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CARLEMAN FORMULA FOR THE PROBLEM OF CONTINUING THE SOLUTION OF A ONE-VARIABLE METAANALYTIC EQUATION FROM A PART OF THE BOUNDARY OF THE DOMAIN

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Abstract. *This paper is dedicated to one of the fundamental and complex problems in the theory of partial differential equations and complex analysis: the problem of analytic continuation for the class of n-metaanalytic functions within a bounded domain D of the complex plane. We rigorously investigate the boundary value problem where an n-metaanalytic function $f(z)$ is reconstructed from the known values of its differential operators $L_s f(t)$ specified on a subset E of the boundary ∂D. First, we establish a comprehensive uniqueness theorem in the function class $M_n(D) \cap C^{n-1}(\bar{D})$ by utilizing an auxiliary lemma concerning the properties of the $\bar{\partial}_z$ operator and an inductive method. Furthermore, we prove the severe ill-posedness of this boundary value problem in the sense of Hadamard by explicitly constructing a sequence of counterexamples (Hadamard-type examples) demonstrating exponential instability in the interior of the domain despite vanishing boundary data as $m \to \infty$. To overcome this instability and construct a stable solution, we derive an explicit Carleman-type formula for n-metaanalytic functions. By leveraging V.V. Pokazeyev's generalized Cauchy integral formula and introducing a specially constructed quenching function $g_\sigma(z) = e^{\sigma h(z)}$ governed by a positive parameter σ, we establish a limiting integral representation. As $\sigma \to \infty$, this formula enables the stable recovery of the function with a controllable error estimate bounded by $O(e^{-\sigma\omega(z)})$, where $\omega(z)$ represents the harmonic measure of the set E. These results provide a robust theoretical framework for treating improperly posed inverse problems associated with higher-order meta-analytic systems.*

Keywords: *n-metaanalytic functions, analytic continuation, ill-posed problems, Carleman formula, Cauchy integral formula, harmonic measure, generalized Cauchy-Riemann system.*

BIR O'ZGARUVCHILI METAANALITIK TENGLAMA YECHIMINI SOHA CHEGARASINING QISMIDAN DAVOM ETTIRISH MASALASI UCHUN KARLEMAN FORMULASI

Annotatsiya. *Ushbu maqola differensial tenglamalar nazariyasi va kompleks analizning murakkab masalalaridan biri bo'lib, kompleks tekisligining chegaralangan D sohasidagi n-metaanalitik funksiyalar sinfi uchun analitik davom ettirish masalasiga bag'ishlangan. Biz n-metaanalitik $f(z)$ funksiyani uning differensial operatorlari $L_s f(t)$ yordamida ∂D soha chegarasidagi E to'plamda berilgan qiymatlariga ko'ra analitik davom ettirish masalasini qaraymiz. Dastlab $\bar{\partial}_z$ operatorning xossalariga oid yordamchi lemmani va induktiv usulni qo'llab, $M_n(D) \cap C^{n-1}(\bar{D})$ sinfida to'liq yagonalik teoremasini isbotlaymiz. Bundan tashqari Hadamard tipidagi misollar yordamida ushbu chegaraviy masalaning nokorrekt qo'yilganligini isbotlaymiz. Ushbu nokorrektlikni bartaraf etish uchun n-metaanalitik funksiyalar uchun Karleman tipidagi formulani keltiramiz. Shuningdek V.V Pokazeyevning umumlashgan Koshi integral formulasidan va musbat σ delta parametrga bog'liq maxsus $g_\sigma(z) = e^{\sigma h(z)}$ so'ndiruvchi funksiyadan foydalanib, $\sigma \to \infty$ da limitga o'tib, $O(e^{-\sigma\omega(z)})$ bilan chegaralangan nokorrektlikni korreksiyalash imkonini beradi, bunda $\omega(z)$ funksiya E to'plamning garmonik o'lchovini ifodalaydi. Olingan natijalar yuqori tartibli n-metaanalitik sistemalar bilan bog'liq nokorrekt qo'yilgan teskari masalalarni yechish uchun mustahkam nazariy asos yaratadi.*

Kalit so'zlar: *n-metaanalitik funksiyalar, analitik davom ettirish, nokorrekt masalalar, Karleman formulasi, Koshi tipidagi integral formulasi, garmonik o'lchov, umumlashgan Koshi-Riman sistemasi.*

ФОРМУЛА КАРЛЕМАНА ДЛЯ ЗАДАЧИ АНАЛИТИЧЕСКОГО ПРОДОЛЖЕНИЯ РЕШЕНИЯ ОДНО ПЕРЕМЕННОГО МЕТААНАЛИТИЧЕСКОГО УРАВНЕНИЯ С ЧАСТИ ГРАНИЦЫ ОБЛАСТИ

Аннотация. *Настоящая работа посвящена одной из фундаментальных и сложных проблем теории дифференциальных уравнений с частными производными и комплексного анализа — проблеме аналитического продолжения для класса n-метааналитических функций в ограниченной области D комплексной плоскости. Мы строго исследуем краевую задачу, в которой n-метааналитическая функция $f(z)$ восстанавливается по известным значениям своих дифференциальных операторов $L_s f(t)$, заданных на подмножестве E границы ∂D. Сначала мы устанавливаем полную теорему единственности в классе функций $M_n(D) \cap C^{n-1}(\bar{D})$, используя вспомогательную лемму о свойствах оператора $\bar{\partial}_z$ и индуктивный метод. Кроме того, мы доказываем сильную некорректность этой краевой задачи в смысле Адамара, явно строя последовательность контрпримеров (примеров типа Адамара), демонстрирующих экспоненциальную неустойчивость внутри области, несмотря на стремление к нулю граничных данных при $m \to \infty$. Чтобы преодолеть эту неустойчивость и построить устойчивое решение, мы выводим явную формулу типа Карлемана для n-метааналитических функций. Используя обобщённую интегральную формулу Коши В.В. Показеева и вводя специально сконструированную «гасящую» функцию $g_\sigma(z) = e^{\sigma h(z)}$, зависящую от положительного параметра σ, мы устанавливаем предельное интегральное представление. При $\sigma \to \infty$ эта формула обеспечивает устойчивое восстановление функции с контролируемой оценкой погрешности, ограниченной величиной $O(e^{-\sigma\omega(z)})$, где $\omega(z)$ представляет собой гармоническую меру множества E. Полученные результаты предоставляют надёжную теоретическую основу для решения некорректно поставленных обратных задач, связанных с n-метааналитическими системами высших порядков.*

Ключевые слова: *n-метааналитические функции, аналитическое продолжение, некорректные задачи, формула Карлемана, интегральная формула Коши, гармоническая мера, обобщённая система Коши–Римана.*

Introduction and Formulation of the Problem. The theory of generalized analytic functions, initiated and extensively developed by I.N. Vekua and other prominent mathematicians, serves as a powerful apparatus for investigating various problems in mathematical physics, continuum mechanics, and complex analysis. Among these generalized classes, ***n***-analytic and ***n***-metaanalytic functions occupy a special place due to their natural appearance in higher-order elliptic systems.

Let D be a bounded simply connected domain in the complex plane $\mathbb{C}$ with a sufficiently smooth boundary ∂D. We consider the class of n-metaanalytic functions denoted by $M_n(D)$. The primary objective of this study is to examine the analytical continuation problem: given the values of the differential operators $L_s f(t)$ on a measurable subset $E \subset \partial D$ of positive measure, recover the n-metaanalytic function $f(z)$ throughout the entire domain D.

Formally, the boundary conditions are stated as follows:

$$L_s f(t) = h_s(t), \quad (s = 0,1,2,\dots,n-1), \quad t \in E,$$

where $L_0 f(t) = f(t) = h_0(t)$, and $h_s(t)$ are prescribed continuous functions on E. When the parameters vanish identically, i.e., $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$, problem (1)-(2) naturally degenerates into the classical boundary value analytic continuation problem for n-analytic functions, which has been thoroughly analyzed in foundational works such as [1]. However, the presence of non-zero parameters λ_s introduces substantial analytical complications, requiring the development of new functional-theoretic tools.

2.Auxiliary Results and Uniqueness Theorem

To establish the uniqueness of the solution to the stated continuation problem, we first formulate and prove an essential auxiliary lemma regarding the solutions of the first-order foundational differential equation.

Lemma 1. *Let $w(z) \in C(D)$ be a continuous function in D satisfying the equation $\bar{\partial}_z w - \lambda w = 0, \quad z \in D$, where λ is a complex constant. If the boundary values of $w(z)$ vanish on a non-trivial arc $E \subset \partial D$, then $w(z) \equiv 0$ everywhere in D.*

Proof. We formally integrate equation (3) by rewriting it in terms of logarithmic derivatives:

$$\frac{\bar{\partial}_z w}{w} = \lambda \Longrightarrow \bar{\partial}_z(\ln w(z) - \lambda \bar{z}) = 0.$$

This implies that the function $f(z) = w(z)e^{-\lambda\bar{z}}$ satisfies the homogeneous Cauchy-Riemann equation $\bar{\partial}_z f = 0$ in D, meaning that $f(z)$ is an arbitrary holomorphic function in D. Consequently, the general solution of equation (3) can be represented in the form:

$$w(z) = f(z)e^{\lambda\bar{z}},$$

where $f(z) \in H(D)$. According to the hypothesis of the lemma, the boundary values satisfy $w(t) = 0$ for $t \in E$, which yields:

$$f(t)e^{\lambda\bar{t}} = 0 \Rightarrow f(t) = 0, \quad t \in E.$$

Since $f(z)$ is a holomorphic function in D and vanishes on a set of positive measure E on the boundary, by applying the generalized Riss theorem (or uniqueness theorem for analytic functions), we obtain $f(z) \equiv 0$ in D. Hence, it directly follows that $w(z) \equiv 0$ throughout D. The lemma is fully proven. □

Using this lemma, we can now state and prove the uniqueness theorem for the main problem in the class $M_n(D) \cap C^{n-1}(\bar{D})$.

Theorem 1. *Let $w(z) \in M_n(D) \cap C^{n-1}(\bar{D})$ be an n-metaanalytic function satisfying the homogeneous boundary conditions on the subset $E \subset \partial D$: $L_s w(t) = (\bar{\partial}_t - \lambda_s)w(t) = 0$, $(s = 0,1,2,\dots,n-1)$. Then, the function vanishes identically in the domain, i.e., $w(z) \equiv 0$ for all $z \in D$.*

Proof. The proof is carried out via mathematical induction with respect to the order of the operators. For $s = n-1$, consider the function $L_{n-1}w(z)$. By the definition of n-metaanalytic functions, it satisfies the first-order equation:

$$(\bar{\partial}_z - \lambda_n)L_{n-1}w(z) = 0, \quad z \in D.$$

In view of boundary condition (4), its trace on $E \subset \partial D$ is equal to zero. Applying Lemma 1 to $L_{n-1}w(z)$, we immediately conclude that:

$$L_{n-1}w(z) = 0, \quad z \in D.$$

Next, descending to the subsequent step, the function $L_{n-2}w(z)$ satisfies the equation governed by λ_{n-1} due to relation (5):

$$(\bar{\partial}_z - \lambda_{n-1})L_{n-2}w(z) = 0, \quad z \in D.$$

Since condition (4) dictates that the boundary values of $L_{n-2}w(z)$ also vanish on E, Lemma 1 ensures that $L_{n-2}w(z) = 0$ in D. Continuing this downward induction process for all indices down to $s = 0$, we ultimately arrive at $w(z) \equiv 0$ throughout the domain D. The theorem is proven.

3.Analysis of Ill-Posedness: Hadamard-Type Example

It is a well-established fact in the theory of partial differential equations that Cauchy-type problems for elliptic equations are typically improperly posed (ill-posed) in the sense of Hadamard, meaning that solutions do not depend continuously on the input data. To demonstrate that problem (1)-(2) shares this property, we construct an explicit counterexample.

Example. Consider the sequence of functions defined in the upper half-plane by:

$$w_m(z) = \frac{e^{-imz}}{m}\sum_{k=1}^{n} e^{\lambda_k\bar{z}}, \quad m = 1,2,3,\dots$$

It can be readily verified that these functions satisfy the n-metaanalytic equation system. Let us compute their s-order differential operators:

$$L_s w_m(z) = \frac{e^{-imz}}{m}\sum_{k=s+1}^{n}(\lambda_k - \lambda_s)(\lambda_k - \lambda_{s-1})\dots(\lambda_k - \lambda_1)e^{\lambda_k\bar{z}}, \quad (s = 0,1,\dots,n-1).$$

For $y = 0$ (on the real axis boundary), evaluating the limit as $m \to \infty$ yields:

$$|L_s w_m(z)|_{y=0} = \frac{e^{-imx}}{m}\sum_{k=s+1}^{n}\prod_{l=1}^{s}(\lambda_k - \lambda_l)e^{\lambda_k x} \to 0, \quad \text{as } m \to \infty.$$

Thus, the boundary data uniformly converges to zero as m grows large. However, inside the domain where $y > 0$, the behavior of $w_m(z)$ is governed by the factor $e^{-imz} = e^{-im(x+iy)} = e^{my}e^{-imx}$, which implies that:

$$|w_m(z)| \to \infty, \quad \text{as } m \to \infty \quad (\text{for } y > 0).$$

This drastic instability explicitly confirms that arbitrarily small perturbations on the boundary can lead to unbounded deviations in the interior of the domain. Consequently, problem (1)-(2) is severely ill-posed, necessitating the construction of regularized solutions via Carleman formulas.

4.Cauchy and Carleman Integral Formulas

In the theory of analytic and generalized analytic functions, integral representations play an instrumental role. For n-metaanalytic functions, the generalized Cauchy integral formula was originally established by V.V. Pokazeyev [2].

For any function $w(z) \in M_n(D) \cap C^{n-1}(\bar{D})$, the following Cauchy formula holds:

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \sum_{k=1}^{n} L_{k-1} f(\tau) \sum_{\nu=1}^{k} K_\nu (z-\tau) \prod_{l=1}^{k} \alpha_{\nu l} \frac{d\tau}{\tau - z}, \quad z \in D,$$

where $\alpha_{ll} = 1$, $\alpha_{\nu l} = (\lambda_\nu - \lambda_l)^{-1}$ for $\nu \neq l$, and the kernel function is defined as:

$$K_\nu(z-\tau) = \exp\big(\lambda_\nu \overline{(z-\tau)} - \bar{\lambda}_\nu (z-\tau)\big).$$

To construct the Carleman formula—which provides a stable analytical continuation operator from the subset $E \subset \partial D$ to the interior domain D—we utilize a specially designed quenching (damping) function. Let us introduce:

$$g_\sigma(z) = e^{\sigma h(z)},$$

where $\sigma > 0$ is a large regularization parameter, and $h(z)$ is an analytic function in D such that its real part satisfies $\operatorname{Re} h(z) = \omega(z)$, where $\omega(z)$ is the harmonic measure of the set E relative to the domain D.

For any function $w(z) \in M_n(D)$, the auxiliary product function $F_\sigma(z) = g_\sigma(z) w(z)$ also belongs to the class $M_n(D)$. Applying Cauchy's formula (6) to $F_\sigma(z)$, we write:

$$F_\sigma(z) = \frac{1}{2\pi i}\int_{\partial D} \sum_{k=1}^{n} L_{k-1} F_\sigma(\tau) \sum_{\nu=1}^{k} K_\nu (z-\tau) \prod_{l=1}^{k} \alpha_{\nu l} \frac{d\tau}{\tau - z}.$$

By transforming this expression and isolating the integration over the subset E versus the remaining boundary $\partial D \backslash E$, we evaluate the approximation error at an interior point $z \in D$:

$$|w(z) - w_\sigma(z)| = \left| \frac{1}{2\pi i}\int_{\partial D \backslash E} \sum_{k=1}^{n} L_{k-1} w(\tau) \sum_{\nu=1}^{k} K_\nu (z-\tau) \prod_{l=1}^{k} \alpha_{\nu l}\, e^{\sigma[\alpha(\tau)-\alpha(z)]} \frac{d\tau}{\tau - z} \right| \leq$$

$$\leq e^{-\sigma\omega(z)} aK,$$

where parameters a and K denote respective bounds:

$$a = \sum_{\nu=1}^{n} \prod_{l=1}^{k} |\alpha_{\nu l}|, \quad K = \max_{\tau \in \partial D \backslash E} |L_{k-1} w(\tau)|, \quad (1 \leq k \leq n).$$

Inequality (8) directly implies the main structural convergence theorem of our paper.

Theorem 2. *For any function $w(z)$ satisfying the conditions of the problem within $M_n(D) \cap C^{n-1}(\bar{D})$, the stable reconstruction formula via the limiting process holds: $w(z) = \lim_{\sigma\to\infty} \frac{1}{2\pi i}\int_E \sum_{k=1}^{n} f_{k-1}(\tau) \sum_{\nu=1}^{k} e^{\lambda_\nu \overline{(z-\tau)} - \bar{\lambda}_\nu (z-\tau)} \prod_{l=1}^{k} (\lambda_\nu - \lambda_l)^{-1} e^{\sigma[\alpha(\tau)-\alpha(z)]} \frac{d\tau}{\tau - z}$, valid for every interior point $z \in D$, ensuring uniform convergence on compact subsets of D.*

Conclusion. In this comprehensive study, the boundary value problem of analytic continuation for the class of n-metaanalytic functions in a bounded domain has been thoroughly investigated. The primary theoretical and constructive findings of this work are summarized as follows:

- **Uniqueness Criteria:** By formulating an auxiliary proposition for the first-order $\bar{\partial}_z$ operator and executing a rigorous downward mathematical induction, we established a definitive uniqueness theorem confirming that vanishing boundary data on a subset $E \subset \partial D$ forces the n-metaanalytic function to be identically zero throughout D.
- **Ill-Posedness Proof:** Through the explicit construction of a Hadamard-type counterexample sequence $w_m(z)$, we demonstrated that the studied boundary value problem is severely unstable and improperly posed in the classical sense, as boundary perturbations grow exponentially inside the domain.
- **Carleman Representation:** Utilizing Pokazeyev's generalized Cauchy formula and implementing an exponential quenching function governed by harmonic measures, we successfully derived an explicit Carleman-type formula for n-metaanalytic functions.
- **Error Estimation and Stability:** We established that the convergence rate of the proposed regularization method is controlled by an error bound of order $O(e^{-\sigma\omega(z)})$ as $\sigma \to \infty$, providing a constructive tool for solving practical inverse problems in mathematical physics.

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