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ON THE BEHAVIOR OF UNBOUNDED SOLUTIONS TO NONLINEAR NON-DIVERGENCE PARABOLIC EQUATIONS

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Abstract. *In this article, the qualitative properties of nonlinear mathematical models of diffusion processes described by parabolic and non-divergent equations are studied, as well as their practical applications. In particular, the asymptotic behavior of solution for non-divergent diffusion equations with sources or absorption is analyzed, estimates of the solutions are obtained, and software tools facilitating the investigation of these nonlinear models are developed.*

Keywords: *Mathematical modeling, self-similar solution, parabolic equation, non-divergence form, subsolution, supersolution.*

NODIVERGENT KO'RINISHDAGI NOCHIZIQLI PARABOLIK TENGLAMALAR CHEGARALANMAGAN YECHIMLARINI TAHLIL QILISH

Annotatsiya. *Ushbu maqolada parabolik va nodivergent tenglamalar orqali tavsiflanadigan nochiziqli matematik diffuziya modellarining sifat xossalari hamda ularning amaliy qo'llanilishi tadqiq etiladi. Xususan, manbali yoki yutilishli nodivergent diffuziya tenglamalari yechimlarining asimptotik xatti-harakati tahlil qilinadi, yechimlar uchun baholar olinadi hamda mazkur nochiziqli modellarni tadqiq etishni yengillashtiruvchi dasturiy vositalar ishlab chiqiladi.*

Kalit so'zlar: *matematik modellashtirish, o'z-o'ziga o'xshash yechim, parabolik tenglama, nodivergent ko'rinish, quyi yechim, yuqori yechim.*

О ПОВЕДЕНИИ НЕОГРАНИЧЕННЫХ РЕШЕНИЙ НЕЛИНЕЙНЫХ НЕДИВЕРГЕНТНЫХ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ

Аннотация. *В данной статье исследуются качественные свойства нелинейных математических моделей диффузионных процессов, описываемых параболическими и недивергентными уравнениями, а также рассматриваются их практические приложения. В частности, анализируется асимптотическое поведение решений недивергентных уравнений диффузии с источниками или поглощением, получены оценки решений и разработаны программные средства, облегчающие исследование данных нелинейных моделей.*

Ключевые слова: *математическое моделирование, самоподобное решение, параболическое уравнение, недивергентная форма, нижнее решение, верхнее решение.*

Introduction. In this work, in the domain

$$Q = (0,T) \times R$$

we consider the Cauchy problem of nonlinear parabolic equation not in divergence form:

$$-\frac{\partial u}{\partial t} + u^{\alpha} \frac{\partial}{\partial x}\left(\left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x} \right) + \lambda u^{q-1} = 0, \tag{1}$$

$$u\big|_{t=o} = u_0(x) \geq 0, \quad x \in R, \tag{2}$$

where $\lambda = 1$, $p > 2$, $q > 2$, $\alpha > 2$ –numerical parameters, $u = u(t,x) \geq 0$ the sought solutions of the problem. Some qualitative properties of a p-Laplacian equation in non-divergent form were studied by

Raimbekov [1] and [2]. Asymptotic solution of the Cauchy problem was also analysed at $t \to \infty$ for slow and fast diffusion cases. Jin and Yin [3] considered the doubly degenerate diffusion Cauchy problem

$$u_t = u^m div\left(\left|\nabla u\right|^{p-2}\nabla u\right) + \lambda u^q,\ \ (x,t) \in \Omega \times (0,T),$$

$$u = 0,\ (x,t) \in \partial\Omega \times (0,T),\ \ u(x,0) \in u_0(x),\ x \in \Omega,$$

where $m > 1,\ p > 1.$

The critical exponent $q_c = p + m - 1$ was obtained, namely, the solutions are global if $q < q_c$, and there exist both global and blow-up solutions if $q > q_c$. In the critical case $q = q_c$ solutions are dictated by the size of the domain. In [4], Zhou and Zheng investigate the positive solutions to the degenerate parabolic equation in the non-divergence form

$$u_t = u^p \nabla u + u^q \left|\nabla u\right|^2 - u^r$$

with zero Dirichlet boundary condition. The existence of global solutions and the large time needed for solving them is studied. The main effort goes into obtaining uniform asymptotic profiles for the decay solutions, under various dominations of the nonlinear diffusion or absorption. It is shown that the large time property of the solution behaves just like

$$\left(1 + (r-1)t\right)(-1/r - 1)$$

if the decay is governed by the nonlinear absorption with

$$1 < r < p + 1.$$

Porous medium equations with local sources or with non-local sources subject to non-local boundary conditions were studied; (see [5], [6] and [7]). The conditions of existence and blow-up are discussed. In [8], Sun et al. study the non-existence and long-time behavior of a weak solution for the degenerate parabolic equation

$$\partial_t u^n = u^m div\left(\left|\nabla u^m\right|^{p-2}\nabla u^m\right) + \gamma\left|\nabla u^m\right|^p + u^n$$

with zero boundary condition. The blow-up time is detected when the blow-up does occur. Zhou et al. [9] investigate the positive solutions to the doubly degenerate parabolic equation in the non-divergence form with the gradient term

$$u_t = u^m div\left(\left|\nabla u\right|^{p-2}\nabla u\right) + \gamma u^r \left|\nabla u\right|^p + \beta u^q$$

and with zero Dirichlet boundary condition. Zhou et al. first established the local existence of weak solutions to the problem, and then determined in what way the gradient term affects the behavior of the solutions. Wang [10] studies the non-existence and the behavior of the non-negative solution to the nonlinear degenerate parabolic equation with the Dirichlet boundary condition and the initial condition

$$u_t = u^m div\left(\left|\nabla u\right|^{p-2}\nabla u\right) + \gamma u^r \left|\nabla u\right|^p + \beta u^q,$$

where $m \in [1;2),\ p > 1,\ \gamma > 0$. The above equation, as a typical divergent equation, is known as a fast diffusive one for $0 < n < 1$, where

$$n = \frac{1}{1-m},\ m < 0 \text{ and } n > 1,$$

which corresponds to $0 < m < 1$.

When $1 < m < 2$, the above equation is a non-divergent one, which has been proposed as a mathematical model of physical problems in many fields such as the resistive diffusion of a force-free magnetic field, dynamics of biological populations, etc. Many authors studied the above equation with $p = 2$ (see, for example, [11] and references therein). Recently, the authors [12] have discussed the case $p > 2,\ m = 1,\ \gamma = 0$ and have constructed different solutions. In [13], the non-existence and the extended time behavior of the solutions for $m = 1$ are studied. In the general case, the authors [12] have discussed the existence of weak solutions to the initial boundary value problems. However they did not study the non-existence and the extended time behavior of the solution. The global solvability of the Cauchy problem for nonlinear parabolic system not in divergence form with a variable density and a source and the asymptotic self-similar solution are studied in many references.

The self-similar analysis of the Nonlinear Equations. We have established the existence and uniqueness off with compact carriers, which imply that the self-similar solution is narrowed. Based on this, we also established the convergent velocity of these solutions at the carrier boundary. To construct a self-similar

system (1), a nonlinear splitting algorithm is proposed, for which the solution of system (1) is sought in the form

$$u(x,t)=\bar{u}(t)\omega(x,\tau(t)). \tag{3}$$

Let $u(x,t)=\bar{u}(t)$ be in the form:

$$\bar{u}(t)=A(T-t)^{\gamma},$$

where

$$\gamma=\frac{1}{2-q} \quad \text{and} \quad A=\left(\frac{1}{q-2}\right)^{\frac{1}{q-2}}.$$

Using (3) the system (1) can be reduced to the following form:

$$\frac{\partial\omega}{\partial\tau}=\omega^{\alpha}\frac{\partial}{\partial x}\left(\left|\frac{\partial\omega}{\partial x}\right|^{p-2}\frac{\partial\omega}{\partial x}\right)+\frac{1}{\alpha+p-q}\tau^{-1}(\omega^{q-1}-\omega), \tag{4}$$

where

$$\tau(t)=-A^{\alpha+p-2}\frac{(T-t)^{\gamma(\alpha+p-2)+1}}{\gamma(\alpha+p-2)+1}.$$

In the future, the equation (1) is investigated when the condition is met. We consider the problem for the case $\alpha+p-q>0,\ q>2,\ p>2$. We seek the self-similar solution of equation (4) in the following form:

$$\omega(x,\tau)=f(\xi) \quad \text{and} \quad \xi=\frac{x}{\tau^{\beta}} \tag{5}$$

By substituting (5) into equation (4), we obtain the following self-similar solution:

$$f^{\alpha}\frac{d}{d\xi}\left(\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right)+\frac{1}{\alpha+p-q}(f^{q-1}-f)+\frac{1}{p}\xi\frac{df}{d\xi}=0. \tag{6}$$

A subsolution of the problem. Suppose that of equation (6) the following conditions are fulfilled:

$$Lu_{-}(t,x)\geq 0. \tag{7}$$

Then the function

$$f(\xi)=B(a+\xi^{n})^{m}, \tag{8}$$

where

$$a>0,\ B=\left(\frac{(\alpha+p-2)^{p-1}}{p^{p-1}(\alpha+p-q)}\right)^{\frac{1}{\alpha+p-2}}$$

satisfy the equation (6). Then, the subsolution theorem holds.

Theorem 1. Let $q\geq\alpha$ and $u_{-}(0,x)\leq u(0,x),\ x\in R$. Then, for problem (1)–(2) , a global solution exists in Q and the following estimate is satisfied:

$$u_{-}(0,x)\leq u(0,x),\ x\in R,$$

where $u_{-}(t,x)=\bar{u}(t)\bar{f}(\xi)$.

Proof. To prove the theorem we use the method of comparison of solution. Hence, comparing solution methods [9] it is taken the function $u_{-}(t,x)$. Substituting (8) in (1) the following inequality can be obtained:

$$f^{\alpha}\frac{d}{d\xi}\left(\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right)+\frac{1}{\alpha+p-q}(f^{q-1}-f)+\frac{1}{p}\xi\frac{df}{d\xi}\geq 0. \tag{9}$$

If the specific form (7) is given for the function $\bar{f}(\xi)$ inequality (9) can be rewritten as follow:

$$\left(B^{\alpha+p-2}n^{p-1}(m)^{p-1}-\frac{1}{\alpha+p-q}\right)+\frac{1}{\alpha+p-q}B^{q-2}(a+\xi^{n})^{\frac{pq-q-p+\alpha}{\alpha+p-2}}\geq 0. \tag{10}$$

It is known that $\overline{f}(\xi)$ positive function and $(q-\alpha)(p-1)\geq 0$. Then, according to the theorem, for the comparison of the solutions to problem. (1)–(2), there exists a global solution in Q and the following estimate is satisfied:

$$u_{-}(0,x)\leq u(0,x).$$

The proof of the theorem is completed.

A supersolution of the problem. Suppose that of equation (6) the following condition is fulfilled:

$$Lu_{-}(t,x)\leq 0.$$

Then, the function $f(\xi)=B(a-\xi^{n})^{m}$ satisfy the equation (6), where $a>0$, and

$$B=\left(\frac{(\alpha+p-2)^{p}}{p^{p-1}(\alpha-2)(1-p)(q-\alpha-p)}\right)^{\frac{1}{\alpha+p-2}}.$$

Under these conditions, the supersolution theorem holds.

Theorem 2. Let the following conditions be satisfied:

a) $q>3$ and $1<\alpha<2$;

b) $q\leq 3$ and $\alpha>2$;

c) $u(0,x)\leq u_{+}(0,x),\quad x\in R$.

Then, there exists a global solution to problem (1)–(2) in Q which satisfies the following estimate:

$$u(0,x)\leq u_{+}(0,x),\ x\in R,$$

where

$$u_{+}(t,x)=\overline{u}(t)\overline{f}(\xi).$$

Proof. To prove the theorem, we employ the comparison principle for solutions. Following the comparison methods established in [9], we consider the function $u_{+}(t,x)$. Substituting (8) into (1), we obtain the following inequality:

$$f^{\alpha}\frac{d}{d\xi}\left(\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right)+\frac{1}{\alpha+p-q}(f^{q-1}-f)+\frac{1}{p}\xi\frac{df}{d\xi}\leq 0. \tag{11}$$

Given the specific form of the function $\overline{f}(\xi)$ in (7), inequality (9) can be rewritten as follows:

$$\frac{1}{\alpha+p-q}B^{q-\alpha-p}(a)^{\frac{(p-1)(q-2)}{\alpha+p-2}}-\left(\frac{p-1}{\alpha+p-2}\right)^{p}(1-\alpha)\leq 0. \tag{12}$$

It is known that $\overline{f}(\xi)$ is a positive function and $q-\alpha-p\leq 0$. Consequently, according to the comparison theorem, there exists a global solution to problem (1)–(2) in Q such that the estimate $u(0,x)\leq u_{+}(0,x)$ holds. The proof of the theorem is completed.

Asymptotic of the self-similar solutions.

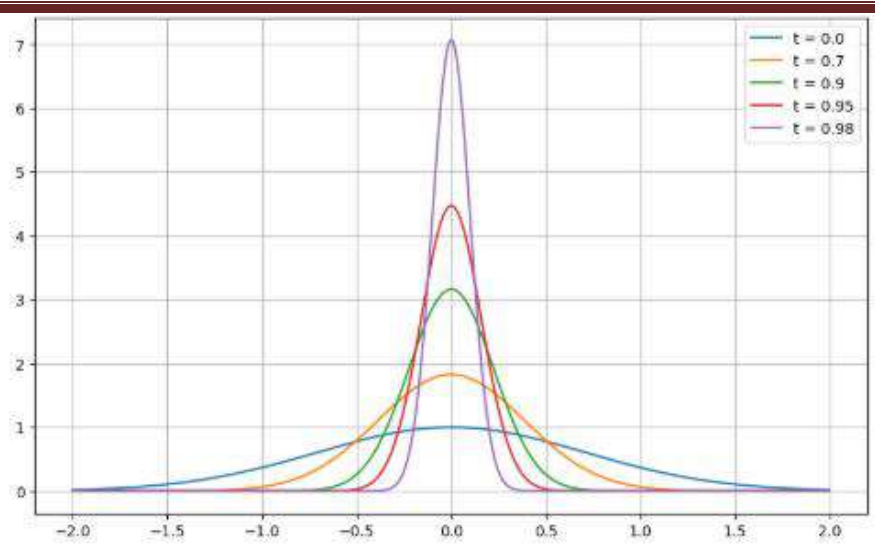
Theorem 3. Let the conditions

$$\alpha+p-q<0\quad\text{and}\quad\frac{(p-2)(q-2)}{\alpha+p-2}<0$$

be satisfied. Then the blow-up solution of equation (1) has the following asymptotic behavior as $t\to T$:

$$u(t,x)\to\left(\frac{1}{q-2}\right)^{\frac{1}{q-2}}(T-t)^{\frac{1}{q-2}}\cdot\left(a+\left((x)(T-t)^{-\frac{\alpha+p-q}{p(2-q)}}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{\alpha+p-2}}.$$

When the exact solution cannot be obtained, deriving an estimate or determining its asymptotic behavior is still considered a significant result. The constructed lower and upper solutions provide preliminary information about the qualitative properties of the exact solution.



Conclusion. The Cauchy problem for a non-divergent nonlinear parabolic equation with an absorption term, describing combustion processes, was investigated using the comparison method. A fundamental contribution of this work is the construction of both subsolution and supsolution corresponding to the problem, which allowed for the establishment of rigorous lower and upper bounds for the main solution. Furthermore, the asymptotic representations of self-similar solutions were derived, and their qualitative evolution was analyzed. It was established that the coefficients of the leading terms of the asymptotic expansion satisfy a specific nonlinear algebraic equation. These results underscore the critical role of nonlinear diffusion and heat generation in combustion dynamics. The obtained findings significantly contribute to the theoretical framework of nonlinear parabolic models used in heat transfer and chemical kinetics.

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