UDC 517.984:530.145

SPECTRAL PROPERTIES OF A TWO-PARTICLE SCHRÖDINGER OPERATOR WITH FINITE-RANGE INTERACTIONS

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Abstract. *We study the spectral properties of the operator $H_{\mu_1\mu_2\mu_3}(K)$, K is the total quasi-momentum, corresponding to a system of two particles with on-site, nearest-neighbor, and next-nearest-neighbor interactions along the coordinate axes on the three-dimensional lattice, with the strengths μ_1, μ_2 and μ_3, respectively. For $K = 0$, we identify an invariant subspace of the operator. On this subspace, the restricted operator depends only on the parameters μ_2 and μ_3. We determine the number of its eigenvalues lying below the essential spectrum. In particular, the (μ_2, μ_3)-parameter plane is partitioned by two hyperbolic curves into three unbounded regions, in each of which the number of eigenvalues below the essential spectrum remains constant and equals* 0, 1 *or* 2.

Keywords: *Two-particle system, lattice Schrödinger operator, essential spectrum, eigenvalues, Fredholm determinant, spectral analysis.*

CHEKLI TA'SIR RADIUSIGA EGA IKKI ZARRACHALI SHREDINGER OPERATORINING SPEKTRAL XOSSALARI

Annotatsiya. *Biz uch o'lchamli panjarada koordinata o'qlari bo'ylab nuqtada, eng yaqin qo'shni tugunlar va keyingi eng yaqin qo'shni tugunlarda mos ravishda μ_1, μ_2 va μ_3 kuchlar bilan ta'sirlashuvchi ikki zarrachali Shredinger operatorining spektral xossalarini o'rganamiz. $K = 0$ hol uchun berilgan operatorning invariant qism fazosi aniqlangan. Bu qism fazodagi operator toraytmasi faqat μ_2 va μ_3 parameterlarga bog'liq. Biz ushbu operator toraytmasining muhim spektrdan pastda joylashgan xos qiymatlari sonini aniqlaymiz. Xususan, ikkita giperbolik egri chiziq (μ_2, μ_3)-parametrlar tekisligini uchta chegaralanmagan sohaga ajratadi. Har bir sohada muhim spektrdan pastda joylashgan xos qiymatlar soni o'zgarmas bo'lib, mos ravishda 0, 1 yoki 2 ga teng.*

Kalit so'zlar: *ikki zarrachali sitema, Shredinger operatori, muhim spektr, xos qiymatlar, Fredgolm determinanti, spektral tahlil.*

СПЕКТРАЛЬНЫЕ СВОЙСТВА ДВУХЧАСТИЧНОГО ОПЕРАТОРА ШРЁДИНГЕРА С ВЗАИМОДЕЙСТВИЯМИ КОНЕЧНОГО РАДИУСА

Аннотация. *Исследуются спектральные свойства оператора $H_{\mu_1\mu_2\mu_3}(K)$, где K- полный квазиимпульс, соответствующего системе двух частиц с одноузельным взаимодействием, а также взаимодействиями между ближайшими и следующими за ближайшими соседями вдоль координатных осей с интенсивностями μ_1, μ_2 и μ_3 соответственно. При $K = 0$ выделяется инвариантное подпространство оператора. На этом подпространстве сужение оператора зависит только от параметров μ_2 и μ_3. Определяется число его собственных значений, лежащих ниже существенного спектра. В частности, плоскость параметров (μ_2, μ_3)-разбивается двумя гиперболическими кривыми на три неограниченные области, в каждой из которых число собственных значений, лежащих ниже существенного спектра, остаётся постоянным и равно 0, 1 или 2.*

Ключевые слова: *двухчастичная система, решётчатый оператор Шрёдингера, существенный спектр, собственные значения, определитель Фредгольма, спектральный анализ.*

Introduction. The spectral analysis of lattice Schrödinger operators is a central topic in mathematical physics. As discrete analogues of continuum Schrödinger operators, lattice Hamiltonians provide an effective framework for studying spectral transitions, threshold effects and bound states; see, e.g., [1, 2].

Few-body lattice Hamiltonians with finite-range interactions can often be treated as finite-rank or compact perturbations of the free lattice Hamiltonian. This allows many spectral problems to be reduced to

finite-dimensional matrix-valued functions and Fredholm determinants. Such methods have been successfully applied to two- and three-particle lattice systems in [3, 4, 5, 6, 7, 8, 9].

Unlike the continuum case, lattice Hamiltonians are neither rotationally invariant nor unbounded, and their dispersion relations generate finite spectral bands. This leads to new spectral phenomena, most notably the possibility of bound states not only below but also *above* the essential spectrum, as observed in [5, 10].

In this paper, we study the three-parameter family of two-particle lattice Schrödinger operators

$$H_{\mu_1\mu_2\mu_3}(K) = H_0(K) + V_{\mu_1\mu_2\mu_3},$$

acting in $L^{2,\mathrm{e}}(\mathbb{T}^3)$, where $K \in \mathbb{T}^3$ is the total quasi-momentum and $V_{\mu_1\mu_2\mu_3}$ describes on-site (μ_1), nearest-neighbor (μ_2), and next-nearest-neighbor (μ_3) interactions.

For $K = 0$, the operator has an invariant subspace and we determine the number of eigenvalues of the restriction of $H_{\mu_1\mu_2\mu_3}(0)$ to this subspace, lying below the essential spectrum. In particular, this restriction depends only on μ_2 and μ_3.

We show that the parameter plane (μ_2, μ_3) is partitioned by two hyperbolic curves into regions with a constant number of eigenvalues below the essential spectrum.

These results extend earlier one- and two-dimensional studies [11, 12] to the three-dimensional case. Moreover, in [13], the discrete spectrum of the same operator was studied in another invariant subspace.

The paper is organized as follows. Section 2 introduces the model and describes its essential spectrum and symmetry reduction. In Section 3, the Fredholm determinant and its threshold asymptotics are studied. The main spectral results are proved in Section 4.

Hamiltonian of a two-boson system on the lattice $\mathbb{Z}^3$

2.1 The position-space representation

Let $\mathbb{Z}^3 := \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ be the three-dimensional integer lattice and let $\ell^{2,s}(\mathbb{Z}^3 \times \mathbb{Z}^3)$ denote the Hilbert space of square-summable symmetric functions on $\mathbb{Z}^3 \times \mathbb{Z}^3$. We study two identical spinless bosons on the three-dimensional lattice $\mathbb{Z}^3$ with finite-range pair interactions supported on the on-site, nearest-neighbor, and next-to-nearest-neighbor sites along the coordinate axes, with coupling constants μ_1, μ_2, and μ_3, respectively. The corresponding Hamiltonian in the position-space representation is introduced on $\ell^{2,s}(\mathbb{Z}^3 \times \mathbb{Z}^3)$ as

$$\widehat{\mathbb{H}}_{\mu_1\mu_2\mu_3} = \widehat{\mathbb{H}}_0 + \widehat{\mathbb{V}}_{\mu_1\mu_2\mu_3}, \qquad \mu_1, \mu_2, \mu_3 \in \mathbb{R}.$$

Here, $\widehat{\mathbb{H}}_0$ denotes the Hamiltonian of the corresponding non-interacting two-boson system:

$$(\widehat{\mathbb{H}}_0\hat{f})(x_1, x_2) = \sum_{y\in\mathbb{Z}^3} \hat{\varepsilon}(x_1 - y)\hat{f}(y, x_2) + \sum_{y\in\mathbb{Z}^3} \hat{\varepsilon}(x_2 - y)\hat{f}(x_1, y), \quad x_1, x_2 \in \mathbb{Z}^3,$$

where

$$\hat{\varepsilon}(s) = \begin{cases} 6 & \text{if} |s| = 0, \\ -1 & \text{if} |s| = 1, \\ 0 & \text{if} |s| > 1, \end{cases}$$

and $|s| = |s_1| + |s_2| + |s_3|$ for $s = (s_1, s_2, s_3) \in \mathbb{Z}^3$. The interaction operator $\widehat{\mathbb{V}}_{\mu_1\mu_2\mu_3}$ acts as multiplication by a function $\hat{v}_{\mu_1\mu_2\mu_3}(\cdot)$,

$$(\widehat{\mathbb{V}}_{\mu_1\mu_2\mu_3}\hat{f})(x, y) = \hat{v}_{\mu_1\mu_2\mu_3}(x - y)\hat{f}(x, y),$$

where

$$\hat{v}_{\mu_1\mu_2\mu_3}(x) = \begin{cases} \mu_1 & |x| = 0, \\ \mu_2 & |x| = 1, \\ \mu_3 & x \in \{(\pm 2,0,0), (0, \pm 2,0), (0,0, \pm 2)\}, \\ 0 & \text{otherwise.} \end{cases}$$

2.2 The quasimomentum-space representation.

By $\mathbb{T}^3$ we denote the three-dimensional torus, $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3 \equiv [-\pi, \pi)^3$. The torus $\mathbb{T}^3$ coincides with the Pontryagin dual group of $\mathbb{Z}^3$, equipped with the Haar measure $\mathrm{d}p$. Let $L^{2,s}(\mathbb{T}^3 \times \mathbb{T}^3)$ be the Hilbert space of square-integrable symmetric functions on $\mathbb{T}^3 \times \mathbb{T}^3$. Let $\mathcal{F}$ denote the standard Fourier transform from $\ell^2(\mathbb{Z}^3)$ to $L^2(\mathbb{T}^3)$,

$$(\mathcal{F}\hat{f})(p) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{x\in\mathbb{Z}^3} \hat{f}(x)e^{ip\cdot x}, \qquad p \in \mathbb{T}^3,$$

where $p \cdot x := p_1x_1 + p_2x_2 + p_3x_3$. Its inverse is given by

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{T}^3} f(p)e^{-ip\cdot x}dp.$$

Under the unitary map $\mathcal{F} \otimes \mathcal{F}$, the Hamiltonian becomes

$$\mathbb{H}_{\mu_1\mu_2\mu_3} := (\mathcal{F} \otimes \mathcal{F})\widehat{\mathbb{H}}_{\mu_1\mu_2\mu_3}(\mathcal{F} \otimes \mathcal{F})^{-1} := \mathbb{H}_0 + \mathbb{V}_{\mu_1\mu_2\mu_3}.$$

The free Hamiltonian is the multiplication operator:

$$\mathbb{H}_0 f(p,q) = [\varepsilon(p) + \varepsilon(q)]f(p,q),$$

where

$$\varepsilon(p) := 2\sum_{i=1}^{3} (1 - \cos p_i), \quad p = (p_1, p_2, p_3) \in \mathbb{T}^3,$$

is the *dispersion relation* of a single boson.

The interaction $\mathbb{V}_{\mu_1\mu_2\mu_3}$ is given by

$$\mathbb{V}_{\mu_1\mu_2\mu_3} f(p,q) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} v_{\mu_1\mu_2\mu_3}(p-u)f(u, p+q-u)du,$$

where $v_{\mu_1\mu_2\mu_3}$ is the Fourier transform of $\hat{v}_{\mu_1\mu_2\mu_3}$:

$$v_{\mu_1\mu_2\mu_3}(p) = \mu_1 + 2\mu_2 \sum_{i=1}^{3} \cos p_i + 2\mu_3 \sum_{i=1}^{3} \cos 2p_i.$$

2.3 Decomposition with respect to the total quasimomentum

Introducing the total quasimomentum

$$K := p + q \in \mathbb{T}^3,$$

we obtain the von Neumann decomposition

$$\mathbb{H}_{\mu_1\mu_2\mu_3} = \int_{\mathbb{T}^3}^{\oplus} H_{\mu_1\mu_2\mu_3}(K)\, dK.$$

Define the unitary map

$$\Upsilon: L^{2,s}(\mathbb{T}^3 \times \mathbb{T}^3) \to \int_{\mathbb{T}^3}^{\oplus} L^{2,\mathrm{e}}(\mathbb{T}^3)\, dK, \qquad (\Upsilon f)(K,k) := f\left(\frac{K}{2} + k, \frac{K}{2} - k\right),$$

where $k \in \mathbb{T}^3$ is the relative momentum. The fiber operator $H_{\mu_1\mu_2\mu_3}(K)$, $K \in \mathbb{T}^3$, is a self-adjoint operator on $L^{2,\mathrm{e}}(\mathbb{T}^3)$ of the form

$$H_{\mu_1\mu_2\mu_3}(K) := H_0(K) + V_{\mu_1\mu_2\mu_3},$$

where

$$(H_0(K)f)(p) = \mathcal{E}_K(p)f(p), \qquad \mathcal{E}_K(p) := 4\sum_{i=1}^{3} \left(1 - \cos\frac{K_i}{2}\cos p_i\right)$$

and

$$\left(V_{\mu_1\mu_2\mu_3} f\right)(p) = \frac{\mu_1}{8\pi^3}\int_{\mathbb{T}^3} f(q)dq + \frac{\mu_2}{4\pi^3}\sum_{i=1}^{3} \cos p_i \int_{\mathbb{T}^3} \cos q_i f(q)dq + \frac{\mu_3}{4\pi^3}\sum_{i=1}^{3} \cos 2p_i \int_{\mathbb{T}^3} \cos 2q_i f(q)dq. \tag{1}$$

Both the operators $H_0(K)$ and $V_{\mu_1\mu_2\mu_3}$, which is a rank-seven operator, are bounded and self-adjoint.

2.4 The essential spectrum of the fiber Schrödinger operators

Depending on $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ the rank of $V_{\mu_1\mu_2\mu_3}$ varies but never exceeds seven. Hence, by the classical Weyl theorem (see [14, Theorem XIII.14]), for any $K \in \mathbb{T}^3$, the essential spectrum $\sigma_{\mathrm{ess}}(H_{\mu_1\mu_2\mu_3}(K))$ of the fiber Schrödinger operator $H_{\mu_1\mu_2\mu_3}(K)$ coincides with the spectrum of $H_0(K)$:

$$\sigma_{\text{ess}}(H_{\mu_1\mu_2\mu_3}(K)) = \sigma(H_0(K)) = [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)],$$

where

$$\mathcal{E}_{\min}(K) := \min_{p\in\mathbb{T}^3} \mathcal{E}_K(p) = 4\sum_{i=1}^{3}(1-\cos\frac{K_i}{2}) \geq \mathcal{E}_{\min}(0) = 0,$$

$$\mathcal{E}_{\max}(K) := \max_{p\in\mathbb{T}^3} \mathcal{E}_K(p) = 4\sum_{i=1}^{3}(1+\cos\frac{K_i}{2}) \leq \mathcal{E}_{\max}(0) = 24.$$

2.5 Invariant subspace of the lattice Schrödinger operator $H_{\mu_1\mu_2\mu_3}(0)$

Lemma 2.1 *The subspace $L^{2,\text{e,m}}(\mathbb{T}^3) \subset L^{2,\text{e}}(\mathbb{T}^3)$ is invariant under $H_{\mu_1\mu_2\mu_3}(0)$, where*

$$L^{2,\text{e,m}}(\mathbb{T}^3) = \{f \in L^{2,\text{e}}(\mathbb{T}^3): f(p_1,p_2,p_3) = f(p_2,p_1,p_3), \quad f = f_1 + f_2,$$
$$f_1(p_1,p_2,p_3) = -f_1(p_3,p_2,p_1), \quad f_2(p_1,p_2,p_3) = -f_2(p_1,p_3,p_2)\}.$$

The restriction of $H_{\mu_1\mu_2\mu_3}(0)$ onto this subspace is:

$$H^{\text{m}}_{\mu_2\mu_3}(0) := H_{\mu_1\mu_2\mu_3}|_{L^{2,\text{e,m}}(\mathbb{T}^3)} = H_0(0) + V^{\text{m}}_{\mu_2\mu_3},$$

with

$$V^{\text{m}}_{\mu_2\mu_3} := V_{\mu_1\mu_2\mu_3}|_{L^{2,\text{e,m}}(\mathbb{T}^3)} = \mu_2(\alpha^{\text{m}}_1,\cdot)\alpha^{\text{m}}_1 + \mu_3(\alpha^{\text{m}}_2,\cdot)\alpha^{\text{m}}_2,$$

where

$$\alpha^{\text{m}}_n(p) = \frac{\cos(np_1)+\cos(np_2)-2\cos(np_3)}{\sqrt{24\pi^3}}, n = 1,2.$$

Proof. Let $f \in L^{2,\text{e,m}}(\mathbb{T}^3)$. Since $H_0(0)$ is the multiplication operator by the symmetric function $\mathcal{E}_0(p)$ under all permutations of (p_1,p_2,p_3), it follows immediately that the subspace $L^{2,\text{e,m}}(\mathbb{T}^3)$ is invariant under $H_0(0)$.

We now consider the operator $V_{\mu_1\mu_2\mu_3}$. We know that f is symmetric under the transposition $p_1 \leftrightarrow p_2$ and $f = f_1 + f_2$, where f_1 and f_2 are antisymmetric with respect to the transposition $p_1 \leftrightarrow p_3$ and $p_2 \leftrightarrow p_3$, respectively. Therefore,

$$\int_{\mathbb{T}^3} f(q)\mathrm{d}q = 0.$$

Using the symmetry property of f, we obtain

$$\int_{\mathbb{T}^3}\cos(nq_1)f(q)\mathrm{d}q = \int_{\mathbb{T}^3}\cos(nq_2)f(q)\mathrm{d}q, \quad n = 1,2. \tag{2}$$

Moreover, the antisymmetry properties of f_1 and f_2 imply that

$$\int_{\mathbb{T}^3}\cos(nq_1)f_2(q)\mathrm{d}q = 0, \quad \int_{\mathbb{T}^3}\cos(nq_2)f_1(q)\mathrm{d}q = 0, \quad n = 1,2.$$

Using these equalities together with the decomposition $f = f_1 + f_2$ and the antisymmetry properties of f_1 and f_2, we obtain

$$\int_{\mathbb{T}^3}\cos(nq_3)f(q)\mathrm{d}q = -2\int_{\mathbb{T}^3}\cos(nq_1)f(q)\mathrm{d}q, \quad n = 1,2. \tag{3}$$

Substituting the identities (2) and (3) into the definition of $V_{\mu_1\mu_2\mu_3}$, we obtain

$$\left(V_{\mu_1\mu_2\mu_3}f\right)(p) = \frac{\mu_2}{4\pi^3}(\cos p_1 + \cos p_2 - 2\cos p_3)\int_{\mathbb{T}^3}\cos q_1 f(q)\mathrm{d}q \tag{4}$$
$$+\frac{\mu_3}{4\pi^3}(\cos(2p_1)+\cos(2p_2)-2\cos(2p_3))\int_{\mathbb{T}^3}\cos(2q_1)f(q)\mathrm{d}q.$$

Since $\cos(np_1)+\cos(np_2)-2\cos(np_3) \in L^{2,\text{e,m}}(\mathbb{T}^3)$, $n = 1,2$, it follows that $L^{2,\text{e,m}}(\mathbb{T}^3)$ is invariant under $V_{\mu_1\mu_2\mu_3}$. Thus, the subspace $L^{2,\text{e,m}}(\mathbb{T}^3)$ is invariant under $H_{\mu_1\mu_2\mu_3}(0) = H_0(0) + V_{\mu_1\mu_2\mu_3}$. Furthermore, the relations (2) and (3) yield

$$\int_{\mathbb{T}^3}\cos(nq_1)f(q)\mathrm{d}q = \frac{1}{6}\int_{\mathbb{T}^3}(\cos(nq_1)+\cos(nq_2)-2\cos(nq_3))f(q)\mathrm{d}q.$$

Combining this identity with (4), we obtain the restriction $V^{\text{m}}_{\mu_2\mu_3}$ of $V_{\mu_1\mu_2\mu_3}$ onto $L^{2,\text{e,m}}(\mathbb{T}^3)$:

$$(V^{\text{m}}_{\mu_2\mu_3}f)(p) = \mu_2(\alpha^{\text{m}}_1,f)\alpha^{\text{m}}_1(p) + \mu_3(\alpha^{\text{m}}_2,f)\alpha^{\text{m}}_2(p). \tag{5}$$

Therefore,

$$H^{\text{m}}_{\mu_2\mu_3}(0) = H_0(0) + V^{\text{m}}_{\mu_2\mu_3}.$$

This completes the proof. ■

Lippmann–Schwinger operator and asymptotics of its determinant

3.1 The Lippmann–Schwinger operator

For $z \in \mathbb{C}\backslash[0,24]$ we define the Lippmann-Schwinger operator (see, e.g., [15]) as

$$B^{\mathrm{m}}_{\mu_2\mu_3}(0,z) = -V^{\mathrm{m}}_{\mu_2\mu_3} R_0(0,z),$$

where $R_0(0,z) := [H_0(0) - zI]^{-1}, z \in \mathbb{C}\backslash[0,24]$, is the resolvent of the operator $H_0(0)$.

Lemma 3.1 *Let $\mu_2, \mu_3 \in \mathbb{R}$. A number $z \in \mathbb{R}\backslash[0,24]$ is an eigenvalue of the self-adjoint operator $H^m_{\mu_2\mu_3}(0)$ if and only if z is a zero of the function*

$$\Delta_{\mu_2\mu_3}(z) := \det[I - B^{\mathrm{m}}_{\mu_2\mu_3}(0,z)] = \begin{vmatrix} 1+\mu_2 a_{11}(z) & \mu_3 a_{12}(z) \\ \mu_2 a_{12}(z) & 1+\mu_3 a_{22}(z) \end{vmatrix},$$

where

$$a_{ij}(z) := \int_{\mathbb{T}^3} \frac{\alpha_i^{\mathrm{m}}(p)\alpha_j^{\mathrm{m}}(p)\mathrm{d}p}{\varepsilon_0(p) - z}, \ i,j = 1,2 \tag{6}$$

Proof. Let $z \in \mathbb{R}\backslash[0,24]$. The eigenvalue equation

$$H^{\mathrm{m}}_{\mu_2\mu_3}(0) f(p) = z f(p)$$

is equivalent to the Lippmann-Schwinger equation

$$f_z(p) = -R_0(0,z) V^{\mathrm{m}}_{\mu_2\mu_3} f(p).$$

Define $g_z := V^{\mathrm{m}}_{\mu_2\mu_3} f_z = \mu_2 c_1(z)\alpha_1^{\mathrm{m}} + \mu_3 c_2(z)\alpha_2^{\mathrm{m}}$, with $c_i = (f_z, \alpha_i^{\mathrm{m}}), \ i = 1,2$. Substituting g_z into Lippmann-Schwinger equation yields

$$(I - B^{\mathrm{m}}_{\mu_2\mu_3}(0,z)) g_z = 0$$

which reduces to a 2×2 homogeneous linear system for $(c_1(z), c_2(z))$:

$$\begin{cases} (1+\mu_2 a_{11}(z))c_1(z) + \mu_3 a_{12}(z) c_2(z) = 0 \\ \mu_2 a_{12}(z) c_1(z) + (1+\mu_3 a_{22}(z)) c_2(z) = 0 \end{cases} \tag{7}$$

The system (7) has a non-trivial solution if and only if $\Delta_{\mu_2\mu_3}(z) = 0$. ■

Proposition 3.2 *Let $a_{ij}(z), \ i,j = 1,2$ be defined by (6). Then the following properties hold:*

- All functions $a_{ij}(z)$ are real-valued and analytic in $\mathbb{R}\backslash[0,24]$.

- The functions $a_{ij}(\cdot), i,j = 1,2$ are strictly increasing on the interval $(-\infty, 0]$, and satisfy:

$$a_{ij}(z) > 0, \quad a_{11}(z)a_{22}(z) - a_{12}^2(z) > 0 \quad \text{for} \quad z \le 0,$$

where $a_{ij}(0) := \lim_{z \mathrm{Z} 0} a_{ij}(z)$.

- The following asymptotic relations hold:

$$a_{ij}(z) = a_{ij}(0) + \mathcal{O}(|z|), \quad z \mathrm{Z} 0.$$

Proof. The proof of Proposition 3.2 proceeds similarly to the proof of [13, Lemma 3]. ■

Lemma 3.3 *The function $\Delta_{\mu_2\mu_3}(z)$ is real-valued and analytic for $z \in \mathbb{R}\backslash[0,24]$. Moreover, the following asymptotic relations hold:*

- $$\lim_{|z|\to\infty} \Delta_{\mu_2\mu_3}(z) = 1.$$

- $$\Delta_{\mu_2\mu_3}(z) = A^-(\mu_2,\mu_3) + \mathcal{O}(|z|), \quad z \mathrm{Z} 0,$$

where

$$A^-(\mu_2,\mu_3) = 1 + a_{11}(0)\mu_2 + a_{22}(0)\mu_3 + (a_{11}(0)a_{22}(0) - a_{12}^2(0))\mu_2\mu_3, \tag{8}$$

The next lemma describes the geometry of the zeros of the polynomials $A^-(\mu_2,\mu_3)$ and the induced partition of the coupling-constant plane.

Lemma 3.4

The zero set of $A^-(\mu_2,\mu_3)$ is the graph of the function

$$\mu_3^-(\mu_2) := -\frac{1+a_{11}(0)\mu_2}{a_{22}(0)+(a_{11}(0)a_{22}(0)-a_{12}^2(0))\mu_2},$$

with the horizontal asymptote $\mu_3=-\mu_3^0$ and the vertical asymptote $\mu_2=-\mu_2^0$, where

$$\mu_3^0=\frac{a_{11}(0)}{a_{11}(0)a_{22}(0)-a_{12}^2(0)}>0,\quad \mu_2^0=\frac{a_{22}(0)}{a_{11}(0)a_{22}(0)-a_{12}^2(0)}>0.$$

This graph consists of two disjoint connected branches, given by

$$\tau_0^-=\{(\mu_2,\mu_3):\mu_3=\mu_3^-(\mu_2),\quad \mu_2>-\mu_2^0\},\quad \tau_1^-=\{(\mu_2,\mu_3):\mu_3=\mu_3^-(\mu_2),\quad \mu_2<-\mu_2^0\},$$

The branches $\tau_0^-\cup\tau_1^-$ divide the plane $\mathbb{R}^2$ into three unbounded regions

$$\mathbb{R}^2\backslash(\tau_0^-\cup\tau_1^-)=\mathcal{A}_0^-\cup\mathcal{A}_1^-\cup\mathcal{A}_2^-,$$

where

$$\begin{aligned}\mathcal{A}_0^- &= \{(\mu_2,\mu_3):\mu_3>\mu_3^-(\mu_2),\ \mu_2>-\mu_2^0\},\\ \mathcal{A}_1^- &= \{(\mu_2,\mu_3):\mu_3<\mu_3^-(\mu_2),\mu_2>-\mu_2^0\}\cup\{(-\mu_2^0,\mu_3)\in\mathbb{R}^2\}\\ &\quad\cup\{(\mu_2,\mu_3):\mu_3>\mu_3^-(\mu_2),\quad \mu_2<-\mu_2^0\},\\ \mathcal{A}_2^- &= \{(\mu_2,\mu_3):\mu_3<\mu_3^-(\mu_2),\quad \mu_2<-\mu_2^0\}.\end{aligned}\tag{9}$$

Proof. By the representation (8), the equation $A^-(\mu_2,\mu_3)=0$ can be solved uniquely for μ_3, which defines the function $\mu_3=\mu_3^-(\mu_2)$. Since the function $\mu_3^-(\mu_2)$ is defined on $(-\infty,-\mu_2^0)\cup(-\mu_2^0,\infty)$, its graph consists of two disjoint unbounded connected branches τ_0^- and τ_1^-.

These branches divide the plane $\mathbb{R}^2$ into three unbounded regions $\mathcal{A}_j^-$, $j=0,1,2$, which are characterized by the sign of the polynomial $A^-(\mu_2,\mu_3)$ and are given explicitly by (9). ■

The main results and their proofs

Recall that the main purpose of this work is to determine the exact number of eigenvalues of $H^{\mathrm{m}}_{\mu_2\mu_3}(0)$ below its essential spectrum.

We will now determine the exact number of eigenvalues of the operator $H^{\mathrm{m}}_{\mu_2\mu_3}(0)$ lying below the essential spectrum within the regions $\mathcal{A}_\zeta^-$, $\zeta=0,1,2$.

Theorem 4.1 *Let $\mu_2,\mu_3\in\mathbb{R}$ and $\zeta=0,1,2$. If $(\mu_2,\mu_3)\in\mathcal{A}_\zeta^-$, the operator $H^{\mathrm{m}}_{\mu_2\mu_3}(0)$ has ζ eigenvalues below its essential spectrum.*

Proof. The Fredholm determinant $\Delta_{\mu_2\mu_3}(z)$ is analytic in $z\in\mathbb{C}\backslash[0,24]$ and jointly analytic in the coupling constants (μ_2,μ_3). Hence, by Rouché's theorem, the number of its zeros in $(-\infty,0)$ is locally constant. Since each region $\mathcal{A}_\zeta^-$, $\zeta=0,1,2$, is connected, this number is constant throughout $\mathcal{A}_\zeta^-$.

It therefore suffices to compute the number of negative eigenvalues $n_-(H^{\mathrm{m}}_{\mu_2\mu_3}(0))$ for a single representative point in each region.

Case $\mathcal{A}_0^-$. Lemma 3.4 yields that $(0,0)\in\mathcal{A}_0^-$. Since $H^{\mathrm{m}}_{00}(0)=H_0(0)$, the operator $H^{\mathrm{m}}_{00}(0)$ has no eigenvalues below the essential spectrum. Therefore $n_-(H^{\mathrm{m}}_{\mu_2\mu_3}(0))=0$ for all $(\mu_2,\mu_3)\in\mathcal{A}_0^-$.

Case $\mathcal{A}_1^-$. For $(-\mu_2^0,0)\in\mathcal{A}_1^-$, the determinant reduces to

$$\Delta^{\mathrm{m}}_{(-\mu_2^0)0}(z)=1-\mu_2^0a_{11}(z),$$

which is strictly decreasing on $(-\infty,0)$ and changes sign at the endpoints. Thus it has exactly one zero in $(-\infty,0)$. By Lemma 3.1, the operator $H^{\mathrm{m}}_{(-\mu_2^0)0}(0)$ has exactly one negative eigenvalue and therefore $n_-(H^{\mathrm{m}}_{\mu_2\mu_3}(0))=1$ for all $(\mu_2,\mu_3)\in\mathcal{A}_1^-$.

Case $\mathcal{A}_2^-$. Let $(\mu_2^{(1)},\mu_3^{(1)})$ be a point in $\mathcal{A}_2^-$. Then by the definition of $\mathcal{A}_2^-$, we have

$$\mu_2^{(1)}<-\mu_2^{(0)}<0,\quad \mu_3^{(1)}<-\mu_3^{(0)}<0$$

and

$$A^-(\mu_2^{(1)},\mu_3^{(1)})>0.$$

Since $(\mu_2^{(1)}, 0) \in \mathcal{A}_1^-$, the determinant $\Delta_{\mu_2^{(1)}0}(z)$ has a unique zero $z(\mu_2^{(1)})$ in $(-\infty, 0)$. Hence

$$1 + \mu_2^{(1)} a_{11}(z(\mu_2^{(1)})) = 0.$$

Using this relation, we obtain

$$\Delta_{\mu_2^{(1)}\mu_3^{(1)}}(z(\mu_2^{(1)})) = -\mu_2^{(1)}\mu_3^{(1)} a_{12}^2(z(\mu_2^{(1)})) < 0.$$

Moreover,

$$\lim_{z\to-\infty} \Delta_{\mu_2^{(1)}\mu_3^{(1)}}(z) = 1, \quad \lim_{z\mathrm{Z}0} \Delta_{\mu_2^{(1)}\mu_3^{(1)}}(z) = A^-(\mu_2^{(1)}, \mu_3^{(1)}) > 0.$$

Thus, $\Delta_{\mu_2^{(1)}\mu_3^{(1)}}(z)$ has at least two zeros in $(-\infty, 0)$. Hence, by Lemma 3.1, the operator $H^{\mathrm{m}}_{\mu_2^{(1)}\mu_3^{(1)}}(0)$ has at least two eigenvalues below the essential spectrum. Since the perturbation is of rank two, $H^{\mathrm{m}}_{\mu_2^{(1)}\mu_3^{(1)}}(0)$ has exactly two negative eigenvalues.

This completes the proof. ■

Acknowledgments.

This work was supported by the Foundation for Fundamental Research of the Republic of Uzbekistan (grant No. FL-9524115052).

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