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EXACT BOUND-STATE AND THRESHOLD ANALYSIS FOR A RANK-ONE TWO-FERMION LATTICE HAMILTONIAN

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Abstract. *We study a one-parameter family of discrete Schrödinger operators associated with two identical fermions on the one-dimensional lattice. After the Floquet–Bloch decomposition, the fiber Hamiltonian at total quasi-momentum $k \in \mathbb{T}$ acts in the odd subspace of $L^2(\mathbb{T})$ and is a rank-one perturbation of a multiplication operator. The perturbation is generated by the second-neighbour form factor $\sin 2q$. We determine the essential spectrum, derive the Birman–Schwinger–Fredholm determinant, and evaluate it explicitly below the spectral band. For every $k \in (-\pi, \pi)$ we identify the critical coupling $\mu_*(k) = 1/\pi \cos k/2$. No eigenvalue occurs below the essential spectrum for $0 < \mu \le \mu_*(k)$, whereas for $\mu < \mu_*(k)$ there is exactly one simple eigenvalue, given in closed form. At the critical coupling the lower band edge is a threshold resonance rather than an L^2 eigenvalue. We formulate this statement in a scale of weighted Hilbert spaces adapted to the two threshold points of the dispersion relation. The degenerate fiber $k = \pi$ is treated separately: its free dispersion is constant, and every attractive coupling produces one simple eigenvalue. We also obtain the exact threshold expansion and the strong-coupling asymptotics of the bound-state energy*

Keywords: *two-fermion lattice system, discrete Schrödinger operator, rank-one perturbation, essential spectrum, bound state, threshold resonance, Fredholm determinant.*

RANGI BIRGA TENG IKKI FERMIONLI PANJARA HAMILTONIANI UCHUN BOG'LANGAN HOLATLAR VA BO'SAG'AVIY HODISALARNING ANIQ TAHLILI

Annotatsiya. *Biz bir o'lchamli panjaradagi ikkita bir xil fermion bilan bog'liq bo'lgan diskret Schrödinger operatorlarining bir parametrli oilasini o'rganamiz . Floquet–Bloch ajratishidan so'ng, to'la kvaziimpulsi $k \in \mathbb{T}$ bo'lgan tola Hamiltoniani $L^2(\mathbb{T})$ fazosining toq qismida ta'sir qiladi va ko'paytirish operatorining bir rangli qo'zg'atilishi hisoblanadi. Qo'zg'atilish ikkinchi qo'shni shakl funksiyasi $\sin 2q$ orqali hosil qilinadi. Biz muhim spektrni aniqlaymiz, Birman–Schwinger–Fredholm determinantini keltirib chiqaramiz va uni spektral yo'lakdan pastda oshkor ko'rinishda hisoblaymiz. Har bir $k \in (-\pi, \pi)$ uchun kritik bog'lanish koeffitsiyentini $\mu_*(k) = 1/\pi \cos k/2$ ko'rinishida topamiz. $0 < \mu \le \mu_*(k)$ bo'lganda muhim spektrdan pastda hech qanday xos qiymat mavjud emas. Ammo, $\mu > \mu_*(k)$ bo'lganda, muhim spektrdan pastda aniq bitta oddiy xos qiymat mavjud bo'ladi va u oshkor formula orqali ifodalanadi. Kritik bog'lanish holatida quyi spektral yo'lak chegarasi L^2 fazosidagi xos qiymat emas, balki bo'sag'aviy rezonans hisoblanadi. Biz bu holatni dispersiya munosabatining ikkita chega nuqtasiga moslashtirilgan vaznli Hilbert fazolari shkalasi yordamida ifodalaymiz. Aynigan $k = \pi$ tolasi alohida ko'rib chiqiladi: bu holda erkin dispersiya doimiy bo'ladi va har qanday tortishuvchi bog'lanish bitta sodda xos qiymatni hosil qiladi. Shuningdek, bog'langan holat energiyasining bo**'sag'a** yaqinidagi aniq yoyilmasi hamda kuchli bog'lanishdagi asimptotik xulqini ham olamiz.*

Kalit so'zlar: *ikki fermionli sistema, diskret Schrödinger operatori, bir rangli qo'zg'alish, muhim spektr, bog'langan holat , bo'sag'aviy rezonans, Fredholm determinanti.*

ТОЧНЫЙ АНАЛИЗ СВЯЗАННЫХ СОСТОЯНИЙ И ПОРОГОВЫХ РЕЗОНАНСОВ ДЛЯ РЕШЁТОЧНОГО ГАМИЛЬТОНИАНА ДВУХ ФЕРМИОНОВ РАНГА ОДИН

Аннотация. *Мы исследуем однопараметрическое семейство дискретных операторов Шрёдингера, связанных с двумя идентичными фермионами на одномерной решётке. После разложения Флоке–Блоха фибровый гамильтониан при полном квазиимпульсе $k \in \mathbb{T}$ действует в нечётном подпространстве пространства $L^2(\mathbb{T})$ и представляет собой возмущение ранга один оператора умножения. Возмущение порождается форм-фактором второй соседней связи $\sin 2q$. Мы определяем существенный спектр, выводим определитель Бирмана–Швингера–Фредгольма и вычисляем его в явном виде ниже спектральной зоны. Для каждого $k \in (-\pi, \pi)$ мы находим критическую константу связи $\mu_*(k) = 1/\pi \cos k/2$. При $0 < \mu \leq \mu_*(k)$ ниже существенного спектра собственные значения отсутствуют. В то же время при $\mu > \mu_*(k)$ существует ровно одно простое собственное значение, которое задаётся в явном виде. При критическом значении константы связи нижняя граница спектральной зоны является не собственным значением в пространстве L^2, а пороговым резонансом. Это утверждение формулируется в шкале взвешенных гильбертовых пространств, адаптированной к двум пороговым точкам дисперсионного соотношения. Вырожденный случай $k = \pi$ рассматривается отдельно: свободная дисперсия является постоянной, и любое притягивающее взаимодействие порождает одно простое собственное значение. Кроме того, получены точное пороговое разложение и асимптотика энергии связанного состояния при сильной связи.*

Ключевые слова: *двухфермионная решёточная система, дискретный оператор Шрёдингера, возмущение ранга один, существенный спектр, связанное состояние, пороговый резонанс, определитель Фредгольма.*

Introduction. Discrete Schrödinger operators provide a natural framework for quantum particles moving in crystalline and optical-lattice media. In contrast with continuum models, lattice Hamiltonians have bounded dispersion relations, and translation invariance leads to a Floquet–Bloch decomposition into fiber operators depending nontrivially on the conserved total quasi-momentum. This dependence is responsible for several lattice-specific spetral phenomena, including quasi-momentum-dependent binding thresholds and threshold states; see, for example, [1–5]. For two identical fermions the relative wave function is odd. Consequently, an on-site interaction does not contribute in the antisymmetric sector, while interactions supported on nonzero lattice separations give rise, after Fourier transformation, to odd trigonometric form factors. Finite-range interactions therefore become finite-rank perturbations of the fiber kinetic-energy operator. This reduction makes it possible to replace the spectral problem outside the essential spectrum by a finite-dimensional Birman–Schwinger equation. The present paper considers the simplest nontrivial rank-one case generated by the secondneighbour form factor $\sin 2q$. The resulting fiber Hamiltonian is

$$h_\mu(k) = h_0(k) - \mu v, \qquad \mu > 0,$$

acting in $L^2_{\mathrm{odd}}(\mathbb{T})$, where

$$\varepsilon_k(q) = 2 - 2\cos\frac{k}{2}\cos 2q$$

is the free dispersion and

$$(vf)(q) = \sin 2q \int_{\mathbb{T}} \sin 2t\, f(t)\mathrm{d}t.$$

The minus sign corresponds to an attractive interaction. Since v has rank one, there can be at most one eigenvalue below the spectral band. The main questions are therefore whether such an eigenvalue exists, where it is located, and what happens at the critical coupling at which it reaches the lower band edge.

Our analysis gives complete answers. For $k \in (-\pi, \pi)$ the lower and upper band edges are

$$\varepsilon_{\min}(k) = 2\left(1 - \cos\frac{k}{2}\right), \qquad \varepsilon_{\max}(k) = 2\left(1 + \cos\frac{k}{2}\right).$$

The critical coupling is $\mu_*(k) = \pi^{-1}\cos\frac{k}{2}$. For $\mu > \mu_*(k)$ the unique eigenvalue is

$$z_\mu(k) = 2 - \frac{\cos^2(k/2) + \pi^2\mu^2}{\pi\mu},$$

and its distance from the lower band edge admits the exact identity

$$z_\mu(k) - \varepsilon_{\min}(k) = -\frac{(\pi\mu - \cos(k/2))^2}{\pi\mu}.$$

At $\mu = \mu_*(k)$ the threshold solution is proportional to $\cot q$. It does not belong to the physical space $L^2(\mathbb{T})$, but it belongs to every weighted space whose weight vanishes at the threshold set with exponent greater than

$1/2$. This gives a precise weighted-space characterization of the threshold resonance. The exceptional fiber $k = \pi$ has the one-point essential spectrum $\{2\}$ and the eigenvalue $2 - \pi\mu$ for every $\mu > 0$.

The paper is organized as follows. Section 2 introduces the fiber Hamiltonian and determines its essential spectrum. Section 3 establishes the rank-one Birman–Schwinger principle and evaluates the Fredholm determinant. Section 4 contains the complete eigenvalue classification. Threshold resonances and asymptotic formulas are discussed in Sections 5 and 6.

The fiber Hamiltonian and its essential spectrum

Let $\mathbb{T} = (-\pi, \pi]$ be the one-dimensional torus equipped with Lebesgue measure. Set

$$L^2_{\mathrm{odd}}(\mathbb{T}) := \{f \in L^2(\mathbb{T}): f(-q) = -f(q) \text{ for a. e. } q \in \mathbb{T}\}.$$

This is a closed subspace of $L^2(\mathbb{T})$and hence a Hilbert space.

For $k \in \mathbb{T}$, define

$$\varepsilon_k(q) := 2 - 2c_k \cos 2q\,, \qquad c_k := \cos\frac{k}{2} \geq 0. \tag{1}$$

The free fiber operator $h_0(k)$ is multiplication by ε_k:

$$(h_0(k)f)(q) = \varepsilon_k(q)f(q).$$

Let $\phi(q) := \sin 2q$ and define

$$vf = (f, \phi)\phi, \;\; (vf)(q) = \sin 2q \int_{\mathbb{T}} \sin 2t\, f(t)\mathrm{d}t. \tag{2}$$

Since $\|\phi\|^2 = \int_{-\pi}^{\pi} \sin^2 2q\, \mathrm{d}q = \pi$, the operator v is nonnegative, self-adjoint, and of rank one. For $\mu > 0$ we consider

$$h_\mu(k) := h_0(k) - \mu v. \tag{3}$$

Proposition 2.1. For every $k \in \mathbb{T}$ and $\mu > 0$, the operator $h_\mu(k)$ is bounded and self-adjoint in $L^2_{\mathrm{odd}}(\mathbb{T})$.

Proof. The function $\boldsymbol{\varepsilon_k}$ is real-valued and bounded, so $h_0(k)$ is bounded and self-adjoint. The operator $v = |\phi\rangle\langle\phi|$ is bounded and self-adjoint. Hence so is $h_\mu(k)$.

Proposition 2.2. For every $k \in \mathbb{T}$ and $\mu > 0$,

$$\sigma_{\mathrm{ess}}\left(h_\mu(k)\right) = \sigma\big(h_0(k)\big) = [\varepsilon_{\min}(k), \varepsilon_{\max}(k)], \tag{4}$$

where

$$\varepsilon_{\min}(k) = 2(1 - c_k), \quad \varepsilon_{\max}(k) = 2(1 + c_k). \tag{5}$$

For $k = \pi$ the interval degenerates to the singleton $\{2\}$.

Proof. The spectrum of a multiplication operator equals the essential range of its multiplier. Since $\boldsymbol{\varepsilon_k}$ is continuous and $\cos 2q$ ranges over $[-1,1]$, formulas (5) follow. The perturbation $-\mu v$ has finite rank and is therefore compact. Weyl's theorem gives (4).

Birman–Schwinger reduction and the Fredholm determinant

Fix $k \in \mathbb{T}$ and let

$$z \in \mathbb{C}\backslash[\varepsilon_{\min}(k), \varepsilon_{\max}(k)]$$

Set

$$d(k, z) := \int_{\mathbb{T}} \frac{\sin^2 2q}{\varepsilon_k(q) - z}\mathrm{d}q \tag{6}$$

and

$$\Delta_\mu(k, z) := 1 - \mu d(k, z). \tag{7}$$

Lemma 3.1 (Rank-one Birman–Schwinger principle). A number $z \notin [\varepsilon_{\min}(k), \varepsilon_{\max}(k)]$ is an eigenvalue of $h_\mu(k)$ if and only if $\Delta_\mu(k, z) = 0$.

Every such eigenvalue is simple, and an associated eigenfunction is

$$f_{\mu,k,z}(q) = C\frac{\sin 2q}{\varepsilon_k(q) - z}, \qquad C \neq 0 \tag{8}$$

Proof. The equation $\big(h_\mu(k) - z\big)f = 0$ is equivalent to

$$f = \mu(h_0(k) - z)^{-1}(f, \phi)\phi.$$

If $(f, \phi) = 0$, then $f = 0$, so a nonzero solution must satisfy $(f, \phi) \neq 0$. Taking the scalar product with ϕ gives $1 = \mu d(k, z)$, which is (7). Conversely, if (7) vanishes, then (8) is a nonzero eigenfunction. The eigenspace is contained in the one-dimensional range of
$(h_0(k) - z)^{-1}v$, so the eigenvalue is simple.■

For $z < \varepsilon_{\min}(k)$ the function $d(k, z)$ is positive and strictly increasing in z, since

$$\partial_z(k, z) = \int_{\mathbb{T}} \frac{\sin^2 2q}{(\varepsilon_k(q) - z)^2}\, dq > 0. \tag{9}$$

Consequently, $\Delta_\mu(k,z)$ is strictly decreasing, with

$$\lim_{z\to-\infty}\Delta_\mu(k,z)=1.$$

Lemma 3.2 (Explicit determinant). Let $k\in(-\pi,\pi)$, so that $c_k>0$, and let $z<\varepsilon_{\min}(k)$. Put

$$A=A(k,z):=\frac{2-z}{2c_k}>1.$$

Then

$$d(k,z)=\frac{\pi}{c_k}\left(A-\sqrt{A^2-1}\right)=\frac{\pi}{2{c_k}^2}\left(2-z-\sqrt{(2-z)^2-4{c_k}^2}\right) \qquad (10)$$

Proof. Using the π - periodicity of the integrand and the substitution $x=2q$ on one period, we obtain

$$d(k,z)=\frac{1}{2c_k}\int_{-\pi}^{\pi}\frac{\sin^2 2q}{A-\cos x}\,\mathrm{d}x.$$

Since

$$\frac{\sin^2 2q}{A-\cos x}=A+\cos x-\frac{A^2-1}{A-\cos x}$$

and

$$\int_{-\pi}^{\pi}\frac{dx}{A-\cos x}=\frac{2\pi}{\sqrt{A^2-1}},$$

we find

$$\int_{-\pi}^{\pi}\frac{\sin^2 2q}{A-\cos x}\,\mathrm{d}x=2\pi\left(A-\sqrt{A^2-1}\right),$$

which proves (10).■

Taking $z\uparrow\varepsilon_{\min}(k)$ in (10) yields

$$d\big(k,\varepsilon_{\min}(k)\big):=\lim_{z\uparrow\varepsilon_{\min}(k)}d(k,z)=\frac{\pi}{c_k}. \qquad (11)$$

This motivates the critical coupling

$$\mu_*(k):=\frac{c_k}{\pi},\qquad k\in(-\pi,\pi) \qquad (12)$$

Exact classification of the discrete spectrum

Theorem 4.1. Let $k\in(-\pi,\pi)$ and $\mu>0$.

(i) If $0<\mu\le\mu_*(k)$, then

$$\sigma_{\mathrm{disc}}\left(h_\mu(k)\right)\cap\left(-\infty,\varepsilon_{\min}(k)\right)=\emptyset.$$

(ii) If $\mu>\mu_*(k)$, then $h_\mu(k)$ has exactly one eigenvalue below its essential spectrum. It is simple and equals

$$z_\mu(k)=2-\frac{{c_k}^2+\pi^2\mu^2}{\pi\mu} \qquad (13)$$

Moreover,

$$z_\mu(k)-\varepsilon_{\min}(k)=-\frac{(\pi\mu-c_k)^2}{\pi\mu}<0 \qquad (14)$$

An associated eigenfunction is

$$f_{\mu,k}(q)=C\,\frac{\sin 2q}{\varepsilon_k(q)-z_\mu(k)},\qquad C\neq 0. \qquad (15)$$

There are no eigenvalues above $\varepsilon_{\max}(k)$.

Proof. By (11),

$$\lim_{z\uparrow\varepsilon_{\min}(k)}\Delta_\mu(k,z)=1-\frac{\mu}{\mu_*(k)}.$$

Since $\Delta_\mu(k,\cdot)$ is strictly decreasing on $\left(-\infty,\varepsilon_{\min}(k)\right)$ and tends to 1 at $-\infty$, it has no zero when $\mu\le\mu_*(k)$ and exactly one zero when $\mu>\mu_*(k)$.

To compute the zero, write the equation $1=\mu d(k,z)$ using (10):

$$A-\sqrt{A^2-1}=\frac{c_k}{\pi\mu}.$$

Since $\mu>c_k/\pi$, the right-hand side lies in $(0,1)$. Solving gives

$$A=\frac{1}{2}\left(\frac{\pi\mu}{c_k}+\frac{c_k}{\pi\mu}\right).$$

As $z = 2 - 2c_k A$, formula (13) follows. Identity (14) is immediate. For $z > \varepsilon_{\max}(k)$ one has $d(k,z) < 0$, whereas $\mu > 0$; hence $1 - \mu d(k,z) > 1$ and the determinant cannot vanish.

Remark 4.2. Because the perturbation is rank one and attractive, the absence of an upper eigenvalue is also a direct consequence of the min–max principle.

4.1 The degenerate fiber $k = \pi$.

At $k = \pi$ one has $c_k = 0$ and $\varepsilon_\pi(q) \equiv 2$. Thus $h_0(\pi) = 2I$ and

$$h_\mu(\pi) = 2I - \mu|\phi\rangle\langle\phi|.$$

Theorem 4.3. For every $\mu > 0$,

$$\sigma_{\text{ess}}\left(h_\mu(\pi)\right) = \{2\},$$

and $h_\mu(\pi)$ has exactly one simple eigenvalue

$$z_\mu(\pi) = 2 - \pi\mu. \tag{16}$$

The corresponding eigenspace is span$\{\sin 2q\}$.

Proof. The vector ϕ satisfies $v\phi = \|\phi\|^2\phi = \pi\phi$, so $h_\mu(\pi)\phi = (2 - \pi\mu)\phi$. On $\phi^\perp$ the operator equals $2I$. The assertion follows.■

Weighted Hilbert spaces and the threshold resonance

For $k \in (-\pi, \pi)$ the minimum of the dispersion relation is attained at the threshold set

$$M_- := \{q \in T : \cos 2q = 1\} = \{0, \pi\},$$

where the points π and $-\pi$ are identified on the torus. A convenient weight measuring the distance from this set is

$$\rho(q) := |\sin q|. \tag{17}$$

For $s \in R$ define

$$\mathcal{H}_s := L^2_s(\mathbb{T}) := \{f : \mathbb{T} \to \mathbb{C} : \ \rho^s f \in L^2(\mathbb{T})\}, \qquad \|f\|_s := \|\rho^s f\|_{L^2(\mathbb{T})}. \tag{18}$$

Functions are identified modulo equality almost everywhere. With the inner product

$$(f,g)_s := \int_{\mathbb{T}} \rho(q)^{2s} f(q)\overline{g(q)}\,\mathrm{d}q,$$

$\mathcal{H}_s$ is a Hilbert space. Notice that $\mathcal{H}_0 = L^2(\mathbb{T})$, and, for $s_2 > s_1$, one has the continuous inclusion $\mathcal{H}_{s_1} \subset \mathcal{H}_{s_2}$. We write

$$\mathcal{H}_s^{\text{odd}} := \mathcal{H}_s \cap L^0_{\text{odd}}(\mathbb{T}),$$

Where $L^0_{\text{odd}}(\mathbb{T})$ denotes the measurable odd functions on the torus.

Definition 5.1 (Lower-threshold resonance). Let $k \in (-\pi, \pi)$. The lower edge $\varepsilon_{\min}(k)$ is called a threshold resonance of $h_\mu(k)$ if there exists a nonzero measurable odd function f such that

(i) $f \in \mathcal{H}_s^{\text{odd}}$ for every $s > \frac{1}{2}$, but $f \notin L^2_{\text{odd}}(\mathbb{T})$;

(ii) the scalar integral $\int_{\mathbb{T}} \sin 2t\, f(t)\,\mathrm{d}t$ exists;

(iii) the threshold equation

$$\big(\varepsilon_k(q) - \varepsilon_{\min}(k)\big)f(q) - \mu \sin 2q = 0 \tag{19}$$

holds for almost every $q \in \mathbb{T}$. If a nonzero solution of (19) belongs to $L^2_{\text{odd}}(\mathbb{T})$, then $\varepsilon_{\min}(k)$ is instead called a threshold eigenvalue.

The exponent $1/2$ in Definition 5.1 is dictated by the one-dimensional singularity $|q - q_0|^{-1}$ at each point $q_0 \in M_-$. Thus the definition distinguishes the borderline non-square-integrable state from an actual eigenfunction.

Theorem 5.2. Let $k \in (-\pi, \pi)$. Then the following assertions hold.

(i) At $\mu = \mu_*(k)$, the lower edge $\varepsilon_{\min}(k)$ is a threshold resonance of $h_{\mu_*(k)}(k)$.

(ii) The corresponding resonant state is unique up to multiplication by a nonzero constant and is given by

$$f_*(q) = \frac{C}{2c_k}\cot q\,, \qquad C \neq 0. \tag{20}$$

It satisfies

$$f_* \in \mathcal{H}_s^{\text{odd}} \text{ for every } s > \tfrac{1}{2}\,,\ f_* \notin L^2_{\text{odd}}(\mathbb{T}). \tag{21}$$

(iii) The lower edge is not a threshold eigenvalue. For $\mu \neq \mu_*(k)$ there is no nonzero solution of (19) of Birman–Schwinger form.

Proof. Since

$$\varepsilon_k(q) - \varepsilon_{\min}(k) = 2c_k(1 - \cos 2q) = 4c_k \sin^2 2q, \tag{22}$$

any nonzero threshold solution obtained from the rank-one equation must be of the form

$$f(q) = C\frac{\sin 2q}{\varepsilon_k(q) - \varepsilon_{\min}(k)} = \frac{C}{2c_k}\cot q.$$

For this function the interaction functional is well defined because

$$\sin 2q\, f_*(q) = \frac{C}{c_k}\cos^2 2q \in L^1(\mathbb{T}).$$

Moreover,

$$\int_{\mathbb{T}} \sin 2q\, f_*(q)\mathrm{d}q = \frac{C}{c_k}\int_{-\pi}^{\pi} \cos^2 2q\mathrm{d}q = \frac{C\pi}{c_k}.$$

Substituting into (19) gives

$$C\sin 2q - \mu\frac{C}{c_k}\sin 2q = 0$$

for almost every q. Hence a nonzero threshold solution exists precisely when

$$\mu = \frac{c_k}{\pi} = \mu_*(k).$$

It remains to determine its weighted integrability. Since $|\cot q| \asymp |\sin q|^{-1}$ near both points of M_-,

$$\rho(q)^{2s}|f_*(q)|^2 \asymp |\sin q|^{2s-2}.$$

The latter function is integrable near a zero of $\sin q$ if and only if $2s - 2 > -1$, equivalently $s > 1/2$. This proves (21). In particular, the choice $s = 0$ is not admissible, so $f_* \notin L^2(\mathbb{T})$ and the lower edge is not an eigenvalue. Finally, the rank-one threshold equation determines every nonzero solution up to a scalar multiple, and the preceding compatibility condition forces $\mu = \mu_*(k)$. The proof is complete.■

Remark 5.3. The particular choice $\rho(q) = |\sin q|$ is not essential. Any bounded nonnegative weight that is comparable to the distance from M_- in neighbourhoods of 0 and π produces the same resonance classification and the same critical exponent $s = 1/2$.

Threshold and strong-coupling asymptotics

The exact identity (14) yields the threshold asymptotics without an implicit-function argument.

Corollary 6.1 (Threshold expansion). Let $k \in (-\pi, \pi)$ and put $c_k = \cos(k/2)$. For

$$\delta := \mu - \mu_*(k) > 0, \qquad |\delta| < \frac{c_k}{\pi},$$

one has the convergent expansion

$$z_\mu(k) = \varepsilon_{\min}(k) + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{\pi^n}{{c_k}^{n-1}}\delta^n. \tag{23}$$

In particular,

$$z_\mu(k) - \varepsilon_{\min}(k) = -\frac{\pi^2}{c_k}\delta^2 + \frac{\pi^3}{{c_k}^2}\delta^3 + O\delta^4, \delta \downarrow 0. \tag{24}$$

Thus the eigenvalue bifurcates from the lower threshold quadratically.

Proof. By (14) and $\pi\mu = c_k + \pi\delta$,

$$z_\mu(k) - \varepsilon_{\min}(k) = -\frac{\pi^2\delta^2}{c_k + \pi\delta}.$$

Expanding $(c_k + \pi\delta)^{-1}$ as a geometric series proves the claim.■

Corollary 6.2 (Strong-coupling asymptotics). For every $k \in \mathbb{T}$,

$$z_\mu(k) = -\pi\mu + 2 + O(\mu^{-1}), \qquad \mu \to +\infty$$

In particular,

$$\lim_{\mu\to+\infty}\frac{z_\mu(k)}{\mu} = -\pi.$$

Concluding remarks. The model admits a complete spectral solution because the fermionic interaction is rank one in each fiber. The total quasi-momentum changes the width of the free spectral band through $c_k = \cos(k/2)$ and simultaneously determines the binding threshold $\mu_*(k) = c_k/\pi$. The critical coupling tends to zero as $k \to \pi$, consistently with the collapse of the free band to the point 2. At nondegenerate quasi-momenta the critical state lies in every weighted space $\mathcal{H}_s^{\mathrm{modd}}$ with $s > 1/2$ but not in the physical space $L^2_{\mathrm{modd}}(\mathbb{T})$. Thus it is a genuine threshold resonance rather than a threshold eigenstate. Above the critical coupling a unique simple bound state appears with a quadratic bifurcation law. The same strategy extends to several finite-range fermionic form factors. In that case the scalar determinant is replaced by a finite-dimensional Fredholm determinant, and the number of eigenvalues can be studied through the inertia and threshold limits of the corresponding Birman–Schwinger matrix.

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