UDC 517.956.6

SAMARSKII-IONKIN TYPE NONLOCAL BOUNDARY VALUE PROBLEMS FOR A THREE-DIMENSIONAL EQUATION OF MIXED TYPE WITH TWO SINGULAR COEFFICIENTS

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Abstract. *Four nonlocal boundary value problems for a three-dimensional equation of mixed type with two singular coefficients in an unbounded parallelepiped are investigated.. Unlike the earlier periodic problem [1], the matching conditions in x are here of non-self-adjoint (Samarskii–Ionkin) type. The Fourier method is used to construct, for each problem, a system of eigen- and associated functions; biorthogonality with the corresponding adjoint system is proved, the resonant ODE system for the Fourier coefficients is derived and solved explicitly by differentiating the Macdonald function with respect to the spectral parameter, and uniqueness and existence theorems for a regular solution are established.*

Keywords: *biorthogonal system, Samarskii-Ionkin type problem, small denominators, spectral analysis method, nonlocal problem, Bessel operator, attached function, singular coefficient, mixed-type equation.*

IKKITA SINGULYAR KOEFFITSIYENTLI ARALASH TIPDAGI UCH O'LCHOVLI TENGLAMA UCHUN SAMARSKIY-IONKIN TIPIDAGI NOLOKAL CHEGARAVIY MASALALAR

Annotatsiya. *Ushbu ishda cheksiz parallelepipedda ikkita singulyar koeffitsiyentli aralash tipdagi uch o'lchovli tenglama uchun to'rtta nolokal chegaraviy masala o'rganilgan. Ilgari o'rganilgan [1] davriy masaladan farqli ravishda, x o'zgaruvchisi bo'yicha nolokal shartlari o'z-o'ziga qo'shma bo'lmagan (Samarskiy-Ionkin tipidagi) xarakterga ega. Furye usuli yordamida har bir masala uchun xos va qo'shma funksiyalar sistemasi qurilgan, ularning mos qo'shma sistemalar bilan biortogonalligi isbotlangan, Furye koeffitsiyentlari uchun bog'langan oddiy differensial tenglamalar sistemasi keltirib chiqarilgan va Makdonald funksiyasini spektral parametr bo'yicha differensiallash orqali uning aniq yechimi qurilgan. Regulyar yechimning yagonaligi va mavjudligi haqidagi teoremalar isbotlandi.*

Kalit so'zlar: *biortogonal sistema, Samarskiy-Ionkin tipidagi masala, kichik maxrajlar, spektral analiz usuli, nolokal masala, Bessel operatori, biriktirilgan funksiya, singular koeffitsiyent, aralash tipdagi tenglama.*

НЕЛОКАЛЬНЫЕ КРАЕВЫЕ ЗАДАЧИ ТИПА САМАРСКОГО-ИОНКИНА ДЛЯ ТРЁХМЕРНОГО УРАВНЕНИЯ СМЕШАННОГО ТИПА С ДВУМЯ СИНГУЛЯРНЫМИ КОЭФФИЦИЕНТАМИ

Аннотация. *Исследуются четыре нелокальные краевые задачи для трёхмерного уравнения смешанного типа с двумя сингулярными коэффициентами в неограниченном параллелепипеде. В отличие от ранее изученной периодической задачи [1], условия склеивания по x здесь носят несамосопряжённый характер (типа Самарского–Ионкина). Методом Фурье для каждой задачи построена система собственных и присоединённых функций. Доказана её биортогональность соответствующей сопряжённой системе. Выведена резонансная система обыкновенных дифференциальных уравнений для коэффициентов Фурье, и найдено её явное решение путём дифференцирования функции Макдональда по спектральному параметру. Установлены теоремы единственности и существования регулярного решения.*

Ключевые слова: *биортогональная система, задача типа Самарского-Ионкина, малые знаменатели, метод спектрального анализа, нелокальная задача, оператор Бесселя, присоединённая функция, сингулярный коэффициент, уравнение смешанного типа.*

Introduction. Boundary value problems with nonlocal matching conditions (in particular, periodic conditions and conditions of Samarskii-Ionkin type) imposed with respect to one of the spatial variables constitute an important area in the theory of mixed-type equations. Such conditions arise naturally in mathematical models of processes in media with internal sources, in heat-conduction problems governed by an integral energy-conservation law, and in problems possessing axial or helical symmetry. The classical results in this direction are due to N. I. Ionkin [9, 10], who proved the existence and uniqueness of a solution to a nonlocal heat-conduction problem with the condition $u(0,t)$ by the spectral analysis method, and to N. I. Ionkin and E. I. Moiseev [11], who extended this method to a broad class of two-point conditions

$$a_1u_x(0,t)+b_1u_x(1,t)+a_0u(0,t)+b_0u(1,t)=0,$$

$$c_1u_x(0,t)+d_1u_x(1,t)+c_0u(0,t)+d_0u(1,t)=0.$$

The essential feature of such problems is that the corresponding one-dimensional spectral problem is generally non-self-adjoint. Some eigenvalues have algebraic multiplicity greater than their geometric multiplicity; consequently, the classical system of eigenfunctions is insufficient for expanding an arbitrary function and must be augmented by associated functions. The resulting system is only a Riesz basis (a complete and minimal system), rather than an orthogonal basis, and the expansion coefficients are determined by means of a biorthogonal adjoint system.

For multidimensional equations of mixed (elliptic-hyperbolic) type with singular coefficients degenerating on coordinate planes, the spectral analysis method was developed in the works of M.S. Salakhitdinov and A.K. Urinov [8], A.K. Urinov and K.T. Karimov [4], and K.T. Karimov [5, 6, 7]. In those works, solutions of the corresponding three-dimensional boundary value problems in a semi-infinite parallelepiped were constructed and justified under local conditions with respect to the regular variable. The direct predecessor of the present article is the work of K.T. Karimov and D.S. Olimova [1], in which a problem for equation (1) below in the domain $\Omega^{\infty}=\{(x,y,z):x\in(0,a),y\in(-b,+\infty),z\in(0,c)\}$, $a,b,c\in R^{+}$, subject to the periodic conditions $u(0,y,z)=y(a,y,z)$, $u_x(0,y,z)=u_x(a,y,z)$, was constructed and justified. The corresponding spectral problem with respect to x is self-adjoint, and the system $\{1,\cos(2\pi nx/a),\sin(2\pi nx/a)\}$ is a classical orthogonal basis.

The present paper continues the investigations in [1, 2, 3] and considers the case in which the matching condition with respect to x is a non-self-adjoint nonlocal condition of Ionkin type. It is shown that the spectral problem in x then acquires associated functions, whereas the dependence on y is governed not by a homogeneous equation but by a resonantly perturbed Bessel-type equation. An exact solution of this equation is constructed by differentiating the Macdonald function with respect to the spectral parameter. To the best of our knowledge, this technique has not previously been applied to equations of this class with singular coefficients and is of independent methodological interest.

Literature review. Nonlocal conditions for mixed-type equations first appeared in F. I. Frankl's problem [16] concerning flow around a profile with a local supersonic region. The spectral analysis method for nonlocal conditions was subsequently developed in the works of N.I. Ionkin [9, 10], N.I. Ionkin and E. I. Moiseev [11], E.I. Moiseev [12], M.E. Lerner and O.A. Repin [13], Yu.K. Sabitova [14], and K.B. Sabitov [15]. The general theory of nonlocal problems for principal-type and mixed-type equations is presented in the monographs by A.M. Nakhushev [17] and Z.A. Nakhusheva [18]. The special functions and asymptotic formulas for cylindrical functions used below are taken from G.N. Watson's monograph [19].

The aim of this paper is to construct and justify classical solutions of problems $B_1^{\infty}-B_4^{\infty}$ by extending to the non-self-adjoint case the spectral analysis method developed in [1] for the periodic case.

Research methodology and problem statements

In the domain Ω^{∞}, consider the mixed-type equation with two singular coefficients

$$u_{xx}+\operatorname{sgn} y\cdot u_{yy}+u_{zz}+\frac{2\beta}{|y|}u_y+\frac{2\gamma}{z}u_z=0,\ \beta,\gamma\in(0,1/2). \tag{1}$$

Equation (1) is elliptic in $\Omega_+^\infty = \Omega^\infty \cap \{y > 0\}$ and hyperbolic in $\Omega_-^\infty = \Omega^\infty \cap \{y < 0\}$; the planes $y = 0$ (the type-change plane) and $z = 0$ are the singularity planes of the coefficients.

Throughout the paper, $\bar{\Omega}^\infty = \{(x,y,z): x \in [0,a], y \in [-b,+\infty), z \in [0,c]\}$, $\tau(x,z)$ – is a given sufficiently smooth function, and the following conditions are imposed:

$$u \in C\left(\bar{\Omega}^\infty\right) \cap C_{x,y,z}^{2,2,2}\left(\Omega_+^\infty \cup \Omega_-\right),\ u_x, z^{2\gamma} u_z \in C\left(\bar{\Omega}^\infty\right); \tag{2}$$

$$u(x,y,0) = 0,\ u(x,y,c) = 0,\ x \in [0,a],\ y \in [0,b]; \tag{3}$$

$$u(x,0,z) = \tau(x,z),\ \lim_{y \to +\infty} u(x,y,z) = 0,\ x \in [0,a],\ z \in [0,c]; \tag{4}$$

$$\lim_{y \to -0} (-y)^{2\beta} u_y(x,y,z) = \lim_{y \to +0} y^{2\beta} u_y(x,y,z),\ x \in (0,a), z \in (0,c). \tag{5}$$

Problem B_1^∞**.** Find a function $u(x,y,z)$ that satisfies equation (1) in $\Omega_+^\infty \cup \Omega_-$, conditions (2)-(5), and the condition

$$u(0,y,z) = u(a,y,z),\ u_x(0,y,z) = 0,\ y \in [0,b],\ z \in [0,c]. \tag{6}$$

Problem B_2^∞**.** Impose conditions (1)-(5), but replace (6) by $u(0,y,z) = u(a,y,z),\ u_x(a,y,z) = 0$.

Problem B_3^∞**.** Impose conditions (1)-(5), but replace (6) by $u(0,y,z) = 0,\ u_x(0,y,z) = u_x(a,y,z)$.

Problem B_4^∞**.** Impose conditions (1)-(5), but replace (6) by $u(a,y,z) = 0,\ u_x(0,y,z) = u_x(a,y,z)$.

The Fourier method, with $u(x,y,z) = X(x)Q(y)Z(z)$, reduces problems $B_k^\infty\left(k = \overline{1,4}\right)$, apart from conditions (4) and (5), to the spectral problem

$$Z''(z) + \frac{2\gamma}{z} Z'(z) + (\lambda - \mu) Z(z) = 0,\ Z(0) = 0,\ Z(c) = 0, \tag{7}$$

to the equation

$$\operatorname{sgn} y \cdot Q''(y) + \frac{2\beta}{|y|} Q'(y) - \lambda Q(y) = 0,\ y \in (-b,0) \cup (0,+\infty), \tag{8}$$

and to the one-dimensional spectral problem with respect to x

$$X''(x) + \mu X(x) = 0 \tag{9}$$

subject to the condition

$$X(0) = X(a), X'(0) = 0, \tag{9_1}$$

$$X(0) = X(a),\ X'(a) = 0, \tag{9_2}$$

$$X(0) = 0,\ X'(0) = X'(a), \tag{9_3}$$

$$X(a) = 0,\ X'(0) = X'(a), \tag{9_4}$$

where $\mu, \lambda \in R$ – are separation constants.

Problems (7) and (8) do not depend on the choice of the condition with respect to x and coincide completely with the corresponding problems in [1]. We reproduce the formulas needed below. The eigenvalues and eigenfunctions of problem (7) are

$$Z_m(z)=z^{1/2-\gamma}J_{1/2-\gamma}\left(\sigma_{\gamma m}z/c\right),\ m\in N, \tag{10}$$

where $\sigma_{\gamma m}$ is the m-th positive root of the equation $J_{1/2-\gamma}(z)=0$, and the eigenvalue of problems (7)-(8) corresponding to the pair (n,m) is $\lambda_{nm}=\mu_n+\left(\sigma_{\gamma m}/c\right)^2$,, where μ_n- are the eigenvalues of problem (9). The unique, up to a constant factor, solution of equation (8) that is bounded as $y\to\infty$, satisfies matching condition (5), and corresponds to $\lambda=\lambda_{nm}$ will be denoted by

$$Q(y,\lambda_{nm})=\begin{cases}y^{1/2-\beta}K_{1/2-\beta}\left(\sqrt{\lambda_{nm}}y\right), & y\ge 0,\\ (-y)^{1/2-\beta}\,\bar{Y}_{1/2-\beta}\left(\sqrt{\lambda_{nm}}(-y)\right), & y\le 0,\end{cases} \tag{11}$$

where $\bar{Y}_{1/2-\beta}(\xi)=\dfrac{\pi}{2\cos\beta\pi}\left[J_{1/2-\beta}(\xi)+J_{\beta-1/2}(\xi)\right]-$ is the same function as in [1, formulas (14) and (24)] (with the normalization factor equal to one). For every fixed y, function (11) is an entire analytic function of the parameter λ, because the identities ensuring that matching condition (5) is satisfied hold for every $\lambda>0$, not only for the eigenvalues λ_{nm}. This fact is crucially used in Section 3.2.

Analysis and results.

Spectral problems with respect to x: eigenfunctions and associated functions

Lemma 1. Let $k_n=2\pi n/a$. The spectrum of problem (9) subject to condition (9_k), $k=\overline{1,4}$ is purely discrete and is the same for all four problems: $\mu_n=(2\pi n/a)^2$, $n=0,1,2,....$

More precisely:

i) for problem $B_1^\infty: X_0(x)=1$, when $n\ge 1$, the eigenfunction is $X_n(x)=\cos k_n x$; in addition, there is an associated function $\tilde{X}_n(x)=x\sin k_n x$ satisfying the same homogeneous conditions (9_1) and the relation $\tilde{X}_n''(x)+\mu_n\tilde{X}_n(x)=2k_nX_n(x)$;

ii) for problem $B_2^\infty: X_0(x)=1;$, when $n\ge 1$, the eigenfunction is $X_n(x)=\cos k_n x,$; in addition, there is an associated function $\tilde{X}_n(x)=(x-a)\sin k_n x$, satisfying the same homogeneous conditions (9_2) and the relation $\tilde{X}_n''(x)+\mu_n\tilde{X}_n(x)=2k_nX_n(x)$;

iii) for problem $B_3^\infty:\ X_0(x)=x;$, when $n\ge 1$, the eigenfunction is $X_n(x)=\sin k_n x,$; in addition, there is an associated function $\tilde{X}_n(x)=x\cos k_n x$, satisfying the same homogeneous conditions (9_3) and the relation $\tilde{X}_n''(x)+\mu_n\tilde{X}_n(x)=-2k_nX_n(x)$;

iv) for problem $B_4^\infty:\ X_0(x)=x-a;$, when $n\ge 1$, the eigenfunction is $X_n(x)=\sin k_n x,$; in addition, there is an associated function $\tilde{X}_n(x)=(x-a)\cos k_n x$, satisfying the same homogeneous conditions (9_4) and the relation $\tilde{X}_n''(x)+\mu_n\tilde{X}_n(x)=-2k_nX_n(x)$;

For no n does the problem have any other linearly independent solutions besides those indicated above. In particular, when $n=0$, there is no associated function.

Proof. The general solution of equation (9) is (9_1). Substitution into conditions (9_1) yields $X'(0)=Bk=0\Rightarrow B=0,\ k\ne 0$ and $X(0)=X(a)\Rightarrow A(1-\cos ka)=0;$. The nontriviality condition gives $\cos ka=1,$, that is, $k=k_n, X_n(x)=\cos k_n x,\ n\ge 1$. When $k=0$, the solution $X(x)=A+Bx$ with $B=0$ yields $X_0=1$. Since, for $\mu=\mu_n, n\ge 1$, each of the two fundamental solutions $\cos k_n x,\sin k_n x$ separately

satisfies the condition $X(0)=X(a)$ (namely, $0=0$ and $\cos k_n a=1=\cos 0$, respectively), the dimension of the solution space of equation (9) subject only to this first condition increases abruptly from one to two precisely when $\mu=\mu_n$. This is the source of the associated function. We seek it in the form $\tilde{X}_n(x)=x\sin k_n x+A\cos k_n x+B\sin k_n x$. The particular solution $x\sin k_n x$ of the equation $\tilde{X}_n''(x)+\mu_n\tilde{X}_n(x)=2k_n\cos k_n x$ is verified by direct differentiation. The condition $\tilde{X}_n(0)=\tilde{X}_n(a)$ is satisfied identically in A,B (both sides equal A, since $\sin k_n a=0$), whereas the condition $\tilde{X}_n'(0)=0$ gives $B=0$. Setting, without loss of generality, $A=0$, we obtain $\tilde{X}_n(x)=x\sin k_n x$. Cases ii)-iv) are treated in exactly the same manner; for iii) and iv), a resonant particular solution of the equation $\tilde{X}_n''(x)+\mu_n\tilde{X}_n(x)=-2k_n\sin k_n x$ is taken in the form $x\cos k_n x$. Uniqueness, that is, the absence of other solutions, follows because, in constructing each associated function, the arbitrary constant A (respectively, B) can be eliminated by adding a multiple of $X_n(x)$, which does not produce a new linearly independent solution. The lemma is proved.

Lemma 2 (Biorthogonality). *The operators $-d^2/dx^2$ whose domains correspond to conditions (9_1) and (9_4) are mutually adjoint in the sense of LaGrange's identity; the same holds for the pair (9_2), (9_3). The corresponding systems are biorthogonal: for all $n,m\ge 1$*

$$\left(X_n^{(1)},X_m^{(4)}\right)=0,\ \left(X_n^{(1)},\tilde{X}_m^{(4)}\right)=-\left(a^2/4\right)\delta_{nm},\ \left(\tilde{X}_n^{(1)},X_m^{(4)}\right)=\left(a^2/4\right)\delta_{nm},\ \left(\tilde{X}_n^{(1)},\tilde{X}_m^{(4)}\right)=0, \qquad (12)$$

$$\left(X_n^{(2)},X_m^{(3)}\right)=0,\ \left(X_n^{(2)},\tilde{X}_m^{(3)}\right)=\left(a^2/4\right)\delta_{nm},\ \left(\tilde{X}_n^{(2)},X_m^{(3)}\right)=-\left(a^2/4\right)\delta_{nm},\ \left(\tilde{X}_n^{(2)},\tilde{X}_m^{(3)}\right)=0, \qquad (13)$$

and $\left(X_0^{(1)},\tilde{X}_0^{(4)}\right)=-a^2/2,\ \left(X_0^{(2)},X_0^{(3)}\right)=a^2/2$, *where* $(f,g)=\int_0^a f(x)g(x)dx$.

Proof. The adjointness of the conditions is verified by Lagrange's identity $\int_0^a\left(X''(x)Y(x)-X(x)Y''(x)\right)dx=\left[X'(x)Y(x)-X(x)Y'(x)\right]_0^a$:. If $X(x)$ satisfies (9_1), then the right-hand side vanishes for every $Y(x)$ satisfying the condition $Y(a)=0,\ Y'(0)=Y'(a)$, which is precisely condition (9_4); the pair (9_2), (9_3) is treated similarly. Equalities (12)-(13) and the values for $n=0$ are verified by direct integration by parts, using $\sin 2\pi n=0,\cos 2\pi n=1$. Because the calculations are lengthy, they are omitted; all integrals in (12)-(13) and the relations in Lemma 1 were additionally verified by the authors using a computer algebra system.

Remark 1. According to the general theory of nonlocal Ionkin-Moiseev spectral problems [9, 11], the biorthogonality in (12)-(13) of two complete systems implies that each system $\left\{X_n^{(k)},\tilde{X}_n^{(k)}\right\}_{n\ge 0},k=\overline{1,4}$, forms a Riesz basis in $L_2(0,a)$. It is obtained from the classical orthonormal basis $\{1,\cos k_n x,\sin k_n x\}$ by a linear transformation that preserves the structure of the eigenspaces: the eigenfunction remains unchanged, whereas the sine or cosine is replaced by an associated function. The inverse transformation is determined precisely by the adjoint system (12)-(13), and its boundedness follows from the fact that the coefficients in (12)-(13) are uniformly separated from zero with respect to n. In particular, each system is complete and minimal in $L_2(0,a)$; hence, every function $f\in L_2(0,a)$ has a unique expansion in that system, and the expansion coefficients are given by formulas (14) below.

Lemma 2 yields the inversion formulas for $f \in C^1[0,a]$, $f(x) = c_0 X_0(x) + \sum_{n=1}^{\infty}\left(c_n X_n(x) + \tilde{c}_n \tilde{X}_n(x)\right)$. For problem B_1^{∞} (the expansion of $\tau(\cdot, z)$ in the system $\left\{X_n^{(1)}(x), \tilde{X}_n^{(1)}(x)\right\}$, with the coefficients determined through the adjoint system $\left\{X_n^{(4)}(x), \tilde{X}_n^{(4)}(x)\right\}$):

$$c_0 = \frac{2}{a^2}\int_0^a f(x)(a-x)dx,\ c_n = \frac{4}{a^2}\int_0^a f(x)(a-x)\cos k_n x dx,\ \tilde{c}_n = \frac{4}{a^2}\int_0^a f(x)\sin k_n x dx. \tag{14}$$

For problem B_4^{∞} (with adjoint system $\left\{X_n^{(1)}, \tilde{X}_n^{(1)}\right\}$):

$$c_0 = \frac{2}{a^2}\int_0^a f(x)x dx,\ c_n = \frac{4}{a^2}\int_0^a f(x)\sin k_n x dx,\ \tilde{c}_n = -\frac{4}{a^2}\int_0^a f(x)(a-x)\cos k_n x dx. \tag{15}$$

For problems $B_2^{\infty}, B_3^{\infty}$, the formulas are obtained from (14) and (15) by replacing $(a-x) \to x$ with $x \to (a-x)$ and changing the sign for $\tilde{c}_n$, in accordance with (13).

Construction of the solution

We seek the solution of each problem B_k^{∞} in the form of the series

$$u(x,y,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} Z_m(z)\left[\omega_{nm}(y)X_n(x) + \tilde{\omega}_{nm}(y)\tilde{X}_n(x)\right] \quad \left(\tilde{X}_0(x) \equiv 0\right). \tag{16}$$

Substituting (16) into (1), using Lemma 1 (the relation $\tilde{X}_n''(x) + \mu_n \tilde{X}_n''(x) = \chi_n X_n(x)$, where $\chi_n = 2k_n$ for problems B_1^{∞}, B_2^{∞} and $\chi_n = -2k_n$ for problems $B_3^{\infty}, B_4^{\infty}$), and using equation (7) for Z_m, we conclude, by the linear independence of the system $\left\{X_n, \tilde{X}_n\right\}-$, that the coefficients of $X_n(x)Z_m(z)$ and $\tilde{X}_n(x)Z_m(z)$ must vanish separately:

$$\begin{cases} \operatorname{sgn} y \cdot \tilde{\omega}_{nm}''(y) + \dfrac{2\beta}{|y|}\tilde{\omega}_{nm}'(y) - \lambda_{nm}\tilde{\omega}_{nm}(y) = 0, \\ \operatorname{sgn} y \cdot \omega_{nm}''(y) + \dfrac{2\beta}{|y|}\omega_{nm}'(y) - \lambda_{nm}\omega_{nm}(y) = -\chi_n \tilde{\omega}_{nm}(y). \end{cases} \tag{17}$$

The first equation in (17) is homogeneous. By uniqueness, its solution that is bounded as $y \to +\infty$ and satisfies (5) is, up to a constant factor, $\tilde{\omega}_{nm}(y) = \tilde{b}_{nm} Q(y; \lambda_{nm})$, with the function defined by (11). The second equation in (17) is resonantly perturbed: its right-hand side is proportional to solution (11) of the corresponding homogeneous equation. Because the identities ensuring that function (11) satisfies matching condition (5) hold for every $\lambda > 0$ (see the remark following (11)), we differentiate equation (8), treating λ as a free parameter, with respect to λ. It follows that the function $\tilde{Q}(y;\lambda) = \dfrac{\partial}{\partial \lambda} Q(y;\lambda)$ satisfies the nonhomogeneous equation $\operatorname{sgn} y \cdot \tilde{Q}'' + \dfrac{2\beta}{|y|}\tilde{Q}' - \lambda \tilde{Q} = Q(y;\lambda)$ and, as the derivative of identity (5), which holds for every λ, also satisfies matching condition (5). Explicitly, the differentiation formula for the Macdonald function with respect to its argument gives, for $y \geq 0$

$$\tilde{Q}(y;\lambda) = \frac{y^{3/2-\beta}}{2\sqrt{\lambda}} K'_{1/2-\beta}\left(\sqrt{\lambda} y\right), \tag{18}$$

and an analogous formula holds for $y \le 0$, with $K_{1/2-\beta} \to \overline{Y}_{1/2-\beta}$ replaced accordingly. Hence, a particular solution of the second equation in (17) is $-\chi_n \tilde{b}_{nm} \tilde{Q}(y;\lambda_{nm})$, and the general solution that is bounded at infinity and satisfies (5) is

$$\omega_{nm}(y) = b_{nm} Q(y;\lambda_{nm}) - \chi_n \tilde{b}_{nm} \tilde{Q}(y;\lambda_{nm}). \tag{19}$$

As in [1], denote $\Delta_{nm}(b) = Q(-b;\lambda_{nm})$, and also set $\tilde{\Delta}_{nm}(b) = \tilde{Q}(-b;\lambda_{nm})$. Conditions $\tilde{\omega}_{nm}(-b) = \tilde{\tau}_{nm}$ and $\omega_{nm}(-b) = \tau_{nm}$ (where $\tau_{nm}, \tilde{\tau}_{nm}$ – are the Fourier coefficients of the data $\tau(x,z)$, defined by formulas analogous to (14)-(15), with additional integration with respect to z and weight $z^{2\gamma} Z_m(z)$, as in [1, formula (21)]) yield

$$\tilde{b}_{nm} = \frac{\tilde{\tau}_{nm}}{\Delta_{nm}(b)},\ b_{nm} = \frac{\tau_{nm}}{\Delta_{nm}(b)} + \chi_n \frac{\tilde{\Delta}_{nm}(b)}{\left(\Delta_{nm}(b)\right)^2} \tilde{\tau}_{nm}. \tag{20}$$

Formulas (16), (19), and (20), together with Lemma 1, give a formal solution of each problem B_j^{∞}, $j = \overline{1,4}$. When $n = 0$, the second terms in (16) and (19) are absent, and $\omega_{0m}(y) = \tau_{0m} Q(y;\lambda_{0m}) / \Delta_{0m}(b)$, exactly as in the periodic problem [1].

Uniqueness of the solution

Theorem 1. *If a solution of problem $B_k^{\infty}, k = \overline{1,4}$ exists, then it is unique.*

Proof. We give the proof for problem B_1^{∞}; for the other problems, the argument is identical after replacing the pair $\left(X^{(1)}, \tilde{X}^{(1)}\right) \leftrightarrow \left(X^{(4)}, \tilde{X}^{(4)}\right)$ by the corresponding pair. Let $\tau(x,z) \equiv 0$, and let $u(x,y,z)$ be a solution of the homogeneous problem. Introduce the projections onto the system $\left\{X_n^{(4)}, \tilde{X}_n^{(4)}\right\}$ adjoint, by Lemma 2, to the system of problem B_4^{∞}:

$$\begin{cases} \omega_{nm}(y) = d_m \int_0^c \int_0^a u(x,y,z) X_n^{(4)}(x) z^{2\gamma} Z_m(z)\, dx dz, \\ \tilde{\omega}_{nm}(y) = d_m \int_0^c \int_0^a u(x,y,z) \tilde{X}_n^{(4)}(x) z^{2\gamma} Z_m(z)\, dx dz, \end{cases}$$

$d_m = \|Z_m\|^{-2}_{L_{2,\rho}(0,c)}$, $\rho = z^{2\gamma}$. Integrating the term u_{xx} by parts twice and using the fact that $u(x,y,z)$ satisfies homogeneous condition (9_1), whereas $X_n^{(4)}(x), \tilde{X}_n^{(4)}(x)$ satisfies the adjoint homogeneous condition (9_4), we see from Lagrange's identity, as in Lemma 2, that the boundary terms vanish. Exactly as in [1, formulas (17)-(19)], it follows that $\tilde{\omega}_{nm}(y)$ satisfies homogeneous equation (8) with $\lambda = \lambda_{nm}$ and the conditions $\tilde{\omega}_{nm}(-b) = 0, \tilde{\omega}_{nm}(+\infty) = 0$. By uniqueness for this problem ([1, Theorem 3.1], which applies without modification because it does not depend on the condition with respect to x), we have $\tilde{\omega} \equiv 0$ for all n, m. Next, for ω_{nm}, the same calculation, taking into account the relation $\left(\tilde{X}_n^{(4)}\right)'' + \mu_n \tilde{X}_n^{(4)} = -2k_n X_n^{(4)}$, shows that ω_{nm} satisfies equation (8) with right-hand side $2k_n \tilde{\omega}_{nm}(y) \equiv 0$, and hence also the homogeneous equation with the same zero conditions; therefore, $\omega \equiv 0$. Since the system $\left\{X_n^{(4)}, \tilde{X}_n^{(4)}\right\}$ is complete in $L_2(0,a)$ (see the remark following Lemma 2), the relations $\omega_{nm} \equiv \tilde{\omega}_{nm} \equiv 0$ for all $n \ge 0, m \ge 1$, together with

the completeness of $\{Z_m\}$ in $L_{2,\rho}(0,c), \rho = z^{2\gamma}$ [1], imply that $u(x,y,z) \equiv 0$ in $\bar{\Omega}^{\infty}$. The theorem is proved.

Existence of the solution and small denominators

The denominators $\Delta_{nm}(b)$ in formulas (20) are exactly the same as those in problem A^{∞}, because they are determined solely by the spectrum $\lambda_{nm} = \mu_n + \left(\sigma_{\gamma m}/c\right)^2$, , which is identical for all four problems by Lemma 1. Therefore, the small-denominator lemma [1, Lemma 4.1] applies without change to lower bounds for $|\Delta_{nm}(b)|$: for every b, except for a countable set of resonant values, and for sufficiently large n,m, $|\Delta_{nm}(b)| \geq C_0 > 0$. For $\tilde{\Delta}_{nm}(b) = \tilde{Q}(-b;\lambda_{nm})$, representation (18) and the asymptotic formulas $K_{\nu}(\xi) \approx \sqrt{\pi/(2\xi)}e^{-\xi}$, $K'_{\nu}(\xi) \approx -\sqrt{\pi/(2\xi)}e^{-\xi}$ as $\xi \to \infty$ ([1, formula (22)], [19]) give an estimate of the same order of growth as that for $\Delta_{nm}(b)$, up to a factor $O\left(\sqrt{\lambda_{nm}}b\right)$. Together with (20), this yields the estimate

$$|b_{nm}| \leq C\left(|\tau_{nm}| + n|\tilde{\tau}_{nm}|\right)/|\Delta_{nm}(b)|, \quad |\tilde{b}_{nm}| \leq |\tilde{\tau}_{nm}|/|\Delta_{nm}(b)|,$$

Combining this estimate with the estimates for $X_n, \tilde{X}_n$ (which grow polynomially in n, by Lemma 1), for Z_m ([1, Lemma 4.3]), and for the coefficients $\tau_{nm}, \tilde{\tau}_{nm}$ obtained by integrating formulas (14)-(15) and (21) in [1] by parts four times, using the sufficient smoothness of $\tau(x,z)$ and the compatibility of its derivatives with the boundary conditions in the variable x on the boundary of the rectangle $[0,a]\times[0,c]$ (the analogue of [1, Lemma 4.7]), shows that series (16) and the series of all its partial derivatives up to and including second order are majorized by a convergent numerical series. This proves the following assertion.

Theorem 2. *Suppose that b is not a resonant value from the small-denominator lemma, and that the function $\tau(x,z)$ satisfies smoothness and compatibility conditions analogous to those in [1, Lemma 4.7], with the periodicity conditions replaced by the conditions with respect to x. Let $n_0, m_0 \in N$ be thresholds such that $|\Delta_{nm}(b)| \geq C_0$ for all indices beyond them. Then series (16), with coefficients (19)-(20), converges in $\bar{\Omega}^{\infty}$ uniformly together with all partial derivatives up to and including second order that occur in the definition of class (2), and its sum $u(x,y,z)$ is the unique regular solution of problem $B_k^{\infty}, k = \overline{1,4}$.*

Conclusions and recommendations.

Four nonlocal boundary value problems of Samarskii-Ionkin type are studied for a three-dimensional mixed-type equation with two singular Bessel coefficients in an unbounded parallelepiped; these problems generalize the periodic problem previously considered in [1]. It is shown that the non-self-adjointness of the matching conditions with respect to x gives rise to associated functions and to a resonantly perturbed system of equations for the Fourier coefficients in the variable y. An exact method for solving this system by differentiating the Macdonald function with respect to the spectral parameter is proposed; the method is also applicable to other nonlocal problems for mixed-type equations with singular coefficients. Uniqueness and existence theorems for a regular solution are proved. The results can be used to study analogous nonlocal problems with several type-change and degeneracy planes and to construct numerical methods for the corresponding non-self-adjoint spectral problems.

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