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STRICTLY NEGATIVE POINCARÉ DISPLACEMENT AND POWER-LAW DECAY IN A FAMILY OF GENERALIZED ABEL EQUATIONS

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Abstract. *We study a one-parameter family of generalized Abel equations with trigonometric coefficients and quadratic and quartic nonlinearities. For $0 < \mu \leq 2$, an exact integral identity establishes strictly negative Poincaré displacement throughout its domain, ruling out positive periodic solutions. Every positive solution defined over the first complete period extends globally forward and converges to zero. An expansion of the return map near the origin then yields algebraic decay with exponent 1/4 and an explicit parameter-dependent leading coefficient. The proof combines a global displacement-sign estimate with a local return-map expansion to determine the decay rate.*

Keywords: *generalized Abel equation, positive periodic solution, rigid planar system, Poincaré return map, displacement function, integral identity, forward convergence, power-law decay, asymptotic formula.*

СТРОГО ОТРИЦАТЕЛЬНОЕ СМЕЩЕНИЕ ПУАНКАРЕ И СТЕПЕННОЕ ЗАТУХАНИЕ В СЕМЕЙСТВЕ ОБОБЩЁННЫХ УРАВНЕНИЙ АБЕЛЯ

Аннотация. *Рассматривается однопараметрическое семейство обобщённых уравнений Абеля с тригонометрическими коэффициентами и нелинейностями второй и четвёртой степеней. Для $0 < \mu \leq 2$ получено точное интегральное представление функции смещения отображения Пуанкаре и установлена её строгая отрицательность на всей области определения. Тем самым исключены положительные периодические решения. Для положительных решений, существующих на первом полном периоде, доказаны продолжимость на всю положительную полуось и сходимость к нулю. Разложение отображения Пуанкаре в окрестности нуля даёт точную степенную асимптотику с показателем 1/4 и ведущим коэффициентом, зависящим от параметра. Глобальная оценка знака смещения и локальное разложение отображения Пуанкаре совместно определяют асимптотическое затухание решений.*

Ключевые слова: *обобщённое уравнение Абеля, положительное периодическое решение, жёсткая система, отображение Пуанкаре, функция смещения, интегральное тождество, сходимость к нулю, степенное затухание, асимптотическая формула.*

UMUMLASHGAN ABEL TENGLAMALARI OILASIDA PUANKARE SILJISHINING QAT'IY MANFIYLIGI VA DARAJALI SO'NISHI

Annotatsiya. *Maqolada davriy trigonometrik koeffitsiyentli, ikkinchi va to'rtinchi darajali nochiziqlikka ega umumlashgan Abel tenglamalarining bir parametrli oilasi tadqiq etiladi. $0 < \mu \leq 2$ da Puankare qaytish akslantirishining siljish funksiyasi uchun aniq integral ifoda olinib, uning butun aniqlanish sohasida qat'iy manfiyligi isbotlanadi. Natijada musbat davriy yechimlarning mavjud emasligi kelib chiqadi. Dastlabki to'liq davrda mavjud bo'lgan musbat yechimlar barcha musbat vaqtlar uchun davom ettirilishi va nolga yaqinlashishi ko'rsatiladi. Qaytish akslantirishining nol yaqinidagi yoyilmasi asosida so'nishning aniq*

darajali asimptotikasi topiladi: ko‘rsatkich 1/4, yetakchi koeffitsiyent esa parametrga bog‘liq. Siljishning global ishora bahosi va lokal qaytish yoyilmasi birgalikda yechimlarning asimptotik so‘nishini aniqlaydi.

Kalit so‘zlar: *umumlashgan Abel tenglamasi, musbat davriy yechim, burchak tezligi o‘zgarmas tekislik sistemasi, Puankare qaytish akslantirishi, siljish funksiyasi, integral ayniyat, nolga yaqinlashish, darajali so‘nish, asimptotik formula.*

MSC 2020: 34C25, 34D05.

Introduction. For planar polynomial rigid systems, the search for limit cycles reduces to the analysis of positive periodic solutions of the associated radial equation. Constant angular velocity makes the Poincaré return map a natural tool: its fixed points correspond to closed trajectories. The radial equations obtained in this way often belong to the class of generalized Abel equations [1].

For generalized Abel equations, nonexistence criteria are available even for sign-changing periodic coefficients [1], and uniqueness criteria for limit cycles have been established [2]. The earlier classification of three-trigonometric-monomial equations [3, Theorem 3.3] is recalled and extended in the broader study of generalized Abel equations [4, Theorem A]. Quartic rigid systems and their planar dynamics are studied in [5].

Existing nonexistence criteria for sign-changing coefficients rely on particular symmetries and sign patterns [1, Theorems 2.4 and 2.8]; related rigid-system conditions appear in [1, Theorem 3.4]. Here we use an antiderivative of the quartic coefficient to express the one-period displacement as an integral whose weight is positive on the open half-period.

The trigonometric-monomial classification [4, Theorem A and Remark 2.3(I.5)] shows that the C, C, S family can exhibit nontrivial limit cycles for suitable coefficients. We examine the fixed one-parameter subfamily with coefficients 1, −μ, and 1.

Our main result concerns this one-parameter subfamily: for $0 < \mu \leq 2$, the Poincaré displacement is strictly negative throughout the domain of the return map. Consequently, positive periodic solutions are absent and admissible trajectories converge to zero. A local expansion additionally gives the exact power-law decay rate.

Problem formulation and domain of the Poincaré return map

$$\frac{dr}{dt} = \sin t\, r^2 + \cos t(\cos^2 t - \mu \sin^2 t) r^4, \qquad r > 0, \quad \mu > 0. \tag{1}$$

Here the quartic coefficient and period are

$$A_\mu(t) = \cos t(\cos^2 t - \mu \sin^2 t), \qquad T = 2\pi.$$

Equation (1) is the radial equation of the planar rigid system

$$\dot{u} = -v + u(v + u^3 - \mu u v^2), \qquad \dot{v} = u + v(v + u^3 - \mu u v^2),$$

Let θ denote the polar angle. Introducing polar coordinates $u = r\cos\theta$, $v = r\sin\theta$, we obtain dθ/dt = 1. With θ(0) = 0, we have θ = t. Accordingly, a positive 2π-periodic solution of the radial equation determines a closed trajectory of the planar system that avoids the origin.

The right-hand side of equation (1) is continuous in t and analytic in r. Hence a unique maximal solution exists for every positive initial value $r(0) = \rho$. Since $r \equiv 0$ is also a solution, a positive trajectory cannot intersect the zero solution in finite time while remaining finite. These initial-value and continuation principles follow from standard ODE theory [6, Chapter 2, Sections 2.2, 2.4, and 2.6].

Let D_μ denote the set of initial values $\rho > 0$ for which the solution remains positive and finite throughout $[0, T]$. For $\rho \in D_\mu$, define the Poincaré return map and displacement function by

$$\Pi_\mu(\rho) = r(T; \rho, \mu), \qquad \Delta_\mu(\rho) = \Pi_\mu(\rho) - \rho. \tag{2}$$

Lemma 1. The set D_μ is nonempty and is an interval of the form $(0, R_\mu)$, where $0 < R_\mu \leq +\infty$. The map Π_μ is continuous on D_μ.

Proof. For sufficiently small $\rho > 0$, continuous dependence on initial data keeps the solution near zero on the fixed interval $[0, T]$, so D_μ is nonempty. Let $\rho \in D_\mu$ and $0 < \sigma < \rho$. Uniqueness for a scalar differential equation preserves the ordering of solutions:

$$0 < r(t;\sigma,\mu) < r(t;\rho,\mu), \qquad 0 \le t \le T,$$

as long as both solutions exist. Since the upper solution is bounded on $[0,T]$, the lower one cannot blow up; uniqueness also prevents it from reaching zero in finite time. Thus $\sigma \in D_\mu$. Continuous dependence makes D_μ open, so it is an interval $(0, R_\mu)$. Continuity of Π_μ follows from the same dependence theorem. ▫

Remark 1. The upper endpoint R_μ is finite. Indeed, $A_\mu(0) = 1$, so there exists $\delta > 0$ such that $A_\mu(t) \ge \frac{1}{2}$ and $\sin t \ge 0$ on $0 \le t \le \delta$. As long as a positive solution exists on this interval, it satisfies $r' \ge \frac{1}{2} r^4$. The comparison equation $z' = \frac{1}{2} z^4$ with $z(0) = \rho$ blows up after time $2/(3\rho^3)$. For sufficiently large ρ, this time is less than δ, so the original solution also cannot survive to δ. Hence $R_\mu < +\infty$; the return map is not defined on the whole positive half-line.

The zero solution is excluded from D_μ. The nonexistence result below concerns positive solutions that remain finite over a complete period.

An integral identity for the one-period displacement

Introduce the reciprocal variable $x(t) = 1/r(t)$ along an arbitrary solution with $\rho \in D_\mu$. Then $x(t) > 0$ and

$$x' = -\sin t - \frac{A_\mu(t)}{x^2}. \tag{3}$$

Define

$$Q_\mu(s) = s - \frac{1+\mu}{3} s^3, \qquad A_\mu(t) = Q_\mu{}'(\sin t)\cos t.$$

Consider the function

$$H(t) = x(t)^3 + 3Q_\mu(\sin t).$$

Differentiating and using (3) gives the exact identity

$$H'(t) = 3x(t)^2 x'(t) + 3Q_\mu{}'(\sin t)\cos t = -3\sin t\, x(t)^2. \tag{4}$$

This identity holds for any solution defined over the period; periodicity of the solution is not assumed.

Lemma 2 (integral representation of the displacement). For every $\rho \in D_\mu$, the following identity holds:

$$x(T)^3 - x(0)^3 = 6 \int_0^\pi e^{L(t)} K(t) Q_\mu(\sin t)\, dt, \tag{5}$$

where, for $0 \le t \le \pi$,

$$X(t) = x(t), \qquad Y(t) = x(T-t).$$

$$K(t) = \frac{3\sin t\,[X(t) + Y(t)]}{X(t)^2 + X(t)Y(t) + Y(t)^2}, \qquad L(t) = \int_0^t K(s)\, ds.$$

For $0 < t < \pi$, we have $K(t) > 0$.

Proof. Set

$$W(t) = H(t) - H(T-t) = X(t)^3 - Y(t)^3 + 6Q_\mu(\sin t).$$

Because $\sin(T-t) = -\sin t$ and Q_μ is odd, this representation holds throughout $[0,\pi]$. Identity (4) gives

$$W'(t) = H'(t) + H'(T-t) = -3\sin t\,[X(t)^2 - Y(t)^2].$$

Using the factorization $X^3 - Y^3 = (X-Y)(X^2 + XY + Y^2)$, we obtain

$$W'(t) = K(t)\left(6Q_\mu(\sin t) - W(t)\right). \tag{6}$$

All denominators are strictly positive because $X, Y > 0$. Thus K is continuous on $[0,\pi]$ and strictly positive in the interior. From (6),

$$\frac{d}{dt}\left(e^{L(t)} W(t)\right) = 6e^{L(t)} K(t) Q_\mu(\sin t).$$

Finally, $W(\pi) = 0$ because $T - \pi = \pi$, whereas $W(0) = x(0)^3 - x(T)^3$. Integrating the last identity from 0 to π yields (5). ▫

Formula (5) determines the displacement sign without explicitly solving (1): the weight $e^{L(t)} K(t)$ is nonnegative for all admissible t and strictly positive on $(0,\pi)$.

Nonexistence of positive periodic solutions and convergence to zero

Theorem 1. Suppose $0 < \mu \leq 2$. Then, for every $\rho \in D_\mu$,

$$0 < \Pi_\mu(\rho) < \rho. \tag{7}$$

In particular, equation (1) admits no positive 2π-periodic solution.

Proof. For $0 \leq s \leq 1$,

$$Q_\mu(s) = s\left(1 - \frac{1+\mu}{3}s^2\right) \geq 0,$$

because $1 + \mu \leq 3$. Moreover, $Q_\mu(\sin t) > 0$ for almost every $t \in (0, \pi)$, including the endpoint case $\mu = 2$. By Lemma 2, the right-hand side of (5) is strictly positive. Hence $x(T)^3 > x(0)^3$, so $x(T) > x(0) > 0$. Since $r = 1/x$, we obtain $0 < r(T) < r(0)$, proving (7). A positive T-periodic solution would satisfy $\Pi_\mu(\rho) = \rho$, contrary to (7). ▫

Theorem 1 therefore excludes positive fixed points of Π_μ and establishes the strict negativity of Δ_μ throughout D_μ. The return map Π_μ itself is increasing with respect to the initial value, as follows from order preservation. The displacement sign also determines the long-time behavior of admissible positive trajectories.

Figure 1 illustrates the sign structure of the auxiliary function Qμ(s) used in the integral criterion.

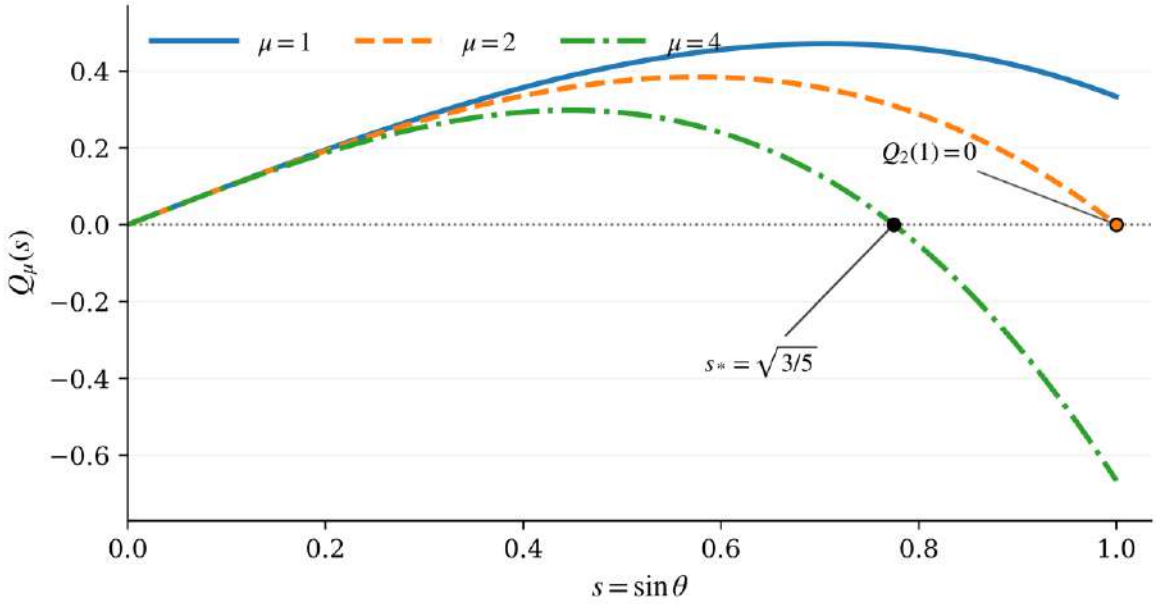


Figure 1. Qμ(s) = s − (1 + μ)s³/3 at μ = 1, 2, 4; Qμ(s) ≥ 0 on [0, 1] for 0 < μ ≤ 2.

Corollary 1. Let $0 < \mu \leq 2$ and $\rho \in D_\mu$. Then the solution of (1) extends to every $t \geq 0$, remains positive, and satisfies

$$\lim_{t \to +\infty} r(t; \rho, \mu) = 0. \tag{8}$$

Proof. Define $\rho_0 = \rho$ and $\rho_{n+1} = \Pi_\mu(\rho_n)$. By Theorem 1, $0 < \rho_{n+1} < \rho_n$. Lemma 1 ensures that each ρ_n belongs to D_μ, so the solution extends successively over all intervals $[nT, (n+1)T]$. This continuation uses the T-periodicity of the right-hand side.

The sequence ρ_n is decreasing and converges to some $\ell \geq 0$. If $\ell > 0$, then $\ell \in D_\mu$; continuity of Π_μ would yield $\Pi_\mu(\ell) = \ell$, contradicting (7). Thus $\rho_n \to 0$. Continuous dependence on the zero solution shows that, for sufficiently small positive initial data, $\max_{0 \leq s \leq T} r(s; \rho_n, \mu) \to 0$. This proves (8). ▫

Thus no positive solution can be periodic with any positive period, and the associated planar rigid system has no nontrivial closed trajectories in the finite plane within this parameter range.

The parameter ranges covered by the sign criterion are summarized in Table 1.

Table 1. Parameter ranges for the condition $Q\mu(s) \geq 0$ and its implications for $\Pi\mu$ ($\rho \in D\mu$)

Parameter range	Sign of $Q\mu(s)$ for $0 < s \leq 1$	Consequence for the Poincaré return map
$0 < \mu < 2$	$Q\mu(s) > 0$	$\Delta\mu(\rho) < 0$; no positive 2π-periodic solution.
$\mu = 2$	$Q_2(s) > 0$ for $s < 1$; $Q_2(1) = 0$	$\Delta_2(\rho) < 0$; no positive 2π-periodic solution.
$\mu > 2$	Sign change at $s^* = \sqrt{[3/(1+\mu)]}$	The criterion does not determine the sign of $\Delta\mu$ across the full return-map domain; it alone gives no conclusion on positive periodic solutions.

Asymptotic decay rate

To quantify the convergence established above, we compute the first nonzero term in the displacement expansion near the zero solution.

Lemma 3. For each fixed $\mu > 0$, as the positive initial value ρ tends to zero,

$$\Pi_\mu(\rho) = \rho - \frac{\pi}{2}(3-\mu)\rho^5 + O(\rho^6). \tag{9}$$

Proof. Fix $\mu > 0$. Choose a constant $C_\mu > 0$ such that, for $0 \leq t \leq T$ and $0 < r \leq 1$,

$$\left|\sin t\, r^2 + A_\mu(t) r^4\right| \leq C_\mu r^2.$$

For sufficiently small $\rho > 0$, comparison with $z' = C_\mu z^2$, $z(0) = \rho$ (using $2\rho \leq 1$ and $C_\mu T\rho \leq 1/2$) gives $0 < r(t;\rho,\mu) \leq 2\rho$ on $[0,T]$. The integral form of equation (1) then provides the uniform estimate

$$|r(t;\rho,\mu) - \rho| \leq 4C_\mu T\rho^2, \qquad 0 \leq t \leq T.$$

Now put $x = 1/r$, with $x(0) = \rho^{-1}$. Since $x(t)^{-2} = r(t)^2 = O(\rho^2)$ uniformly for $t \in [0,T]$, integration of (3) from 0 to t gives

$$x(t) = \rho^{-1} + \cos t - 1 + O(\rho^2), \qquad 0 \leq t \leq T.$$

Hence

$$\frac{1}{x(t)^2} = \rho^2[1 - 2\rho(\cos t - 1) + O(\rho^2)]$$

uniformly on $[0,T]$. Integrating (3) over one period yields

$$x(T) - \rho^{-1} = -\int_0^T \frac{A_\mu(t)}{x(t)^2}\, dt.$$

$$x(T) - \rho^{-1} = -\rho^2 \int_0^T A_\mu(t)\, dt + 2\rho^3 \int_0^T A_\mu(t)(\cos t - 1)\, dt + O(\rho^4).$$

The first integral vanishes because $A_\mu(t) = \frac{d}{dt} Q_\mu(\sin t)$. For the second integral, we find

$$\int_0^T A_\mu(t)(\cos t - 1)\, dt = \int_0^T (\cos^4 t - \mu \sin^2 t \cos^2 t)\, dt = \frac{\pi}{4}(3-\mu).$$

Consequently,

$$x(T) = \rho^{-1} + \frac{\pi}{2}(3-\mu)\rho^3 + O(\rho^4).$$

Taking reciprocals gives (9). ▫

The leading coefficient in (9) also agrees with the Lyapunov-constant formula for quartic rigid systems in [5, proof of Theorem 2.1], specialized to the present parameters.

Theorem 2. Let $0 < \mu \leq 2$ and $\rho \in D_\mu$. Then

$$r(t;\rho,\mu) \sim [(3-\mu)t]^{-1/4}, \qquad t \to +\infty. \tag{10}$$

Equivalently,

$$\lim_{t\to+\infty} t^{1/4} r(t;\rho,\mu) = (3-\mu)^{-1/4}.$$

Proof. Let $\rho_n = r(nT;\rho,\mu)$. Corollary 1 gives $\rho_n \to 0$. Write $c_\mu = \frac{\pi}{2}(3-\mu) > 0$. By Lemma 3,

$$\rho_{n+1} = \rho_n - c_\mu \rho_n^5 + O(\rho_n^6).$$

Raising the ratio $\rho_{n+1}/\rho_n = 1 - c_\mu\rho_n^4 + O(\rho_n^5)$ to the power -4, we obtain

$$\rho_{n+1}^{-4} - \rho_n^{-4} = 4c_\mu + O(\rho_n) = 2\pi(3-\mu) + o(1).$$

Summing these increments and applying the Cesàro mean theorem gives

$$\frac{\rho_n^{-4}}{n} \to 2\pi(3-\mu).$$

It follows that

$$\rho_n \sim [2\pi(3-\mu)n]^{-1/4}. \quad (11)$$

To pass from the discrete return times to arbitrary $t = nT + s$, where $0 \le s \le T$, use the T-periodicity of the right-hand side. On each such interval, the estimate in Lemma 3 applies with initial value ρ_n. For all sufficiently large n,

$$|r(nT+s;\rho,\mu) - \rho_n| \le 4C_\mu T\rho_n^2, \qquad 0 \le s \le T.$$

Consequently,

$$r(nT+s;\rho,\mu) = \rho_n[1 + O(\rho_n)]$$

uniformly in s. Since $t/(nT) \to 1$ as $n \to \infty$, relation (11) implies (10). ▫

Convergence is algebraic rather than exponential: the decay exponent $1/4$ is the same for all $0 < \mu \le 2$, whereas the leading factor depends on $3 - \mu$.

Discussion and conclusions.

The principal result is the strict negativity of the Poincaré displacement, expressed by $\Pi_\mu(\rho) < \rho$ for $0 < \mu \le 2$ throughout the return-map domain. The proof exploits the representation of the quartic coefficient as the derivative of $Q_\mu(\sin t)$ and derives an exact integral formula for the displacement. This excludes positive 2π-periodic solutions and establishes convergence to zero for positive solutions defined over their first complete period.

The integral criterion controls the displacement across all admissible initial values; the local return-map expansion supplies the exact decay rate.

The asymptotic decay law $r(t) \sim [(3-\mu)t]^{-1/4}$ holds for $0 < \mu \le 2$ when the positive solution exists over its first complete period. The case $\mu > 2$ lies outside the established global displacement-sign range.

REFERENCES:

1. Bravo J. L., Torregrosa J. Abel-like differential equations with no periodic solutions. Journal of Mathematical Analysis and Applications. 2008. Vol. 342, no. 2. P. 931–942. DOI: 10.1016/j.jmaa.2007.12.060.

2. Álvarez M. J., Bravo J. L., Fernández M. Abel-like differential equations with a unique limit cycle. Nonlinear Analysis. 2011. Vol. 74. P. 3694–3702. DOI: 10.1016/j.na.2011.02.049.

3. Álvarez M. J., Bravo J. L., Fernández M. Uniqueness of limit cycles for polynomial first-order differential equations. Journal of Mathematical Analysis and Applications. 2009. Vol. 360, no. 1. P. 168–189. DOI: 10.1016/j.jmaa.2009.06.031.

4. Álvarez M. J., Bravo J. L., Fernández M., Prohens R. Characterization of the existence of non-trivial limit cycles for generalized Abel equations. Qualitative Theory of Dynamical Systems. 2021. Vol. 20. Article 15. DOI: 10.1007/s12346-021-00450-4.

5. Álvarez M. J., Bravo J. L., Calderón L. A. Quartic rigid systems in the plane and in the Poincaré sphere. Qualitative Theory of Dynamical Systems. 2024. Vol. 23. Article 226. DOI: 10.1007/s12346-024-01083-z.

6. Teschl G. Ordinary Differential Equations and Dynamical Systems. Graduate Studies in Mathematics, Vol. 140. Providence, RI: American Mathematical Society; 2012. ISBN: 978-0-8218-8328-0.