Lemma 2. *For the SCSO* W (13), *the following assertions true*:

(i) *If* $\mathbf{x}^{(0)}\in\Gamma_{\{2,3\}}$, *then* $\lim\limits_{n\to\infty}W^n(\mathbf{x}^{(0)})=\mathbf{e}_3$,

(ii) *If* $\mathbf{x}^{(0)}\in\Gamma_{\{1,2\}}$, *then* $\lim\limits_{n\to\infty}W^n(\mathbf{x}^{(0)})=\mathbf{e}_1$,

(iii) *If* $\mathbf{x}^{(0)}\in\Gamma_{\{1,3\}}$, *then* $\lim\limits_{n\to\infty}W^n(\mathbf{x}^{(0)})=\mathbf{e}_3$,

(iv) *If* $\mathbf{x}^{(0)}\in\mathrm{int}\,S^2$, *then* $\lim\limits_{n\to\infty}W^n(\mathbf{x}^{(0)})=\mathbf{e}_3$.

Proof: (i) Let $\mathbf{x}^{(0)}=\left(0,x_2^{(0)},x_3^{(0)}\right)\in\Gamma_{\{2,3\}}$. By Lemma 1, the face $\Gamma_{\{2,3\}}$ is an invariant set and the vertexes $\mathbf{e}_2$ and $\mathbf{e}_3$ are fixed points belonging to this face. For the SCSO W (13) is true $x_1^{(n)}=0$, $n=0,1,2,\ldots$ at the face $\Gamma_{\{2,3\}}$. The restriction of the SCSO W (12) on the face $\Gamma_{\{2,3\}}$ has the form

$$\begin{cases} x_2'=x_2\left(1-0,5x_3\right), \\ x_3'=x_3\left(1+0,5x_2\right). \end{cases} \tag{17}$$

From the first equation of (17), one has $x_2^{(n+1)}=x_2^{(n)}\left(1-0,5x_3^{(n)}\right)\le x_2^{(n)}$, $n=0,1,2,\ldots$. Therefore, it follows that there exists the $\lim\limits_{n\to\infty}x_2^{(n)}=x_2^*$. Moreover, from the second equation of (17), one has $x_3^{(n+1)}=x_3^{(n)}\left(1+x_2^{(n)}\right)\ge x_3^{(n)}$, $n=0,1,2,\ldots$. Therefore, it follows that there exists the $\lim\limits_{n\to\infty}x_3^{(n)}=x_3^*$. So there is the limit $\lim\limits_{n\to\infty}W^n\left(\mathbf{x}^{(0)}\right)=\mathbf{x}^*=\left(0,x_2^*,x_3^*\right)$. Since $\mathbf{x}^*$ should be a fixed we get $\mathbf{x}^*=\mathbf{e}_3$. That is, if $x_3^{(0)}>0$ then we have that $\lim\limits_{n\to\infty}W^n\left(\mathbf{x}^{(0)}\right)=\mathbf{e}_3$ for any $\mathbf{x}^{(0)}=\Gamma_{\{2,3\}}\setminus\left\{\mathbf{e}_2\right\}$.

Let $\mathbf{x}^{(0)}\in\Gamma_{\{1,2\}}$. By Lemma 1, the face $\Gamma_{\{1,2\}}$ is an invariant set and the vertexes $\mathbf{e}_1$ and $\mathbf{e}_2$ are fixed points belonging to this face. The restriction of the SCSO W (13) on the face $\Gamma_{\{1,2\}}$ has the form

$$\begin{cases} x_1'=x_1\left(1+x_2\right), \\ x_2'=x_2\left(1-x_1\right). \end{cases} \tag{18}$$

Not that, for the SCSO W (18) is true $x_3^{(n)}=0$, $n=0,1,2,\ldots$ at the face $\Gamma_{\{1,2\}}$. From the first equation of (18), one has $x_1^{(n+1)}=x_1^{(n)}\left(1+x_2^{(n)}\right)\ge x_1^{(n)}$, $n=0,1,2,\ldots$. Therefore, it follows that there exists the $\lim\limits_{n\to\infty}x_1^{(n)}=x_1^*$. Moreover, from the second equation of (18), one has $x_2^{(n+1)}=x_2^{(n)}\left(1-x_1^{(n)}\right)\le x_2^{(n)}$, $n=0,1,2,\ldots$. So we have that there exists the $\lim\limits_{n\to\infty}x_2^{(n)}=x_2^*$. Hence there is the limit $\lim\limits_{n\to\infty}W^n\left(\mathbf{x}^{(0)}\right)=\mathbf{x}^*=\left(x_1^*,x_2^*,0\right)$. Since $\mathbf{x}^*$ should be a fixed point we get $\mathbf{x}^*=\mathbf{e}_1$. That is, if $x_1^{(0)}>0$ then we have that $\lim\limits_{n\to\infty}W^n\left(\mathbf{x}^{(0)}\right)=\mathbf{e}_1$ for any $\mathbf{x}^{(0)}=\Gamma_{\{1,2\}}\setminus\left\{\mathbf{e}_2\right\}$.

Let $\mathbf{x}^{(0)}=\left(x_1^{(0)},0,x_3^{(0)}\right)\in\Gamma_{\{1,3\}}$. By Lemma 1, the face $\Gamma_{\{1,3\}}$ is an invariant set and the vertexes $\mathbf{e}_1$ and $\mathbf{e}_3$ are fixed points belonging to this face. For the SCSO W (13) is true $x_2^{(n)}=0$, $n=0,1,2,\ldots$ at the face $\Gamma_{\{1,3\}}$. The restriction of the SCSO W (13) on the face $\Gamma_{\{1,3\}}$ has the form

$$\begin{cases} x_1'=x_1\left(1-x_3\right), \\ x_3'=x_3\left(1+x_1\right). \end{cases} \tag{19}$$

From the first equation of (19), one has $x_1^{(n+1)}=x_1^{(n)}\left(1-x_3^{(n)}\right)\le x_3^{(n)}$, $n=0,1,2,\ldots$. Therefore, it follows that there exists the $\lim\limits_{n\to\infty}x_1^{(n)}=x_1^*$. Moreover, from the second equation of (19), one has $x_3^{(n+1)}=x_3^{(n)}\left(1+x_1^{(n)}\right)\ge x_3^{(n)}$, $n=0,1,2,\ldots$. Therefore, it follows that there exists the $\lim\limits_{n\to\infty}x_3^{(n)}=x_3^*$. So there is the limit $\lim\limits_{n\to\infty}W^n\left(\mathbf{x}^{(0)}\right)=\mathbf{x}^*=\left(x_1^*,0,x_3^*\right)$. Since $\mathbf{x}^*$ should be a fixed we get $\mathbf{x}^*=\mathbf{e}_3$. That is, if $x_1^{(0)}>0$ then we have that $\lim\limits_{n\to\infty}W^n\left(\mathbf{x}^{(0)}\right)=\mathbf{e}_3$ for any $\mathbf{x}^{(0)}=\Gamma_{\{1,3\}}\setminus\left\{\mathbf{e}_1\right\}$.

Let $\mathbf{x}^{(0)}=\operatorname{int}S^2$. Then, from the third equation of operator (13), one has $x_3^{(n+1)}=x_3^{(n)}\left(1+x_1^{(n)}\right)\left(1+0,5x_2^{(n)}\right)\ge x_3^{(n)}$, $n=0,1,2,\ldots$. So we have that there exists the $\lim\limits_{n\to\infty}x_3^{(n)}=x_3^*$. Moreover, from the third equation of operator (13), one has $x_2^{(n+1)}=x_2^{(n)}\left(1-0,5x_3^{(n)}\right)\left(1-x_1^{(n)}\right)\le x_2^{(n)}$, $n=0,1,2,\ldots$.

Therefore, it follows that there exists the $\lim_{n\to\infty} x_2^{(n)} = x_2^*$. Thus, using $x_1^{(n)} + x_2^{(n)} + x_3^{(n)} = 1$ we have $\lim_{n\to\infty} x_1^{(n)} = x_1^*$.

So there is the limit $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{x}^* = \left(x_1^*, x_2^*, x_3^*\right)$. Since $\mathbf{x}^*$ should be a fixed point we get $\mathbf{x}^* = \mathbf{e}_3$.

Consequently, we have that $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_3$ for any $\mathbf{x}^{(0)} = \operatorname{int} S^2$.

The theorem is proved.

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