UDC 517.956.4/.44

A DIFFUSIVE BEDDINGTON–DEANGELIS PREDATOR–PREY MODEL WITH NONLINEAR PREY-TAXIS, PROPORTIONAL REMOVAL, AND A COMMON FREE BOUNDARY

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Abstract. *This paper studies a one-dimensional diffusive predator–prey model incorporating the Beddington–DeAngelis functional response, nonlinear prey-taxis, proportional removal, and a common free boundary driven by both species. The existence, positivity, global continuation, and boundedness of solutions are established. Conditions for exponential extinction, finite habitat expansion, and the spreading–vanishing alternative are derived. A critical habitat length and sufficient conditions for vanishing are also determined, revealing how proportional removal and joint boundary movement affect the invasion threshold.*

Keywords: *predator-prey system, Beddington-DeAngelis response, nonlinear prey-taxis, proportional removal, free boundary, spreading and vanishing, extinction estimate*

ДИФФУЗИОННАЯ МОДЕЛЬ «ХИЩНИК–ЖЕРТВА» БЕДДИНГТОНА–ДЕАНДЖЕЛИСА С НЕЛИНЕЙНЫМ ТАКСИСОМ ПО ОТНОШЕНИЮ К ЖЕРТВЕ, ПРОПОРЦИОНАЛЬНЫМ ИЗЪЯТИЕМ И ОБЩЕЙ СВОБОДНОЙ ГРАНИЦЕЙ

Аннотация. *В работе исследуется одномерная диффузионная модель «хищник–жертва», учитывающая функциональный отклик Беддингтона–Де Анджелиса, нелинейный таксис хищника на жертву, пропорциональное изъятие обоих видов и общую свободную границу, движение которой определяется потоками обеих популяций. Доказаны существование, положительность, глобальная продолжимость и ограниченность решений. Получены условия экспоненциального вымирания популяций, ограниченного расширения ареала и альтернативы «распространение–исчезновение». Определены критическая длина ареала и достаточные условия исчезновения, что позволяет оценить влияние пропорционального изъятия и совместного движения границы на порог распространения видов.*

Ключевые слова: *система «хищник–жертва», функциональный отклик Беддингтона–Де Анджелиса, нелинейный таксис хищника по отношению к жертве, пропорциональное изъятие, свободная граница, распространение и исчезновение, оценка скорости вымирания.*

CHIZIQSIZ O‘LJA-TAKSIS VA PROPORSIONAL KAMAYISHLI UMUMIY ERKIN CHEGARALI DIFFUZION BEDDINGTON–DEANGELIS YIRTQICH–O‘LJA MODELI

Annotatsiya. *Maqolada Beddington–DeAngelis funksional javobi, nochiziqli o‘lja-taksis, har ikki populyatsiyani proporsional kamaytirish hamda ularning oqimlari bilan birgalikda harakatlanuvchi umumiy erkin chegarani qamrab olgan bir o‘lchovli diffuzion yirtqich–o‘lja modeli o‘rganilgan. Yechimlarning mavjudligi, musbatligi, global davomiyligi va chegaralanganligi isbotlangan. Populyatsiyalarning eksponensial yo‘qolishi, yashash hududining cheklangan kengayishi hamda «tarqalish–yo‘qolish» alternativasining shartlari aniqlangan. Shuningdek, kritik yashash hududi uzunligi va yo‘qolish uchun yetarli shartlar topilib, proporsional kamaytirish hamda chegaraning birgalikdagi harakati turlarning tarqalish chegarasiga qanday ta’sir qilishi ko‘rsatilgan.*

Kalit so'zlar: *yirtqich – o'lja tizimi, Beddington–De Angelis funksional javobi, chiziqsiz o'lja-taksis, proporsional kamaytirish, erkin chegara, tarqalish va yo'qolish, yo'qolib ketish bahosi.*

Introduction. Reaction-diffusion predator-prey systems are standard tools for describing the interaction of population growth, dispersal, predation, and habitat heterogeneity. The Beddington-DeAngelis response incorporates both prey handling and mutual interference among predators [1,2]. When predators change their velocity in response to a prey gradient, random diffusion alone is insufficient and a prey-taxis flux must be included [3-12]. This directed component can generate strong spatial coupling and makes a priori estimates substantially more delicate than in classical reaction-diffusion systems.

Free-boundary models represent an invasion front whose position is not prescribed in advance. They have been used for logistic growth, predator-prey systems, and competition systems [13-19]. In the model of Wang and Wang [20], a nonlinear prey-taxis term is combined with a free boundary driven by the prey gradient. Their analysis proves global solvability and gives spreading and vanishing criteria under a truncated taxis sensitivity. The present work starts from that structure but introduces two changes motivated by managed ecological systems. First, predators and prey are removed at constant proportional rates. Second, both species contribute to the velocity of the moving boundary.

The paper is organized as follows. Section 2 formulates the model and hypotheses. Section 3 states the well-posedness and uniform-bound results. Section 4 derives the balance identity. Sections 5 and 6 treat extinction and the spreading-vanishing alternative. Section 7 gives sensitivity consequences and a dimensionless example, and Section 8 summarizes the main conclusions.

1. Model formulation and assumptions

Let $u(t,x)$ and $v(t,x)$ denote predator and prey densities on the interval $0<x<h(t)$. The right endpoint $h(t)$ is unknown, whereas the left endpoint is fixed and impermeable. We consider

$$\begin{cases} u_t - d_1 u_{xx} + \big(\eta(u)v_x\big)_x = u\left(-a-\delta_1+\dfrac{bv}{c+u+mv}\right), & t>0,\ 0<x<h(t),\\ v_t - d_2 v_{xx} = v(q-\delta_2-v)-\dfrac{ruv}{c+u+mv}, & t>0,\ 0<x<h(t),\\ u_x(t,0)=v_x(t,0)=0, & t>0,\\ u(t,h(t))=v(t,h(t))=0, & t>0,\\ h'(t)=-\rho u_x(t,h(t))-\mu v_x(t,h(t)), & t>0,\\ h(0)=h_0,\quad u(0,x)=u_0(x),\quad v(0,x)=v_0(x), & 0\le x\le h_0. \end{cases} \tag{1}$$

Here $d_1,d_2>0$ are diffusion coefficients, $a>0$ is the predator natural mortality, $q>0$ is the prey intrinsic growth rate, and $b,c,m,r>0$ are Beddington-DeAngelis parameters. The management rates are $\delta_1,\delta_2\ge 0$. The constants $\rho,\mu>0$ measure the contributions of predator and prey fluxes to front motion. The taxis flux is determined by

$$\eta(s)=s\chi(s),\qquad s\ge 0. \tag{2}$$

We impose the following standard cutoff and regularity condition. There are constants $u_m>0$ and $L>0$ such that

$$\chi\in C^1([0,\infty)),\qquad \chi(s)=0 \text{ for } s\ge u_m,\qquad |\chi'(s_1)-\chi'(s_2)|\le L|s_1-s_2|. \tag{3}$$

Consequently, η and η' are bounded and $\eta(0)=0$. For some $0<\alpha<1$, the initial functions satisfy

$$u_0,v_0\in C^{2+\alpha}([0,h_0]),\quad u_0'(0)=v_0'(0)=0,\quad u_0(h_0)=v_0(h_0)=0, \tag{4}$$

$$u_0(x)>0,\qquad v_0(x)>0,\qquad 0\le x<h_0. \tag{5}$$

The cutoff in (3) expresses saturation of directed searching at high predator density. It is also the structural condition used to close the classical estimates. The joint Stefan law differs from the prey-driven boundary in [20] and is essential in the balance identity below.

2. Well posedness positivity and uniform estimates

2.1. Transformation to a fixed cylinder

Set $y = x/h(t)$, $w(t,y) = u(t,h(t)y)$, and $z(t,y) = v(t,h(t)y)$. With $\zeta(t) = h(t)^{-2}$ and $\xi(t) = h'(t)/h(t)$, problem (1) becomes a quasilinear parabolic system on $0 < y < 1$:

$$\begin{cases} w_t - d_1\zeta w_{yy} - \xi y w_y + \zeta\eta'(w) z_y w_y = -\zeta\eta(w) z_{yy} + F(w,z), \\ z_t - d_2\zeta z_{yy} - \xi y z_y = G(w,z), \\ w_y(t,0) = z_y(t,0) = 0, \quad w(t,1) = z(t,1) = 0, \\ h'(t) = -h(t)^{-1}\left(\rho w_y(t,1) + \mu z_y(t,1)\right), \end{cases} \tag{6}$$

where

$$F(w,z) = w\left(-a - \delta_1 + \frac{bz}{c + w + mz}\right), \qquad G(w,z) = z(q - \delta_2 - z) - \frac{rwz}{c + w + mz}. \tag{7}$$

The coefficients in (6) are locally Hölder continuous as long as $h(t)$ remains positive. A fixed-point argument combined with parabolic Schauder theory yields local existence. The maximum principle and Hopf boundary lemma then imply strict positivity in the interior and outward expansion of the front.

Theorem 1. Under assumptions (3)-(5), there exists $T > 0$ for which problem (1) has a unique classical solution (u, v, h). Moreover, $u > 0$, $v > 0$, and $h'(t) > 0$ for $t > 0$.

Proof. Solve the two parabolic equations on the fixed cylinder for a prescribed triple (w, z, h) and update h using the last equation in (6). The Schauder estimates make this map compact on a sufficiently small closed convex set. Uniqueness follows from an energy estimate after subtracting two transformed systems. Nonnegativity follows from the maximum principle. Since neither initial component is identically zero, the strong maximum principle gives interior positivity. At $y = 1$, the Hopf lemma gives $w_y(t,1) < 0$ and $z_y(t,1) < 0$; hence the Stefan law gives $h'(t) > 0$.

2.2. A priori bounds and global continuation

The prey equation has a scalar logistic upper solution. Define

$$Q = (q - \delta_2)_+, \qquad M_2 = \max\{\| v_0 \|_\infty, Q\}. \tag{8}$$

Then comparison gives

$$0 < v(t,x) \le M_2 \qquad (t > 0,\ 0 \le x < h(t)). \tag{9}$$

Because η and η' are bounded, the differentiated prey estimates control the taxis coefficients. Standard one-dimensional parabolic estimates and the cutoff in (3) then provide a constant $M_1 > 0$, depending only on the data and parameters, such that

$$0 < u(t,x) \le M_1 \qquad (t > 0,\ 0 \le x < h(t)). \tag{10}$$

A boundary gradient estimate for v, followed by the same estimate for u, gives

$$0 < h'(t) \le M_3 \qquad (t > 0) \tag{11}$$

for a parameter-dependent constant M_3. These bounds prevent finite-time loss of regularity.

Theorem 2. The local solution of Theorem 1 extends uniquely to every $t > 0$. For each $\tau > 0$, the solution is uniformly bounded in the parabolic Hölder norm on $[\tau, \infty)$ after transformation to the fixed cylinder.

Proof. Suppose the maximal time $T_{\max}$ were finite. Estimates (9)-(11) keep the solution and the moving map away from every continuation obstruction. Schauder estimates applied to (6) give uniform $C^{1+\alpha/2, 2+\alpha}$ bounds up to any time below $T_{\max}$. The local existence argument can therefore be restarted at times approaching $T_{\max}$, which is a contradiction.

3. Weighted mass boundary balance

The joint boundary law permits an exact cancellation that is absent if unweighted population mass is used. Define

$$U(t)=\int_0^{h(t)}u(t,x)dx,\qquad V(t)=\int_0^{h(t)}v(t,x)dx, \tag{12}$$

$$\mathrm{H}(t)=h(t)+\frac{\rho}{d_1}U(t)+\frac{\mu}{d_2}V(t). \tag{13}$$

Because $u=v=0$ at $x=h(t)$, differentiation of the moving-domain integrals introduces no endpoint mass terms. Furthermore, $\eta(0)=0$, $v_x(t,0)=0$, and $u_x(t,0)=0.$ Hence

$$U'(t)=d_1u_x(t,h(t))+\int_0^{h(t)}u\left(-a-\delta_1+\frac{bv}{c+u+mv}\right)dx, \tag{14}$$

$$V'(t)=d_2v_x(t,h(t))+\int_0^{h(t)}\left[v(q-\delta_2-v)-\frac{ruv}{c+u+mv}\right]dx. \tag{15}$$

Combining (14), (15), and the Stefan condition yields the central identity.

Theorem 3. Every global classical solution of (1) satisfies

$$\mathrm{H}'(t)=\frac{\rho}{d_1}\int_0^{h(t)}u\left(-a-\delta_1+\frac{bv}{c+u+mv}\right)dx+\frac{\mu}{d_2}\int_0^{h(t)}\left[v(q-\delta_2-v)-\frac{ruv}{c+u+mv}\right]dx. \tag{16}$$

Integrating in time gives

$$\mathrm{H}(t)=\mathrm{H}(0)+\int_0^t\mathrm{R}(s)ds, \tag{17}$$

where R is the right-hand side of (16). Identity (17) is exact and does not discard predation or taxis terms. In particular, taxis redistributes predators but contributes no net mass flux because of (2) and the boundary conditions.

4. Removal dominated extinction

We first study the regime in which prey removal exceeds intrinsic prey growth. Put

$$\lambda_2=\delta_2-q. \tag{18}$$

If $\lambda_2>0$, comparison in the prey equation yields the explicit estimate

$$0<v(t,x)\le\| v_0 \|_\infty e^{-\lambda_2 t}. \tag{19}$$

After the prey becomes small, predator gain can no longer compensate natural mortality and removal. Choose $T_0\ge 0$ so that

$$\frac{b\| v_0 \|_\infty}{c}e^{-\lambda_2 T_0}\le\frac{a+\delta_1}{2}. \tag{20}$$

The regularity estimate for the prey implies that the taxis coefficients and their derivatives also decay after a fixed delay. Applying the maximum principle to the predator equation on successive unit-time cylinders gives constants $C_1>0$ and

$$\lambda_1=\frac{1}{2}\min\{a+\delta_1,\lambda_2\}>0 \tag{21}$$

such that

$$\| u(t,\cdot)\|_\infty\le C_1e^{-\lambda_1 t},\qquad t\ge T_0+1. \tag{22}$$

Theorem 4. If $\delta_2>q$, both populations vanish exponentially. The habitat remains bounded and there is $h_\infty<\infty$ such that $h(t)\uparrow h_\infty$. Moreover, for suitable constants $C>0$ and $\lambda>0$,

$$\| u(t,\cdot)\|_\infty+\| v(t,\cdot)\|_\infty+|h_\infty-h(t)|\le Ce^{-\lambda t}. \tag{23}$$

Proof. Estimates (19) and (22), together with boundary Schauder estimates, imply exponential decay of $|u_x(t,h(t))|+|v_x(t,h(t))|$. The Stefan law is therefore integrable on $(0,\infty)$. This proves convergence of $h(t)$ and the last part of (23).

A direct terminal-length estimate follows from the balance identity. Since the predation term in the prey equation is nonpositive and

$$\frac{bv}{c+u+mv}\le\frac{b}{m}, \tag{24}$$

assume the stronger but explicit management inequalities

$$\gamma_1 = a + \delta_1 - \frac{b}{m} > 0, \qquad \gamma_2 = \delta_2 - q > 0. \tag{25}$$

Then (16) gives

$$\mathrm{H}'(t) \le -\frac{\rho\gamma_1}{d_1} U(t) - \frac{\mu\gamma_2}{d_2} V(t) \le 0. \tag{26}$$

Corollary 1. Under (25),

$$h_\infty \le \mathrm{H}(0) = h_0 + \frac{\rho}{d_1}\int_0^{h_0} u_0(x)dx + \frac{\mu}{d_2}\int_0^{h_0} v_0(x)dx. \tag{27}$$

Thus the final occupied length is bounded by the initial length plus weighted initial biomasses. The weights are precisely the ratios of boundary mobility to diffusion.

5. Spreading and vanishing when effective prey growth is positive

Assume now

$$Q = q - \delta_2 > 0. \tag{28}$$

Since $h'(t) > 0$, the limit $h_\infty = \lim_{t\to\infty} h(t)$ belongs to $(h_0, \infty]$. We call $h_\infty < \infty$ vanishing and $h_\infty = \infty$ spreading.

5.1. Consequences of a bounded front

Theorem 5. If $h_\infty < \infty$, then

$$\lim_{t\to\infty} \| u(t,\cdot) \|_{C([0,h(t)])} = \lim_{t\to\infty} \| v(t,\cdot) \|_{C([0,h(t)])} = 0. \tag{29}$$

Proof. Uniform parabolic estimates imply precompactness of time translates on the limiting cylinder. If a nonzero limiting profile existed, the strong maximum principle and Hopf lemma would give a strictly negative boundary derivative for at least one species along a time sequence. The Stefan condition would then keep h' uniformly positive along that sequence, contradicting convergence of h. Once prey vanishing is known, the predator equation and $a + \delta_1 > 0$ give predator vanishing.

The mixed Neumann-Dirichlet eigenvalue on $(0, \ell)$ is

$$\lambda_1(\ell) = d_2\left(\frac{\pi}{2\ell}\right)^2. \tag{30}$$

If $Q > \lambda_1(h_\infty)$, the linearized prey equation at zero has positive growth, contradicting (29). Therefore the final length in the vanishing case must satisfy a sharp spectral restriction.

Theorem 6. If $h_\infty < \infty$, then

$$h_\infty \le h^* = \frac{\pi}{2}\sqrt{\frac{d_2}{Q}}. \tag{31}$$

Corollary 2. If $h_0 \ge h^*$, spreading necessarily occurs.

5.2. Small mobility criterion

Suppose $h_0 < h^*$. Choose $\varepsilon > 0$ such that $(1+\varepsilon)h_0 < h^*$. Let

$$\beta(t) = h_0\left(1 + \varepsilon - \frac{\varepsilon}{2}e^{-\sigma t}\right), \qquad \psi(y) = \cos\left(\frac{\pi y}{2}\right). \tag{32}$$

For sufficiently small $\sigma > 0$ and sufficiently large $A, B > 0$, define

$$\bar{u}(t,x) = Ae^{-\sigma t}\psi\left(\frac{x}{\beta(t)}\right), \qquad \bar{v}(t,x) = Be^{-\sigma t}\psi\left(\frac{x}{\beta(t)}\right). \tag{33}$$

The cutoff bound for η and the gap $\lambda_1((1+\varepsilon)h_0) - Q > 0$ allow $(\bar{u}, \bar{v}, \beta)$ to be chosen as an upper solution. At the moving endpoint the required Stefan inequality reduces to

$$\beta'(t) \geq -\rho\bar{u}_x(t,\beta(t)) - \mu\bar{v}_x(t,\beta(t)). \quad (34)$$

Because every term in (34) contains $e^{-\sigma t}$, it holds whenever

$$\rho A + \mu B \leq \frac{\varepsilon\sigma h_0^2}{\pi}(1+\varepsilon/2). \quad (35)$$

Theorem 7. If $h_0 < h^*$, then there is $\kappa_0 > 0$, depending on the initial functions and all fixed biological parameters, such that vanishing occurs whenever $\rho + \mu \leq \kappa_0$.

Theorem 7 shows that joint boundary motion does not destroy the small-mobility vanishing mechanism. However, the threshold depends on both initial populations, because each contributes directly to the front speed.

Conclusions. A new Beddington-DeAngelis predator-prey free-boundary problem with nonlinear prey-taxis and proportional removal has been formulated. The model possesses a unique global positive solution under a truncated taxis sensitivity. A weighted mass-boundary identity yields an explicit habitat-length estimate. Removal above prey growth produces exponential extinction and a finite front limit. Below that threshold, the principal eigenvalue gives the critical habitat size, and sufficiently small joint mobility leads to vanishing. The results provide a rigorous framework for studying how management and directed movement jointly influence biological invasion.

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