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SECOND-ORDER RENEWAL STRUCTURE IN SUBCRITICAL BELLMAN-HARRIS PROCESSES WITH HEAVY-TAILED LIFETIMES

Imomov A'zam Abdurakhimovich,
Karshi State University, Professor of the Department of Algebra and Geometry
Fayzullaeva Sevara Akbar kizi,
Navoi State University
1st-year basic doctoral student

Abstract. *For a subcritical Bellman–Harris process with offspring mean $m \in (0,1)$ and a lifetime law whose tail $\overline{G}(t) = t^{-\alpha} L(t)$ is regularly varying, we obtain second-order asymptotic expansion of the mean population size $A(t) = \mathrm{E}Z(t)$ in the finite-mean regime $\alpha > 1$. The leading term $\overline{G}(t)/(1-m)$ is classical. For $\alpha > 1$ the correction is shown to live at the scale of the lifetime density and to equal, $2m\mu_G g(t)/(1-m)^2$ where μ_G is the mean lifetime. The proofs separate a renewal-theoretic mechanism from a distributional one. On the renewal side we establish an exact two-term transfer law for the defective renewal operator generated by mG : its finite mass reproduces the forcing term, while its regularly varying tail contributes a further term at the density scale, weighted by the total mass of the forcing term. On the distributional side the mean defect is driven by the convolution defect $\overline{G^{*2}} - 2\overline{G}$, Above the threshold that defect carries total mass zero, which removes the second term of the transfer law from the expansion of the mean.*

Keywords: *Bellman–Harris process, age-dependent branching, defective renewal equation, regular variation, subexponential density, convolution defect, Karamata Tauberian theorem, second-order asymptotics.*

OG'IR DUMLI UMR KO'RISH DAVRLARIGA EGA BO'LGAN SUBKRITIK BELLMAN-HARRIS JARAYONLARIDAGI IKKINCHI TARTIBLI TIKLANISH TUZILMASI

Annotatsiya. *Avlod o'rtacha soni $m \in (0,1)$ bo'lgan subkritik Bellman-Harris jarayoni va dumi muntazam o'zgaruvchi bo'lgan umr ko'rish taqsimoti uchun, ya'ni $\overline{G}(t) = t^{-\alpha} L(t)$, bu yerda L(t) _ sekin o'zgaruvchi funksiya, $\alpha > 1$ bo'lgan chekli o'rtacha qiymat rejimida o'rtacha populyatsiya sonining ikkinchi tartibli asimptotik yoyilmasi olinadi. Bu yerda $A(t) = \mathrm{E}Z(t)$ t vaqtdagi o'rtacha populyatsiya sonini ifodalaydi. Yetakchi had $\overline{G}(t)/(1-m)$ klassik natija hisoblanadi. $\alpha > 1$ bo'lganda ikkinchi tartibli tuzatish umr ko'rish taqsimoti zichligi g(t)g(t) shkalasida ekanligi va $2m\mu_G g(t)/(1-m)^2$ ga tengligi ko'rsatiladi. Bu yerda μ_G — umr ko'rishning o'rtacha davomiyligi. Isbotlash jarayonida yangilanish nazariyasiga asoslangan mexanizm taqsimotga oid mexanizmdan ajratiladi. Yangilanish nazariyasi nuqtayi nazaridan mG tomonidan hosil qilinadigan nuqsonli yangilanish operatori uchun aniq ikki hadli uzatish qonuni o'rnatiladi: uning chekli massasi majburlovchi hadni qayta tiklaydi, muntazam o'zgaruvchi dumi esa zichlik shkalasida majburlovchi hadning umumiy massasi bilan og'irliklangan qo'shimcha hadni hosil qiladi. Taqsimot nuqtayi nazaridan esa o'rtacha qiymatning nuqsoni konvolyutsiya nuqsoni $\overline{G^{*2}} - 2\overline{G}$, orqali aniqlanadi. Belgilangan chegaradan yuqorida ushbu nuqsonning umumiy massasi nolga teng bo'lib, natijada uzatish qonunining ikkinchi hadi o'rtacha qiymatning asimptotik yoyilmasidan yo'qoladi.*

Kalit so'zlar*: Bellman-Harris jarayoni, yoshga bog'liq tarmoqlanish, nuqsonli yangilanish tenglamasi, muntazam o'zgarish, subeksponensial zichlik, konvolyutsiya nuqsoni, Karamata Tauberian teoremasi, ikkinchi darajali asimptotiklar.*

СТРУКТУРА ВОССТАНОВЛЕНИЯ ВТОРОГО ПОРЯДКА В ДОКРИТИЧЕСКИХ ПРОЦЕССАХ БЕЛЛМАНА-ХАРРИСА С ТЯЖЁЛЫМИ ХВОСТАМИ РАСПРЕДЕЛЕНИЯ ВРЕМЕНИ ЖИЗНИ

Аннотация. *Для докритического процесса Беллмана-Харриса со средним числом потомков $m \in (0,1)$ и законом распределения времени жизни, «хвост» которого $\overline{G}(t) = t^{-\alpha} L(t)$ является*

правильно меняющейся функцией, мы получаем асимптотическое разложение второго порядка для средней численности популяции $A(t)=\mathrm{E}Z(t)$ *в режиме конечного среднего значения* $\alpha>1$. *Главный член* $\overline{G}(t)/(1-m)$ *является классическим. Показано, что при* $\alpha>1$ *поправка имеет порядок плотности распределения времени жизни и равна* $2m\mu_G g(t)/(1-m)^2$, *где* μ_G *— среднее время жизни. В доказательствах разделяются механизм, основанный на теории восстановления, и механизм, связанный с распределением вероятностей. В рамках теории восстановления мы устанавливаем точный двухчленный закон переноса для дефектного оператора восстановления, порождённого функцией mG: его конечная масса воспроизводит правую часть уравнения, тогда как его правильно меняющийся хвост даёт дополнительный член на масштабе плотности, взвешенный полной массой правой части. С точки зрения распределения дефект среднего значения определяется дефектом свёртки* $\overline{G^{*2}}-2\overline{G}$, *При* $\alpha>1$ *этот дефект имеет нулевую полную массу, в результате чего второй член закона переноса исключается из асимптотического разложения среднего значения.*

Ключевые слова: *процесс Беллмана-Харриса, ветвящийся процесс с зависимостью от возраста, дефектное уравнение восстановления, правильное изменение, субэкспоненциальная плотность, дефект свёртки, тауберова теорема Караматы, асимптотика второго порядка.*

Introduction. In the Bellman–Harris branching process a particle lives for a random time drawn from a fixed lifetime law and, at the end of that time, is replaced by a random number of independent offspring behaving in the same way. The model goes back to [2] and occupies a settled place in the theory of age-dependent branching, where it is treated at length in [7], [10] and [1]. Writing $Z(t)$ for the number of particles alive at time t, $A(t)=\mathrm{E}Z(t)$ for the mean population size, G for the lifetime law and $\overline{G}:=1-G$ for its tail, the mean satisfies a renewal equation, and the whole asymptotic analysis of A is an exercise in renewal theory with $\overline{G}$ as forcing term. In the subcritical case the offspring mean m is less than one and the renewal equation is defective. When the lifetime tail is light, the classical treatment tilts the equation by the Malthusian parameter and the mean decays exponentially. When the tail is heavy, no such parameter exists, the defective renewal measure has finite total mass, and the decay of the mean is polynomial and inherited directly from the tail: if the tail is regularly varying then

$$A(t)\sim\frac{1}{1-m}\overline{G}(t)\qquad as\quad t\to\infty,$$

the branching mechanism entering only through the total mass $1/(1-m)$ of the renewal measure. This first-order statement rests on the subexponentiality of regularly varying tails, established for branching applications in [4], and is classical.

Once the principal term is removed, what remains is a residual whose scale and constant are not decided by the lifetime tail alone, and the present paper is devoted to identifying them. The mean population size turns out to admit a genuine two-term expansion, and for the finite-mean regime $\alpha>1$ we identify the second-order term and show that it is governed by the lifetime density, the quantity that governs the second order is not the tail itself but the amount by which the tail $\overline{G^{*2}}$ of the twofold convolution fails to double it, that is the convolution defect The quantity

$$r_G(t):=\overline{G^{*2}}(t)-2\overline{G}(t).$$

Subexponentiality says exactly that r_G is of smaller order than $\overline{G}$, and it says nothing more; but it is` precisely this small difference that carries the first nonvanishing correction to the mean. Asymptotics of convolution tails at this level of precision have a literature of their own, going back to [9], and in this paper we focus on the finite-mean regime $\alpha>1$, where the convolution defect admits a density-scale asymptotic. Our analysis separates two mechanisms that are usually entangled. The first is renewal-theoretic and asks how the defective renewal operator propagates an asymptotic scale. The second is distributional and asks what the convolution defect actually is. The separation is worth making because the first mechanism turns out to be more delicate than it looks. The renewal measure generated by mG is finite, which invites the belief that it simply multiplies its forcing term by its mass and preserves the scale. That belief is false: the renewal measure has a regularly varying tail of the same order as the lifetime tail, and the correct transfer law carries a second term, produced by that tail and weighted by the total mass of the forcing term. We

establish the law in this exact two-term form in Section 5, and we then show that the Bellman–Harris defect escapes its second term for a reason that is exact rather than asymptotic, namely that the convolution defect carries no mass at all. With the operator understood, the distributional half decides the answer. For a tail index $\alpha > 1$ the mean lifetime μ_G is finite and the convolution defect lives at the scale of the lifetime density g,

$$r_G(t) \sim 2\mu_G g(t) \quad as \quad t \to \infty,$$

the two equal contributions coming from the two ends of the convolution range: displacements of order one, where what survives is the local variation of the tail, and displacements close to t , where the density is seen far out. Combined with the transfer law this yields the first of our two main results,

$$A(t) = \frac{1}{1-m}\overline{G}(t) + \frac{2m\mu_G}{(1-m)^2} g(t) + o\big(g(t)\big),$$

a two-scale expansion in which the leading term is inherited from the tail while the correction is governed by its local variation, and in which the branching mechanism enters the correction only through the universal factor $2m/(1-m)^2$.

MODEL AND RENEWAL EQUATION

A single ancestor is born at time zero. It lives for a random time with distribution function G on [0, ∞), nondegenerate and with G(0) = 0, and at the end of its life it is replaced by a random number of offspring with probability generating function

$$f(z) = \sum_{k\geq 0} p_k z^k, \qquad z \in [0,1],$$

each of which then behaves as the ancestor did, independently of everything else. We write

$$m := f'(1) \in (0,1),$$

so that the process is subcritical throughout, and we let Z(t) denote the number of particles alive at time t and

$$A(t) = \mathrm{E}Z(t), \quad t \geq 0$$

its mean. The lifetime tail is written $\overline{G} := 1 - G$, the n-fold Stieltjes convolution of G with itself is G^{*n}, with G^{*0} the unit point mass at the origin, and the mean lifetime is

$$\mu_G := \int_0^\infty \overline{G}(u)\,du \in (0,\infty].$$

Convolutions of a function against a measure are always written in the order

$$(z * G)(t) = \int_{[0,t]} z(t-u)\, dG(u).$$

Since all the asymptotic hypotheses of the paper are stated in terms of regular variation, we recall the class before using it. A positive measurable function φ on a neighbourhood of infinity belongs to $\Re V_\sigma(\infty)$, the class of regularly varying functions of index σ at infinity, if

$$\varphi(\lambda t)/\varphi(t) \to \lambda^\sigma \ as \ t \to \infty$$

for every λ > 0. The functions of index zero are the slowly varying ones, and every $\varphi \in \Re V_\sigma(\infty)$ has the form $t^\sigma L(t)$ with $L \in \Re V_0(\infty)$. We use throughout the standard properties of these classes as presented in [3]. Conditioning on the lifetime τ of the ancestor gives the mean function at once. On the event $\{\tau > t\}$ the ancestor is still alive and contributes one particle; on the complementary event it has been replaced, at time $u \leq t,$, by an expected number m of independent copies of the whole processstarted afresh. Hence

$$A(t) = \overline{G}(t) + m(A * G)(t), \quad t \geq 0, \tag{2.1}$$

and A is the unique locally bounded solution of (2.1). The kernel of (2.1) is the measure mG, of total mass m < 1, so the equation is a defective renewal equation. Its resolvent is the renewal measure

$$U_m := \sum_{n\geq 0} (mG)^{*n}, \qquad with \quad U_m(\infty) = \frac{1}{1-m}, \tag{2.2}$$

a finite measure whose term with n = 0 is the unit point mass at the origin, and iterating (2.1) gives the resolvent representation

$$A = \overline{G} * U_m. \tag{2.3}$$

Hypothesis (G). The lifetime tail is regularly varying,

$$\overline{G}(t) = t^{-\alpha} L(t), \quad t > 0,$$

where $\alpha > 1$ *and* $L \in \mathcal{V}_0(\infty)$.

Hypothesis (H). The law G is absolutely continuous, with a density g that is locally bounded on [0, ∞) and ultimately monotone.

FIRST-ORDER ASYMPTOTICS

We record the principal term, both because it fixes the object from which the defect is measured and because its proof already displays the mechanism, a geometric mixture of convolution tails, that the rest of the paper refines.

Theorem 1 (First-order asymptotics of the mean function). *Let* $Z(t)$ *be a subcritical Bellman–Harris process with offspring mean* $m \in (0,1)$ *whose lifetime law satisfies* $\overline{G} \in \mathcal{RV}_{-\alpha}(\infty)$ *for some* $\alpha > 0$. *Then*

$$A(t) \sim \frac{1}{1-m} \overline{G}(t) \qquad as \quad t \to \infty \tag{3.1}$$

Proof. By (2.3) and the definition (2.2) of the renewal measure,

$$A(t) = \sum_{n \geq 0} m^n \left(\overline{G} * G^{*n}\right)(t),$$

and each summand telescopes: the convolution is the probability that $S_n \leq t < S_{n+1}$, where S_n is the sum of n independent lifetimes, that is

$$\left(\overline{G} * G^{*n}\right)(t) = \overline{G^{*(n+1)}}(t) - \overline{G^{*n}}(t),$$

with $\overline{G^{*0}}(t) = 0$ *for* $t > 0$. All these differences being nonnegative, the terms may be regrouped, and shifting the index in the first of the two resulting series gives

$$A(t) = (1-m) \sum_{n \geq 1} m^{n-1} \overline{G^{*n}}(t), \tag{3.2}$$

which exhibits the mean as the tail of a geometric mixture: if N is geometric on {1, 2, . . .} with $\mathrm{P}\{\mathrm{N} = n\} = (1-m)m^{n-1}$ and independent of the lifetimes, then $A(t) = \mathrm{P}\{S_N > t\}$.

A regularly varying tail is subexponential, so $\overline{G^{*n}}(t) \sim n\overline{G}(t)$, for every fixed n, and Kesten's bound provides the domination needed to sum these relations: for each $\varepsilon > 0$ there is $K = K(\varepsilon)$ with $\overline{G^{*n}}(t) \leq K(1+\varepsilon)^n \overline{G}(t)$ *for all* $n \geq 1$ and all $t \geq 0$; see [5] or [6]. Choosing ε so small that $m(1+\varepsilon) < 1$ makes the series in (3.2), divided by $\overline{G}(t)$, dominated term by term by the convergent numerical series $K(1-m)\sum_{n\geq 1} m^{n-1}(1+\varepsilon)^n$, and dominated convergence for series gives

$$\frac{A(t)}{\overline{G}(t)} \to (1-m) \sum_{n \geq 1} n m^{n-1} = \frac{1}{1-m} \quad as \quad t \to \infty,$$

which is (3.1).

The proof used regular variation only through subexponentiality, and the branching mechanism only through the mean m. Both facts are worth keeping in view, because the second-order analysis will break this symmetry: the correction term is sensitive to the finer structure of G, and it will feel the renewal measure through more than its total mass.

THE DEFECT EQUATION

We now measure the mean against its principal term. Put

$$\rho(t) := A(t) - \frac{1}{1-m}\overline{G}(t), \qquad t \geq 0 \tag{4.1}$$

and call ρ the mean defect. Theorem 1 says that $\rho = o\left(\overline{G}\right)$, and the object of the paper is to say what ρ actually is. The following lemma reduces that question to a renewal equation of the same defective type as (2.1), but with a forcing term that is one order smaller.

Lemma 1 (The defect equation). The elementary identity

$$\left(\overline{G} * G\right)(t) = \overline{G^{*2}}(t) - \overline{G}(t), \quad t \geq 0, \tag{4.2}$$

holds, and the mean defect satisfies the defective renewal equation

$$\rho = \frac{m}{1-m} r_G + m(\rho * G), \qquad where \quad r_G := \overline{G^{*2}} - 2\overline{G}, \tag{4.3}$$

whose unique locally bounded solution is

$$\rho = \frac{m}{1-m}(r_G * U_m). \tag{4.4}$$

The second-order problem has therefore split into two questions that no longer interfere: how the operator $*U_m$ transforms an asymptotic scale, and what the asymptotic behaviour of r_G is. The first is settled in Section 5, in a form valid for any forcing term, and the second occupies Sections 6 and 7, in the present paper, we treat the finite-mean $\alpha > 1$ in which the convolution defect is of the same scale as the lifetime density

THE TRANSFER MECHANISM

The transfer law. We can now state the exact action of the defective renewal operator on a forcing term living at the density scale. The convolution of a function against a measure is written throughout in the order $(\phi * U_m)(t) = \int_{[0,t]} \phi(t-u)\, dU_m(u)$.

Theorem 2 (Two-term transfer law). *Assume Hypotheses* (G) *and* (H)*, and let* ϕ *be a measurable function on [0, ∞) which is absolutely integrable and satisfies*

$$\frac{\phi(t)}{g(t)} \to c \quad as \quad t \to \infty$$

for some constant c. Put $J := \int_0^\infty \phi(v)\, dv$*. . Then*

$$(\phi * U_m)(t) = \left[\frac{c}{1-m} + \frac{mJ}{(1-m)^2}\right] g(t) + o(g(t)) \quad as \quad t \to \infty. \tag{5.1}$$

The convolution defect carries no mass. The transfer law shows that the second-order be-haviour of the mean defect is decided by two numbers attached to the forcing term: its asymptotic constant at the density scale, and its total mass. For the convolution defect the second of these is not merely small, it vanishes identically. This is the exact cancellation that keeps the Bellman–Harris expansion clean.

Lemma 3 (Vanishing mass of the convolution defect). *Assume Hypothesis (G). Then the convolution defect* $r_G(t)/g(t)$ *is absolutely integrable on [0, ∞) and*

$$\int_0^\infty r_G(v)\, dv = 0 \tag{5.2}$$

Proof. Let τ1 and τ2 be independent lifetimes with law G. Hypothesis (G) gives $\alpha > 1$, hence $\mu_G = \mathrm{E}\tau_1 < \infty$, and integrating the tail of a nonnegative variable yields

$$\int_0^\infty \overline{G^{*2}}(v)\, dv = \mathrm{E}[\tau_1 + \tau_2] = 2\mu_G \quad and \quad \int_0^\infty 2\overline{G}(v)\, dv = 2\mu_G.$$

Both integrals being finite, the difference r_G is absolutely integrable and its integral is the difference of the two values, namely zero, which is (5.2)

Theorem 3 (Transfer to the mean defect). *Assume Hypotheses (G) and (H), and suppose that* $r_G(t)/g(t)$ *tends to a limit cr . Then the mean defect (4.1) satisfies*

$$\rho(t)=\frac{mc_r}{(1-m)^2}g(t)+o(g(t)) \quad as \quad t\to\infty. \tag{5.3}$$

Proof. By the resolvent form of the defect equation, $\rho=(m/(1-m))r_G*U_m$. The forcing term r_G is absolutely integrable by Lemma 3 and satisfies $r_G/g\to c_r$ by hypothesis, so Theorem 2 applies with $c=c_r \; and \; J=\int_0^\infty r_G$. That mass vanishes by (5.2), so the second term of (5.1) is absent and

$$(r_G*U_m)(t)=\frac{c_r}{1-m}g(t)+o(g(t)).$$

Multiplying by $m/1-m$ gives (5.3)

Theorem 3 completes the operator-theoretic half of the analysis. What remains is the purely distributional constant c_r, identified in Section 6, where it is shown that $r_G(t)\sim 2\mu_G g(t)$ and hence that the mean defect obeys $\rho(t)\sim 2\mu_G g(t)/(1-m)^2$. It is worth recording what the two halves contribute separately: the factor $m/(1-m)^2$ in (5.3) is the renewal mass applied twice, once in passing from the mean equation to the defect equation and once in the resolvent, and no trace of the tail of the renewal measure survives in it. Had the forcing term carried a nonzero mass, a second term of the same order would have appeared and the expansion of the mean would have acquired a contribution having nothing to do with the local variation of the lifetime tail.

THE CONVOLUTION DEFECT AND THE SECOND-ORDER EXPANSION

Theorem 3 has reduced the second-order problem to a single distributional constant, the limit of $r_G(t)/g(t)$. Nothing branching-theoretic is left in the question: what remains is to decide by how much the convolution tail $\overline{G^{*2}}$ deviates from its subexponential approximation $2\overline{G}$, and on what scale. We begin with an exact representation which makes the mechanism of the deviation visible, and then evaluate it.

Proposition 1 (The convolution defect at the density scale). *Assume Hypotheses (G) and (H). Then*

$$r_G(t)=\overline{G^{*2}}(t)-2\overline{G}(t)\sim 2\mu_G g(t) \qquad as \quad t\to\infty \tag{6.1}$$

Proof. Being integrable and ultimately monotone, the density g is ultimately decreasing, and by (5.1) it is regularly varying of index $-\alpha-1$. Fix an exponent

$$\beta\in\left(\frac{\alpha+1}{2\alpha},1\right),$$

an interval which is nonempty precisely because $\alpha>1.$, and set $h(t):=t^\beta$, so that $h(t)\to\infty$ and $h(t)=o(t)$. Splitting the integral in (6.2) at $h(t)$ and at $t-h(t)$ we write

$$r_G(t)=\mathrm{I}_1(t)+\mathrm{I}_2(t)+\mathrm{I}_3(t)-\overline{G}^2(t),$$

the three integrals being taken over $[o,h(t)]$, over $[h(t),t-h(t)]$ and over $[t-h(t),t]$ respectively. The two edges will each produce $\mu_G g(t)$, while the middle range and the centring term will be negligible. On the left edge the increment is an integral of the density over a short interval,

$$\overline{G}(t-u)-\overline{G}(t)=\int_{t-u}^{t}g(v)dv, \qquad 0\le u\le h(t),$$

and since $h(t)=o(t)$, the uniform convergence theorem for regularly varying functions [3, Th 1.5.2] gives

$$\sup_{t-h(t)\le v\le t}\left|\frac{g(v)}{g(t)}-1\right|\to 0 \qquad as \quad t\to\infty.$$

Hence $\overline{G}(t-u)-\overline{G}(t)=ug(t)(1+o(1))$ uniformly in $u\in[0,h(t)]$, so that

$$\mathrm{I}_1(t)=g(t)(1+o(1))\int_0^{h(t)}ug(u)du.$$

The mean being finite by Hypothesis (G), the integral on the right converges to

$$\int_0^\infty u g(u)\,du = \mu_G,$$

and therefore $\mathrm{I}_1(t) \sim \mu_G g(t)$.).

On the middle range monotonicity alone suffices. Since $\overline{G}$ is nonincreasing and nonnegative, the integrand of I_2 lies between 0 and $\overline{G}(t-u)$, while $g(u) \le g(h(t))$ there, so that after the substitution $v = t-u$

$$0 \le \mathrm{I}_2(t) \le g(h(t)) \int_{h(t)}^{t-h(t)} \overline{G}(v)\,dv \le g(h(t)) \int_{h(t)}^{\infty} \overline{G}(v)\,dv$$

By Karamata's theorem [3, Th 1.5.11] the last integral is asymptotically $h(t)\overline{G}(h(t))/(\alpha-1)$,

whence $\mathrm{I}_2(t) = O\left(h^{-2\alpha}(t) L^2(h(t))\right)$ and

$$\frac{\mathrm{I}_2(t)}{g(t)} = O\left(t^{\alpha+1-2\alpha\beta} \frac{L^2(t^\beta)}{L(t)} \right).$$

The exponent $\alpha+1-2\alpha\beta$ is strictly negative by the choice of β, and by Potter's bounds [3, Th 1.5.6] the slowly varying quotient is $O(t^\eta)$ for every $\eta > 0$; choosing $\eta < 2\alpha\beta - \alpha - 1$ we conclude that $\mathrm{I}_2(t) = o(g(t))$. On the right edge the substitution $v = t-u$ turns I_3 into

$$\mathrm{I}_3(t) = \int_0^{h(t)} \overline{G}(v)\, g(t-v)\,dv - \overline{G}(t) \int_0^{h(t)} g(t-v)\,dv$$

an expression in which the density is now evaluated far out and the tail on a short range. The uniform convergence theorem applies to g exactly as before and gives $g(t-v) = g(t)(1+o(1))$ uniformly in $v \in [0, h(t)]$, so the first integral is asymptotically $g(t)\int_0^{h(t)} \overline{G}(v)\,dv$ and hence, the mean being finite, equivalent to $\mu_G g(t)$ The second is bounded by $\overline{G}(t)\overline{G}(t-h(t))$, and since $h(t) = o(t)$ this is asymptotically $\overline{G}^2(t)$. Consequently the right edge contributes

$$\mathrm{I}_3(t) = \mu_G g(t) + \mathrm{O}\left(\overline{G}^2(t)\right).$$

Finally, both the centring term and the remainder just produced are of order $\overline{G}^2(t) \approx t^{-2\alpha} L^2(t)$, and by (5.1),

$$\frac{\overline{G}^2(t)}{g(t)} \approx \frac{t^{1-\alpha} L(t)}{\alpha} \to 0 \qquad as \qquad t \to \infty$$

because $\alpha > 1$. Collecting the three ranges therefore gives $r_G(t) = 2\mu_G g(t) + o(g(t))$, which is (6.1).

Theorem 4 (Second-order asymptotics of the Bellman–Harris mean). *Let Z(t) be a subcritical Bellman–Harris process with offspring mean* $m \in (0,1)$ *whose lifetime law satisfies Hypotheses (G) and (H). Then, for* $A(t) = \mathrm{E}Z(t)$,

$$A(t) = \frac{1}{1-m}\overline{G}(t) + \frac{2m\mu_G}{(1-m)^2} g(t) + o g(t) \quad as \ t \to \infty \tag{6.2}$$

and, in terms of the tail parameters alone,

$$A(t) = \frac{1}{1-m} t^{-\alpha} L(t) + \frac{2\alpha m \mu_G}{(1-m)^2} t^{-\alpha-1} L(t) + o\left(t^{-\alpha-1} L(t)\right). \tag{6.3}$$

Proof. Proposition 1 gives $r_G(t)/g(t) \to 2\mu_G$, so Theorem 3 applies with $c_r = 2\mu_G$ and yields

$\rho(t) = 2m\mu_G g(t)/(1-m)^2 + o(g(t))$. Adding the principal term $\overline{G}(t)/(1-m)$ gives (6.2), and (6.3) follows from (5.1).

Remark 3. The expansion (6.2) is two-scale, and the two scales are of different natures. The principal term is inherited from the lifetime tail and the branching mechanism enters it only through the mass $1/(1-m)$ of the defective renewal measure. The correction lives one power of t lower, at the scale of the lifetime density, so it is governed not by the size of the tail but by its local variation; the branching mechanism enters it through the universal factor $m/(1-m)^2$, and the lifetime law through the single number $2\mu_G$. It is worth stressing that the tail of the renewal measure, which by (5.3) is itself of the principal order $\overline{G}(t)$,, leaves no trace in (6.2). Its contribution is removed exactly, by the vanishing mass (5.2) of the convolution defect, and not by any estimate.

A WORKED EXAMPLE

The two expansions have been proved under hypotheses that leave the lifetime law otherwise free, so it is worth exhibiting a family for which every constant is available in closed form and can be compared with the mean function itself. The Pareto family provides an explicit example satisfying Hypotheses (G) and (H), with all constants in the second-order expansion available in closed form.

7.1. The family and its constants. Fix $\alpha > 0$ and take

$$\overline{G}(t) = (1+t)^{-\alpha} \quad with \quad g(t) = \alpha(1+t)^{-\alpha-1}, \quad t \geq 0. \tag{7.1}$$

The law is absolutely continuous, its density is decreasing on the whole half-line, and the tail is of the form $t^{-\alpha}L(t)$ $with$ $L(t) = (1+1/t)^{-\alpha}$, t)−α , which is slowly varying and tends to one. Hypotheses (G) and (H) therefore hold for every $\alpha > 1$,, and Hypothesis (G_0) for every $\alpha \in (0,1)$.. The mean lifetime is

$$\mu_G = \int_0^\infty (1+u)^{-\alpha}\, du = \frac{1}{\alpha - 1} \quad for \ \alpha > 1,$$

and is infinite otherwise. Substituting these expressions into (6.2) gives the following explicit second-order correction:

$$\rho(t) \square \frac{2\alpha m}{(1-m)^2(\alpha-1)}(1+t)^{-\alpha-1} \tag{7.2}$$

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